ON THE CONDITION $(\dagger)_A$ IN KERZ–SAITO–TAMME'S PAPERS

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Abstract. In this short note, we study the condition $(\dagger)_A$ that appeared in the work of Kerz–Saito–Tamme [KST19]. We propose a variant in the non-Noetherian setting, which concerns specifically rigid-analytic varieties over a (not necessarily discretely valued) nonarchimedean field k. We explain that most arguments of Kerz–Saito–Tamme go through verbatim, conditionally on a pro-vanishing result.

Contents

This note is originated in a discussion with Christian Dahlhausen and Can Yaylali, where we attempted to remove the Noetherianness assumption in Kerz–Saito–Tamme's work [KST19, KST23]. In this note, we consider the rigid-analytic case, i.e. for geometric objects over a nonarchimedean field k (or related formal or algebraic schemes over k°). Note that if k is discretely valued, then we are in the Noetherian case as in *op. cit.*

1. The $\mathrm{N}^\pi\mathrm{K}$ vanishes in the semistable case

Let k be a nonarchimedean field, i.e. a complete non-trivially real-valued field, whose ring of integer is denoted by k°, with maximal ideal $k^{\circ\circ} = \mathfrak{m}$. Here, the non-triviality of valuation implies the existence of a topologically nilpotent unit $\pi \in k^\circ$, which we call a *pseudo-uniformiser*[1], which we shall not fix and which depends on the context.

There are two spectra-valued construction, $\mathrm{NK}(-)$ and $\mathrm{N}^\pi\mathrm{K}(-)$, which measure the failure of $\mathbf{A}^1$-homotopy invariance and, respectively, certain "pro"-$\mathbf{A}^1$-homotopy invariance (related to the analytic $\mathbf{A}^1$-homotopy invariance), of a commutative ring (see Definition 1.1.4).

Our main goal of this section is to show that $\mathrm{N}^\pi\mathrm{K}(X) \simeq 0$ for strictly semistable schemes X over k° (Corollary 1.5.6) and a (partially) completed version (Proposition 1.6.6). When the

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[1] The terminology of *pseudo-* reflects the fact that it is not necessarily a uniformiser, i.e. the maximal ideal $\mathfrak{m}$ of k° is not the principal ideal πk°; however, we have $\mathfrak{m} = \sqrt{\pi k^\circ}$.

valuation on k is discrete and π is a *uniformiser*, then X is regular and we have the stronger vanishing $\mathrm{NK}(X) \simeq 0$ by a classical result of Weibel [Wei13, Theorem V.6.3].

Moreover, as the proof strategy will demonstrate, the condition of being "semistable" can be generalised, for example, to being "polystable" (see Remark 1.5.7 (i)) or "with toric l.c.i. singularities" (see Remark 1.5.7 (ii)).

1.1. **Definitions and main theorem.**

1.1.1. **Definition.** A k°-scheme X is called *strictly semistable* if, Zariski-locally, it is étale over a standard semistable k°-algebra

$$S_{d,r} = k^\circ[t_0, t_1, \ldots, t_d]/(t_0 \cdots t_r - \pi), \tag{1.1.2}$$

where $0 \neq \pi \in k^\circ$ is a pseudo-uniformiser and $0 \leq r \leq d$.

A k°-algebra A is called *strictly semistable* if $\mathrm{Spec}(A)$ is a semistable k°-scheme.

We make similary definitions for formal schemes, replacing $k^\circ[t_0, t_1, \ldots, t_d]$ with the convergent power series $k^\circ\langle t_0, t_1, \ldots, t_d\rangle$.

1.1.3. **Remark.** Here, contrary to usual convention of semistability, we adopt a weaker convention, following [Tem17], that π is not necessarily a uniformiser of k, which is very practical in the general setting where k is not discretely valued.

1.1.4. **Definition.** (i) For any ring R, define the spectra

$$\mathrm{NK}(R) \coloneqq \mathrm{fib}(\mathrm{K}(R[T]) \to \mathrm{K}(R)).$$

(ii) Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any $\mathcal{O}$-algebra R, define the pro-spectra

$$\mathrm{N}^\pi\mathrm{K}(R) \coloneqq \mathrm{fib}(\underset{T\mapsto \pi T}{\text{“}\lim\text{”}} \mathrm{K}(R[T]) \to \mathrm{K}(R)).$$

This definition extends to arbitrary functor F on the category of (commutative rings) valued in (pro-)spectra.

1.1.5. **Remark.** If $\mathcal{O} = k^\circ$ for a nonarchimedean field k,which has rank one valuation, the definition $\mathrm{N}^\pi\mathrm{K}(R)$ is independent of the choice of the pseudo-uniformiser π.

In general, $\mathrm{N}^\pi\mathrm{K}(R)$ up to equivalence in $\mathrm{Pro(Sp)}$ depends only on the ideal $\sqrt{\pi\mathcal{O}}$.

1.1.6. **Lemma.** *We have*

$$\mathrm{K}(R[T]) \simeq \mathrm{K}(R) \oplus \mathrm{NK}(R) \quad \textit{in } \mathrm{Sp},$$
$$\underset{T\mapsto \pi T}{\text{“}\lim\text{”}} \mathrm{K}(R[T]) \simeq \mathrm{K}(R) \oplus \mathrm{N}^\pi\mathrm{K}(R) \quad \textit{in } \mathrm{Pro(Sp)}.$$

Proof. Both arrows are induced by the map $T \mapsto 0$, which has a canonical splitting induced by the natural embedding $R \mapsto R[T]$. □

1.1.7. **Lemma.** *We have* $\mathrm{N}^\pi\mathrm{K}(R) \simeq \text{“}\lim\text{”}_{T\mapsto\pi T}\, \mathrm{NK}(R)$ *in* $\mathrm{Pro(Sp)}$.

Proof. It is clear from the definition. □

1.1.8. **Lemma.** *If R has bounded π^∞-torsion, then* $\mathrm{N}^\pi\mathrm{K}(R) \simeq 0$ *in* $\mathrm{Pro(Sp)}$.

Proof. If R has bounded π^∞-torsion, then the filtered system $(R[T])_{T\mapsto\pi T}$ is equivalent to the constant system R. In particular, $\mathrm{N}^\pi\mathrm{K}(R) \simeq 0$ as pro-spectra. □

1.1.9. **Remark.** We have $\mathrm{N}^\pi\mathrm{K}(R) \not\simeq 0$ if $R = \mathcal{O}/\pi$ and $\pi\mathcal{O}$ is not a prime ideal[2]. Indeed, one can then choose a minimal prime ideal $\mathfrak{p}$ over $\pi\mathcal{O}$ and an element $a \in \mathfrak{p}\backslash\pi\mathcal{O}$, and then $1 + aT \in R[T]^\times$ represents a non-zero class in $\mathrm{NK}(R)$.

In order to formulate a variant of [KST19, Lemma 5.13 (ii)], we wish to prove the following theorem.

Theorem (Corollary 1.5.6)**.** *For any strictly semistable k°-algebra A, the pro-spectra $\mathrm{N}^\pi\mathrm{K}(A)$ is weakly contractible, i.e. $\mathrm{N}^\pi\mathrm{K}(A)$ becomes equivalent to zero in* $\mathrm{Pro}(\mathrm{Sp}^+)$.

Then it will be reasonable to consider a variant of the dagger condition (see 1.6.4) and prove a variant of [KST19, Lemma 5.13 (ii)] (see Proposition 1.6.6), which is the key to the works of Kerz–Saito–Tamme [KST19, KST23].

1.2. **Regular case.** Recollections on the regular (in a general sense) case to begin with. We want to go beyond Noetherian cases, so we need to consider the notation of regularity for non-Noetherian rings (see Definition 1.2.3).

Let us start with the baby case. Recall the following theorem:

1.2.1. **Theorem** (Kelly–Morrow, [KM21, Theorem 3.3])**.** *Let $\mathcal{O}$ be a valuation ring. Then*

(i) $\mathrm{K}(\mathcal{O}) \to \mathrm{K}(\mathcal{O}[T_1, \dots, T_d])$ *is an equivalence for any $d \geq 1$;*
(ii) $\mathrm{K}(\mathcal{O}) \to \mathrm{KH}(\mathcal{O})$ *is an equivalence;*
(iii) $\mathrm{K}_n(\mathcal{O}) = 0$ *for all $n < 0$.*

In particular, $\mathrm{NK}(\mathcal{O}) \simeq 0$.

Let us recall its slight generalisation:

1.2.2. **Theorem** (Antieau–Mathew–Morrow, [AMM22, Proposition 2.4])**.** *Let $\mathcal{O}$ be a valuation ring. Let A be a smooth $\mathcal{O}$-algebra. Then*

(i) $\mathrm{K}(A) \to \mathrm{K}(A[T_1, \dots, T_d])$ *is an equivalence for any $d \geq 1$;*
(ii) $\mathrm{K}(A) \to \mathrm{KH}(A)$ *is an equivalence;*
(iii) $\mathrm{K}_n(A) = 0$ *for all $n < 0$.*

In particular, $\mathrm{NK}(A) \simeq 0$.

The proof of it has actually been given Antieau–Mathew–Morrow [AMM22], but let us recall it for our own convenience.

1.2.3. **Definition** (Regularity of non-Noetherian rings)**.** Let R be a ring.

(i) R is called *coherent* if every finitely generated ideal of it is a finitely presented module.
(ii) R is *regular* if every finitely generated ideal (or equivalently, module) of it has finite projective dimension [Gla89, §6.2 Definition, Theorem 6.2.1].
(iii) R is called *stably coherent* if every polynomial extension of it (i.e. $R[T_1, \dots, T_d]$ with $d \geq 1$) is coherent.

1.2.4. **Lemma** ([Gla89, Theorem 6.2.5], [GL24, Lemma A.1])**.** *Let $R \to R'$ be a faithfully flat ring map. If R' is coherent (*resp. *stably coherent, coherent regular, stably coherent regular), so is R.* □

[2]In a valuation ring $\mathcal{O}$, the radical ideal $\sqrt{\pi\mathcal{O}}$ is always a prime ideal. Hence, the condition of $\pi\mathcal{O}$ being prime amounts to asking that $\pi\mathcal{O}$ is radical.

1.2.5. **Lemma** ([GL24, Lemma A.2]). *Let $R \to R'$ be an étale ring map. If R is coherent (*resp.* stably coherent, coherent regular, stably coherent regular), so is R'. The converse is true if the map is an étale covering (or equivalently, faithfully flat).* □

1.2.6. **Lemma** ([Gla89, Corollary 6.2.14]). *If R is stably coherent and regular, then every polynomial extension of R is regular.* □

1.2.7. **Remark.** Coherence is not necessarily stable under polynomial extensions.

1.2.8. **Corollary.** *Let $R \to R'$ be a smooth ring map with R stably coherent. If R is regular, so is R'. The converse is true if the map is a smooth covering (or equivalently, faithfully flat).*

Proof. Any smooth ring map is étale locally on the source and on the target an étale map from a polynomial algebra. The forward implication follows then from Lemma 1.2.5 and Lemma 1.2.6.

The converse follows directly from Lemma 1.2.4. □

Now one can prove Theorem 1.2.2.

1.2.9. **Proposition.** *Every valuation $\mathcal{O}$ is stably coherent regular.*

Proof. It is known that $\mathcal{O}$ is stably coherent [Gla89, Theorem 7.3.3].

Let us show that $\mathcal{O}$ is regular. Valuation rings have weak global dimension at most one since any finitely generated ideal of $\mathcal{O}$ is principal [Gla89, p. 25], and this dimension bounds the projective dimension of all finitely presented ideals (or equivalently, modules) [Gla89, Corollary 2.5.5, Corollary 2.5.6]. By definition, we see that $\mathcal{O}$ is regular. □

Proof of Theorem 1.2.2. [3] As explained by Kelly–Morrow in [KM21, Theorem 3.3], the assertions (i)–(iii) are consequences of the stably coherent regularity of A, which we claim now to be true[4]. By the smooth-local nature of coherent regularity (Corollary 1.2.8), one reduces to the case $A = \mathcal{O}$. Then it is Proposition 1.2.9. □

1.3. **Examples.** Let $\mathcal{O}$ be a valuation ring. In some particular statements below, it might to be necessary to impose $\mathcal{O} = k^\circ$ specifically.

Our goal is to generalise Theorem 1.2.2 to the semistable case.

(A). *Nodal algebras.* Consider first the case of the standard nodal algebra

$$A^{\mathrm{nodal}} \coloneqq \mathcal{O}[t_0, t_1]/(t_0 t_1 - \pi),$$

where $\pi \in k^\circ$ is a pseudo-uniformiser (1.1.2). Étale algebras over A^{nodal}, which provide étale local models of semistable models over k°, will be studied next but one.

1.3.1. **Remark.** If $\mathcal{O}$ is a discrete valuation ring, then A^{nodal} is regular if and only if π is a uniformiser. If $\mathcal{O}$ is the ring of integers k° where k is non-discretely valued, then A^{nodal} is never regular (in the sense of Definition 1.2.3). In fact, this is because A^{nodal} is non-Noetherian and $A^{\mathrm{nodal}} \otimes_{k^\circ} \tilde{k}$ is singular, and by applying [GL24, Proposition 3.9 (ii)][5].

[3] The proof of Kelly–Morrow [KM21, Theorem 3.3] relying on the countability assumption not present here, we follow instead more closely that of Banerjee–Sadhu [BS22, Theorem 1.1], with slight modifications. It turns out that the proof coincides with that of [AMM22, Proposition 2.4].

[4] This is actually the statement of [AMM22, Corollary 2.3]. But we give a proof of it below. At the moment when the proof below was written, I was unaware of *loc. cit.*, knowing only that it could be very elementary and classical.

[5] Maybe it is unnecessary to refer to this as there exists perhaps an simpler argument.

It is no surprise then to see that Theorem 1.2.2 could fail for nodal algebras.

1.3.2. **Theorem.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any smooth $\mathcal{O}$-algebra B, there is a canonical equivalence of spectra*

$$\mathrm{NK}(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq \mathrm{NK}(B/\pi) \tag{1.3.3}$$

and isomorphisms

$$\mathrm{K}_n(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq \mathrm{K}_n(B/\pi), \quad n < 0. \tag{1.3.4}$$

As a result, for A^{nodal}, the $\mathbf{A}^1$-homotopy invariance does not hold in general, while the negative K-groups do vanish. Instead, the *"pro"-$\mathbf{A}^1$-homotopy invariance*, i.e. the contratibility of $\mathrm{N}^{\pi}\mathrm{K}$, holds.

1.3.5. **Corollary.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit.*

(i) $\mathrm{NK}(A^{\mathrm{nodal}}) \simeq 0$ *if and only if* $\pi\mathcal{O} \subset \mathcal{O}$ *is a prime ideal;*
(ii) $\mathrm{N}^{\pi}\mathrm{K}(A^{\mathrm{nodal}}) \simeq 0$ *in* $\mathrm{Pro}(\mathrm{Sp})$*;*
(iii) $\mathrm{K}_0(A^{\mathrm{nodal}}) \simeq \mathbf{Z}$*, and* $\mathrm{K}_n(A^{\mathrm{nodal}}) = 0$ *for all* $n < 0$*;*
(iv) $\mathrm{NK}_n(A^{\mathrm{nodal}}) = 0$ *for all* $n \leq 0$*.*

More generally, for any smooth $\mathcal{O}$-algebra B,

(i') $\mathrm{NK}(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq 0$ *if and only if* $\pi\mathcal{O} \subset \mathcal{O}$ *is a prime ideal;*
(ii') $\mathrm{N}^{\pi}\mathrm{K}(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq 0$ *in* $\mathrm{Pro}(\mathrm{Sp})$*;*
(iii') $\mathrm{K}_n(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) = 0$ *for all* $n < 0$*;*
(iv') $\mathrm{NK}_n(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) = 0$ *for all* $n \leq 0$*.*

Proof. According to Theorem 1.3.2, we have $\mathrm{NK}(A^{\mathrm{nodal}}) \simeq \mathrm{NK}(\mathcal{O}/\pi)$ and $\mathrm{K}_n(A^{\mathrm{nodal}}) \simeq \mathrm{K}_n(\mathcal{O}/\pi)$ for $n < 0$.

(i) The "only if" direction follows from Remark 1.1.9. For the "if" direction, if $\pi\mathcal{O}$ is a prime ideal, Then $\mathcal{O}/\pi$ is still a valuation ring, so that $\mathrm{NK}(\mathcal{O}/\pi)$ is contractible by Theorem 1.2.1.

(ii) This follows from Lemma 1.1.8.

(iii) This is a consequence of the nil-invariance of non-positive K-groups, noticing that $\sqrt{\pi\mathcal{O}}$ is always a prime ideal. More precisely, $(\mathcal{O}/\pi)_{\mathrm{red}}$ is the filtered colimit of nil-thichekings $\mathcal{O}/I$ of $\mathcal{O}/\pi$. By nil-invariance of negative K-groups, we obtain $\mathrm{K}_n((\mathcal{O}/\pi)_{\mathrm{red}}) = \mathrm{colim}_I \mathrm{K}_n(\mathcal{O}/I) \simeq \mathrm{K}_n(\mathcal{O}/\pi)$ for $n \leq 0$. Also, $(\mathcal{O}/\pi)_{\mathrm{red}} \simeq \mathcal{O}/\sqrt{\pi\mathcal{O}}$ is a valuation ring, so $\mathrm{K}_0((\mathcal{O}/\pi)_{\mathrm{red}}) \simeq \mathbf{Z}$ and $\mathrm{K}_n((\mathcal{O}/\pi)_{\mathrm{red}}) = 0$ for $n < 0$.

(iv) By nil-invariance of non-positive K-groups, we have $\mathrm{NK}_n(A^{\mathrm{nodal}}) \simeq \mathrm{NK}_n((\mathcal{O}/\pi)_{\mathrm{red}})$ for $n \leq 0$, as in (iii), which vanishes by Theorem 1.2.1.

For (i')–(iv'), the proof is the same. □

Let us come back to the proof of Theorem 1.3.2. As A^{nodal} is not regular (Remark 1.3.1) in general (if $\pi \in \mathcal{O}$ is not a uniformiser), we are not able to proceed similarly as for Theorem 1.2.2. Instead, we will locate $\mathrm{Spec}(A^{\mathrm{nodal}})$ as an affine open in a quasi-smooth blow-up, and resort to the blow-up formula for algebraic K-theory.

1.3.6 (Quasi-smooth blow-ups and blow-up formula for algebraic K-theory)**.** Recall that a closed immersion $Z \subset X$ is called *quasi-smooth* if Zariski-locally on X, Z is defined by a regular sequence $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$; we then say that this is a closed immersion of codimension c.

In this case, we have a blow-up formula proved by Thomason [Tho93, Théorème 2.1] (or see [Kha20, Corollary 4.4]):

$$\mathrm{K}(\mathrm{Bl}_Z X) \simeq \mathrm{K}(X) \oplus \bigoplus_{k=1}^{c-1} \mathrm{K}(Z). \tag{1.3.7}$$

This holds after any flat base change, since blow-up commutes with flat base change [Sta18, Lemma 0805].

Proof of Theorem 1.3.2. Consider the scheme $X \coloneqq \mathrm{Spec}(\mathcal{O}[u])$ and the locus $Z \coloneqq V((u,\pi)) = \mathrm{Spec}(\mathcal{O}/\pi)$. The closed immersion $Z \hookrightarrow X$ is quasi-smooth of codimension two as it is defined by the regular sequence u, π of $\mathcal{O}[u]$. Let $\widetilde{X}$ be the blow-up $\mathrm{Bl}_Z X$. It is covered by two affine opens

$$\begin{aligned} D_+(u) &\simeq \mathrm{Spec}\Big(\mathcal{O}[u][\frac{\pi}{u}]\Big) \simeq \mathrm{Spec}(\mathcal{O}[u,\bar{u}]/(u\bar{u}-\pi)) \simeq \mathrm{Spec}(A^{\mathrm{nodal}}), && \bar{u} = \frac{\pi}{u}, \\ D_+(\pi) &\simeq \mathrm{Spec}\Big(\mathcal{O}[u][\frac{u}{\pi}]\Big) \simeq \mathrm{Spec}(\mathcal{O}[u']), && u' = \frac{u}{\pi}, \end{aligned}$$

whose intersection is

$$D_+(u) \cap D_+(\pi) \simeq \mathrm{Spec}(\mathcal{O}[u, u'^{\pm 1}]) \simeq \mathrm{Spec}(\mathcal{O}[u'^{\pm 1}]).$$

Now, on the one hand, the Mayer-Vietoris sequence of K-theory implies immediately the following cartesian square of spectra:

$$\begin{tikzcd} \mathrm{K}(\widetilde{X}) \arrow[r] \arrow[d] & \mathrm{K}(A^{\mathrm{nodal}}) \arrow[d] \\ \mathrm{K}(\mathcal{O}[u']) \arrow[r] & \mathrm{K}(\mathcal{O}[u'^{\pm 1}]). \end{tikzcd}$$

By Theorem 1.2.2, we obtain an equivalence of spectra

$$\mathrm{NK}(\widetilde{X}) \xrightarrow{\simeq} \mathrm{NK}(A^{\mathrm{nodal}}) \tag{1.3.8}$$

and surjections on negative K-groups

$$\mathrm{K}_n(\widetilde{X}) \twoheadrightarrow \mathrm{K}_n(A^{\mathrm{nodal}}), \quad n < 0, \tag{1.3.9}$$

which are all isomorphisms. Indeed, the injectivity is clear if $n < 1$; at $n = -1$, the sequence extends to a long exact sequence

$$\mathrm{K}_0(A^{\mathrm{nodal}}) \oplus \mathrm{K}_0(\mathcal{O}[u']) \xrightarrow{\beta} \mathrm{K}_0(\mathcal{O}[u'^{\pm 1}]) \to \mathrm{K}_{-1}(\widetilde{X}) \to \mathrm{K}_{-1}(A^{\mathrm{nodal}}) \to 0, \tag{1.3.10}$$

where β is surjective since $\mathrm{K}_0(\mathcal{O}[u']) \simeq \mathrm{K}_0(\mathcal{O}) \simeq \mathbf{Z}$ by Theorem 1.2.2, and also that there is an equivalence $\mathrm{K}_0(\mathcal{O}[u'^{\pm 1}]) \simeq \mathrm{K}_0(\mathcal{O}) \oplus \mathrm{K}_{-1}(\mathcal{O}) \oplus \mathrm{NK}_0(\mathcal{O}) \oplus \mathrm{NK}_0(\mathcal{O}) \simeq \mathbf{Z}$ by Bass Fundamental Theorem.

On the other hand, the blow-up formula (1.3.7) provides canonical equivalences

$$\begin{gathered} \mathrm{K}(\widetilde{X}) \simeq \mathrm{K}(\mathcal{O}[u]) \oplus \mathrm{K}(\mathcal{O}/\pi), \\ \mathrm{K}(\widetilde{X}[T]) \simeq \mathrm{K}(\mathcal{O}[u][T]) \oplus \mathrm{K}(\mathcal{O}/\pi[T]), \end{gathered}$$

where the latter holds because the blowup commutes with flat base change. Then, by Theorem 1.2.2, we obtain an equivalence of spectra:

$$\mathrm{NK}(A^{\mathrm{nodal}}) \xrightarrow{\simeq} \mathrm{NK}(\mathcal{O}/\pi) \tag{1.3.11}$$

and isomorphisms of negative K-groups

$$\mathrm{K}_n(A^{\mathrm{nodal}}) \xrightarrow{\simeq} \mathrm{K}_n(\mathcal{O}/\pi), \quad n < 0. \tag{1.3.12}$$

Now one can conclude the proof of (1.3.3) and (1.3.4) in the case where $B = \mathcal{O}$.

Now we treat the case of general smooth $\mathcal{O}$-algebras B. Since blow-up commutes with flat base change, and in view of Theorem 1.2.2, the same arguments as above apply to any smooth base change of A^{nodal}, and this completes the proof. □

(B). *Formal completion.* The π-adic completion is a way of producing other sources of rings generalising the ("pro"-)$\mathbf{A}^1$-homotopy invariance and negative K-group vanishing.

1.3.13 (K-theory and analytic isomorphism)**.** Let $A \to B$ be a (commutative) ring map. Let $s \in A$ be a non-zero-divisor in both A and B and such that $A/s^nA \xrightarrow{\simeq} B/s^nB$ for any $n \in \mathbf{N}$. Then $A \to B$ is a so-called *s-analytic isomorphism.* Then by Vorst–Weibel [Wei80, Theorem 1.3 (i)], we have a cartesian square of spectra

$$\begin{array}{ccc} \mathrm{K}(A) & \longrightarrow & \mathrm{K}(A[\frac{1}{s}]) \\ \downarrow & & \downarrow \\ \mathrm{K}(\widehat{A}) & \longrightarrow & \mathrm{K}(\widehat{A}[\frac{1}{s}]). \end{array} \tag{1.3.14}$$

Since $A[T] \to B[T]$ is then also an analytic isomorphism, the above applies to their polynomial extension with one variable, hence as a corollary [Wei80, Corollary 1.4], we have also a cartesian square of spectra

$$\begin{array}{ccc} \mathrm{NK}(A) & \longrightarrow & \mathrm{NK}(A[\frac{1}{s}]) \\ \downarrow & & \downarrow \\ \mathrm{NK}(\widehat{A}) & \longrightarrow & \mathrm{NK}(\widehat{A}[\frac{1}{s}]). \end{array} \tag{1.3.15}$$

1.3.16**.** Let A be a flat k°-algebra. Consider the π-adic completion $A \to \widehat{A}$, which is an analytic isomorphism along π (as $\widehat{A}$ is then also flat over k°).

Moreover, if $A[\frac{1}{\pi}]$ and $\widehat{A}[\frac{1}{\pi}]$ are both noetherian regular, by a classical result of Weibel [Wei13, Theorem V.6.3] they satisfy $\mathbf{A}^1$-homotopy invariance and negative K-group vanishing. As a result, we obtain

$$\mathrm{NK}(A) \xrightarrow{\simeq} \mathrm{NK}(\widehat{A}), \tag{1.3.17}$$

$$\mathrm{K}_n(A) \xrightarrow{\simeq} \mathrm{K}_n(\widehat{A}), \quad n < -1, \tag{1.3.18}$$

while at $n = -1$, there is an exact sequence

$$K_0(A[\tfrac{1}{\pi}]) \oplus \mathrm{K}_0(\widehat{A}) \to \mathrm{K}_0(\widehat{A}[\tfrac{1}{\pi}]) \to K_{-1}(A) \to K_{-1}(\widehat{A}) \to 0. \tag{1.3.19}$$

(C). *Étale algebras over nodal algebras: a counterexample.* There exist counterexamples to the negative K-group vanishing in the semistable generality, even for semistable *affine curves*, due to the potential existence of loops in the dual graph; see Example 1.3.20.

1.3.20. **Example.** For semistable affine curves X, $K_{-1}(X) \simeq H^1(\Gamma; \mathbf{Z})$, where Γ is the dual graph of X. One can construct one such X with non-trivial loops existing in Γ, so that $K_{-1}(X) \not\simeq 0$.

1.4. **Standard semistable case.** For $d \in \mathbf{N}$, let $A_d^{\mathrm{sst}} \coloneqq \mathcal{O}[t_0, \dots, t_d]/(t_0 \dots t_d - \pi)$.

1.4.1. **Theorem.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any smooth $\mathcal{O}$-algebra B, we have*

(i) $\mathrm{N}^\pi\mathrm{K}(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) \simeq 0$;
(ii) $\mathrm{K}_n(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) = 0$ *for* $n < 0$;
(iii) $\mathrm{NK}_n(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) = 0$ *for* $n \leq 0$.

Proof. We argue by induction on $d \geq 1$. (In fact, $d = 0$ corresponds to the regular case (Theorem 1.2.2).) The base case $d = 1$ has been treated with in Theorem 1.3.2 and Corollary 1.3.5. The (iii) would be a consequences of $\mathrm{NK}(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) \simeq \mathrm{NK}(B/\pi)$, which is stronger but may not be true.

Let $d \geq 2$. Consider the scheme $X \coloneqq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u])$ and the regular locus $Z \coloneqq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u]/(u, t_{d-1})) \simeq \mathrm{Spec}(\mathcal{O}/\pi[t_0, \dots, t_{d-2}])$. The closed immersion $Z \hookrightarrow X$ is quasi-smooth of codimension two. Let $\widetilde{X}$ be the blow-up $\mathrm{Bl}_Z X$. It is covered by two affine opens

$$D_+(u) \simeq \mathrm{Spec}\Big(A_{d-1}^{\mathrm{sst}}[u][\frac{t_{d-1}}{u}]\Big) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u, \bar{u}]/(u\bar{u} - t_{d-1})) \simeq \mathrm{Spec}(A_d^{\mathrm{sst}}), \quad \bar{u} = \frac{t_{d-1}}{u},$$
$$D_+(t_{d-1}) \simeq \mathrm{Spec}\Big(A_{d-1}^{\mathrm{sst}}[u][\frac{u}{t_{d-1}}]\Big) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u']), \qquad u' = \frac{u}{t_{d-1}},$$

whose intersection is

$$D_+(u) \cap D_+(t_{d-1}) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u, u'^{\pm 1}]) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]).$$

Now, on the one hand, the Mayer-Vietoris sequence of K-theory implies immediately the following cartesian square of spectra:

$$\begin{tikzcd} \mathrm{K}(\widetilde{X}) \arrow[r] \arrow[d] & \mathrm{K}(A_d^{\mathrm{sst}}) \arrow[d] \\ \mathrm{K}(A_{d-1}^{\mathrm{sst}}[u']) \arrow[r] & \mathrm{K}(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]), \end{tikzcd}$$

which induces a long exact sequence

$$\cdots \to \mathrm{K}_n(\widetilde{X}) \to \mathrm{K}_n(A_d^{\mathrm{sst}}) \oplus \mathrm{K}_n(A_{d-1}^{\mathrm{sst}}[u']) \to \mathrm{K}_n(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]) \to \cdots,$$

and similarly, since blow-up commutes with flat base change, we have the corresponding cartesian diagram and the long exact sequence for NK in place of K. By induction hypotheses for $d - 1$, we obtain:

- an equivalence of spectra

$$\mathrm{N}^\pi\mathrm{K}(\widetilde{X}) \xrightarrow{\simeq} \mathrm{N}^\pi\mathrm{K}(A_d^{\mathrm{sst}}) \tag{1.4.2}$$

  by (i);
- surjections

$$\mathrm{NK}_n(\widetilde{X}) \twoheadrightarrow \mathrm{NK}_n(A_d^{\mathrm{sst}}), \quad n \leq 0 \tag{1.4.3}$$

  by (iii), which are isomorphisms for $n \leq -1$;
- surjections on negative K-groups

$$\mathrm{K}_n(\widetilde{X}) \twoheadrightarrow \mathrm{K}_n(A_d^{\mathrm{sst}}), \quad n < 0 \tag{1.4.4}$$

  by (ii), which are isomorphisms for $n < -1$.

The map (1.4.4) is an isomorphism at $n=-1$, too; indeed, the sequence extends to a long exact sequence

$$\mathrm{K}_0(A_d^{\mathrm{sst}})\oplus \mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u'])\xrightarrow{\beta}\mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u'^{\pm1}])\to \mathrm{K}_{-1}(\widetilde{X})\to \mathrm{K}_{-1}(A_d^{\mathrm{sst}})\to 0, \tag{1.4.5}$$

where β is surjective since $\mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u'])\simeq \mathrm{K}_0(A_{d-1}^{\mathrm{sst}})$ by induction hypothesis (iii), and also that there is an equivalence $\mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u'^{\pm1}])\simeq \mathrm{K}_0(A_{d-1}^{\mathrm{sst}})\oplus \mathrm{K}_{-1}(A_{d-1}^{\mathrm{sst}})\oplus \mathrm{NK}_0(A_{d-1}^{\mathrm{sst}})\oplus \mathrm{NK}_0(A_{d-1}^{\mathrm{sst}})\simeq \mathrm{K}_{2-d}(A_{d-1}^{\mathrm{sst}})$ by Bass Fundamental Theorem, where $\mathrm{K}_{-1}(A_{d-1}^{\mathrm{sst}})=0$ and $\mathrm{NK}_0(A_{d-1}^{\mathrm{sst}})=0$ respectively by induction hypotheses (ii) and (iii).

On the other hand, the blow-up formula (1.3.7) provides canonical equivalences

$$\mathrm{K}(\widetilde{X})\simeq \mathrm{K}(A_{d-1}^{\mathrm{sst}}[u])\oplus \mathrm{K}(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]),$$
$$\mathrm{NK}(\widetilde{X})\simeq \mathrm{NK}(A_{d-1}^{\mathrm{sst}}[u])\oplus \mathrm{NK}(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]),$$

where the latter holds because the blowup commutes with flat base change. Then, by induction hypotheses, we obtain

- an equivalence of spectra:
$$\mathrm{N}^{\pi}\mathrm{K}(\widetilde{X})\simeq 0;$$
- isomorphisms
$$\mathrm{NK}_n(\widetilde{X})\simeq \mathrm{NK}_n(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]),\quad n\le 0;$$
- isomorphisms on negative K-groups
$$\mathrm{K}_n(\widetilde{X})\xrightarrow{\simeq}\mathrm{K}_n(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]),\quad n<0.$$

Now one can conclude the proof of (i)–(iii) in the case where $B=\mathcal{O}$. The (i) is clear from above. For (iii), notice that for $n<0$, $\mathrm{K}_n(\mathcal{O}/\pi[t_0,\dots,t_{d-2}])\simeq \mathrm{K}_n((\mathcal{O}/\pi)_{\mathrm{red}}[t_0,\dots,t_{d-2}])$ by nil-invariance, which vanishes by Theorem 1.2.2; as its quotient (1.4.4), $\mathrm{NK}_n(A_d^{\mathrm{sst}})$ vanishes, too. Likewise for (ii), using (1.4.3). This proves the induction hypotheses for d and $B=\mathcal{O}$.

The case of general smooth $\mathcal{O}$-algebras B is the same, since blow-up commutes with flat base change, and in view of Theorem 1.2.2. This completes the induction step for d. □

1.5. **Semistable case.**

1.5.1 (Standard smooth algebras)**.** Let R be a ring. An R-algebra S is called *standard smooth* if there exists integers $n\ge c\ge 0$ and $f_1,\dots,f_c\in R[x_1,\dots,x_n]$ such that

$$S\simeq R[x_1,\dots,x_n]/(f_1,\dots,f_c)$$

and that the maximal minor $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le c}$ is invertible in S.

For example, every étale R-algebra is standard smooth [Sta18, Lemma 00U9]. The class of standard smooth ring maps is stable under base change and composition [Sta18, Section 00T1].

1.5.2 (Lifting properties)**.** All étale ring maps lift along surjections of rings [Sta18, Lemma 04D1]; more generally, this holds for standard smooth ring maps (see the proof of [Sta18, Lemma 04B1]).

1.5.3. **Definition.** An $\mathcal{O}$-algebra is called *modelled on an $\mathcal{O}$-algebra C* if it is has an étale $C\otimes_{\mathcal{O}}B$-algebra structure for some standard smooth $\mathcal{O}$-algebra B. In particular, it is a standard smooth $\mathcal{O}$-algebra by 1.5.1.

1.5.4. **Example.** For example, any étale algebra over $\mathcal{O}[t_0,\dots,t_d]/(t_0\dots t_r-\pi)$ is modelled over A_r^{sst}. Therefore, such $\mathcal{O}$-algebras provide an affine base of the Zariski topology on strictly semistable k°-schemes.

1.5.5. **Theorem.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any $\mathcal{O}$-algebra A modelled on A_d^{sst}, we have*

(i) $\mathrm{N}^\pi\mathrm{K}(A) \simeq 0$ *in* $\mathrm{Pro}(\mathrm{Sp})$*;*
(ii) $\mathrm{K}_n(A) = 0$ *for* $n < 0$*;*
(iii) $\mathrm{NK}_n(A) = 0$ *for* $n \leq 0$.

1.5.6. **Corollary.** *Let X be a strictly semistable scheme over k°. Then* $\mathrm{N}^\pi\mathrm{K}(X) \simeq 0$ *in* $\mathrm{Pro}(\mathrm{Sp})$.

Proof. The presheaf $X \mapsto \mathrm{N}^\pi\mathrm{K}(X)$ satisfies Zariski descent. Hence we may assume X to be an affine scheme modelled over some A_d^{sst} (Example 1.5.4). Now the vanishing follows from Theorem 1.5.5 (i). □

1.5.7. **Remark** (Slight generalisation)**.** The semistability condition in Theorem 1.5.5 and Corollary 1.5.6 is not optimal. It is our first attempt in extending Theorem 1.2.2 to bad reduction with mild singularities. Here we discuss its potential generalisation.

(i) The semistability condition can be relaxed to a polystability condition. In fact, one checks easily that the same proof, but by induction on the poset of finite tuples of natural numbers, gives the same conclusion under polystability condition, i.e.:
  (a) For any A algebra modelled on $A_{d_1}^{\mathrm{sst}} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} A_{d_n}^{\mathrm{sst}}$ for some tuple $(d_1, \ldots, d_n) \in \mathbf{N}^n$, we have $\mathrm{N}^\pi\mathrm{K}(A) \simeq 0$, $\mathrm{K}_n(A) = 0$ for $n < 0$, and $\mathrm{NK}_n(A) = 0$ for $n \leq 0$.
  (b) For any strictly polystable (or equivalently, Zariski-locally modelled on finite products of standard semistable ones as above) scheme X over k°, we have $\mathrm{N}^\pi\mathrm{K}(X) \simeq 0$.

(ii) The induction process indicates that it be generalised to models with what we call (by ignorance of any literature related to it) *toric l.c.i. singularities*, i.e. models that are Zariski-locally modelled on an $\mathcal{O}$-algebra of the form

$$A_{\underline{M}} := \mathcal{O}[z_1, \ldots, z_d, w_1, \ldots, w_d]/(z_j w_j - M_j, 1 \leq j \leq d),$$

where $d \in \mathbf{N}$ and M_j is a monomial on π and variables with index $\leq j - 1$[6]. It is easily seen that $A_{\underline{M}}[\frac{1}{\pi}] \simeq \mathcal{O}[\frac{1}{\pi}][z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ is a standard toric chart over $\mathcal{O}[\frac{1}{\pi}]$.

Similar considerations apply to the formal case, but we will not repeat this remark and leave the reader (!) to formulate the correct statements.

Let us now prove Theorem 1.5.5.

Proof of Theorem 1.5.5. We argue by induction on $d \geq 0$. The case $d = 0$ corresponds to the regular case $A_0^{\mathrm{sst}} = \mathcal{O}$ (Theorem 1.2.2).

Let $d \geq 1$. We apply the same strategy as in the proof of Theorem 1.4.1.

First, let us show that, if A is base changed from an $\mathcal{O}$-algebra A_0 *modelled on* $A_{d-1}^{\mathrm{sst}}[u]$, then (i)–(iii) hold. Consider the scheme $X := \mathrm{Spec}(A_0)$ and the regular locus $Z := \mathrm{Spec}(A_0/(u, t_{d-1})) \simeq \mathrm{Spec}(\mathcal{O}/\pi[t_0, \ldots, t_{d-2}] \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0)$. The blow-up $\widetilde{X} := \mathrm{Bl}_Z X$ is covered by two affine opens

$$D_+(u) \simeq \mathrm{Spec}(A_0[u, \bar{u}]/(u\bar{u} - t_{d-1})) \simeq \mathrm{Spec}(A_d^{\mathrm{sst}} \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0) = \mathrm{Spec}(A),$$

$$D_+(t_{d-1}) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u'] \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0),$$

[6]This is the mixed characteristic analogue of Nakajima's classification, as described by Dais–Henk [DH03, Theorem 3.1], where here the ring $\mathcal{O}$ plays the role of $k[z_1]$ of *loc. cit.*

whose intersection is

$$D_+(u) \cap D_+(t_{d-1}) \simeq \operatorname{Spec}(A^{\mathrm{sst}}_{d-1}[u'^{\pm 1}] \otimes_{A^{\mathrm{sst}}_{d-1}[u]} A_0).$$

As in the proof of Theorem 1.4.1, by induction hypotheses, we have a cartesian square of spectra:

$$\begin{tikzcd}
\mathrm{K}(\widetilde{X}) \arrow[r] \arrow[d] & \mathrm{K}(A) \arrow[d] \\
\mathrm{K}(D_+(t_{d-1})) \arrow[r] & \mathrm{K}(D_+(u) \cap D_+(t_{d-1})),
\end{tikzcd}$$

and an equivalence

$$\mathrm{K}(\widetilde{X}) \simeq \mathrm{K}(A_0) \oplus \mathrm{K}(Z),$$

and the likewise for NK in place of K. The coordinate rings of $D_+(t_{d-1})$ and $D_+(u) \cap D_+(t_{d-1})$ are both modelled on A^{sst}_{d-1} whenever A_0 is modelled on $A^{\mathrm{sst}}_{d-1}[u]$. Also the coordinate ring of Z is smooth over $\mathcal{O}/\pi$, hence Z_{red} is smooth over the valuation ring $(\mathcal{O}/\pi)_{\mathrm{red}}$. These allow to do induction on d: we obtain an equivalence

$$\mathrm{N}^{\pi}\mathrm{K}(A) \simeq 0,$$

and surjections

$$\begin{aligned}
\mathrm{K}_n(A_0) \oplus \mathrm{K}_n(Z) &\twoheadrightarrow \mathrm{K}_n(A), \qquad n < 0, \\
\mathrm{NK}_n(A_0) \oplus \mathrm{NK}_n(Z) &\twoheadrightarrow \mathrm{K}_n(A), \qquad n \leq 0.
\end{aligned}$$

We conclude as before.

Now we remove the base change assumption on A. Assume A to be *modelled* on A^{sst}_d, which we identify with $A^{\mathrm{sst}}_{d-1}[u, \bar{u}]/(u\bar{u} - t_{d-1})$. Choose a smooth $\mathcal{O}$-algebra B and an étale ring map $A^{\mathrm{sst}}_d \otimes_{\mathcal{O}} B \to A$. In order to simplify the notations, we set

$$R_0 \coloneqq A^{\mathrm{sst}}_{d-1}[u], \quad R \coloneqq A^{\mathrm{sst}}_d = A^{\mathrm{sst}}_{d-1}[u, \bar{u}]/(u\bar{u} - t_{d-1}).$$

Previously, we established (i)–(iii) for A obtained as a base changed along $R_0 \to R$. Beware that $R_0 \to R$ is an isomorphism away from the locus $\{u = 0\}$, over which the fibre is an affine line; hence most A do not arise in this way. One should not expect to prove a descent-to-R_0-type statement. Instead, we try to lift a reduction of this standard smooth algebra.

Observe that there is an isomorphism

$$\mathcal{O}/\pi[t_0, \ldots, t_{d-2}, u] = R_0/t_{d-1} \xrightarrow{\simeq} R/\bar{u}.$$

Consider the reduction $A/\bar{u}$; it is étale over $R_0/t_{d-1} \otimes_{\mathcal{O}} B$, hence it lifts to an étale $R_0 \otimes_{\mathcal{O}} B$-algebra A_0 by 1.5.2. By [Sta18, Lemma 04FV], can complete the following commutative diagram using an étale $R \otimes_{\mathcal{O}} B$-algebra A':

$$\begin{tikzcd}
 & & (A_0 \otimes_{R_0} R)/\bar{u} \arrow[d, "\simeq"] & A_0 \otimes_{R_0} R \arrow[l] \arrow[d] \\
 & & A'/\bar{u} & A' \arrow[l, two heads] \\
A_0 \arrow[r, two heads] \arrow[uurr, bend left] & A_0/t_{d-1} \arrow[r, "\simeq"] & A/\bar{u} \arrow[u, "\simeq"'] & A \arrow[l, two heads] \arrow[u] \\
R_0 \otimes_{\mathcal{O}} B \arrow[r, two heads] \arrow[u] \arrow[rrr, bend right] & R_0/t_{d-1} \otimes_{\mathcal{O}} B \arrow[r, "\simeq"] \arrow[u] & R/\bar{u} \otimes_{\mathcal{O}} B \arrow[u] & R \otimes_{\mathcal{O}} B. \arrow[l, two heads] \arrow[u] \arrow[uuu, bend right]
\end{tikzcd}$$

In fact, A' can be chosen to consist of idempotent component(s) of $(A_0 \otimes_{R_0} R) \otimes_{R \otimes_{\mathcal{O}} B} A$.

We want to know about $\mathrm{K}(A)$. The top right two Nisnevich squares give rise to the following Nisnevich descent cartesian squares of spectra:

$$\begin{tikzcd}
\mathrm{K}(A_0 \otimes_{R_0} R) & K((A_0 \otimes_{R_0} R)[\frac{1}{\bar{u}}]) \arrow[l] \\
\mathrm{K}(A') \arrow[u] \arrow[d] & K(A'[\frac{1}{\bar{u}}]) \arrow[l] \arrow[u] \arrow[d] \\
\mathrm{K}(A) & K(A[\frac{1}{\bar{u}}]) \arrow[l]
\end{tikzcd} \tag{1.5.8}$$

As $R[\frac{1}{\pi}] \simeq \mathcal{O}[\frac{1}{\pi}][t_0^{\pm 1}, \dots, t_{d-2}^{\pm 1}, u^{\pm 1}]$, which is smooth over $\mathcal{O}[\frac{1}{\pi}]$, which is stably coherent regular as $\mathcal{O}$ is stably coherent regular (see Lemma 1.2.5 and the proof of Proposition 1.2.9). Hence, being smooth over $R[\frac{1}{\pi}]$, the rings $A[\frac{1}{\bar{u}}]$, $A'[\frac{1}{\bar{u}}]$ and $(A_0 \otimes_{R_0} R)[\frac{1}{\bar{u}}]$ are all stably coherent regular as well by Lemma 1.2.5, so that their negative K-groups vanish and that their NK are contractible. Therefore, the left vertical maps in (1.5.8) become suitable equivalences, and we get

$$\mathrm{K}_n(A_0 \otimes_{R_0} R) \xleftarrow{\simeq} \mathrm{K}_n(A') \xrightarrow{\simeq} \mathrm{K}_n(A), \quad n < 0,$$
$$\mathrm{NK}(A_0 \otimes_{R_0} R) \xleftarrow{\simeq} \mathrm{NK}(A') \xrightarrow{\simeq} \mathrm{NK}(A),$$

which, combined with the case that we have just seen (for the base changed R-algebra $A_0 \otimes_{R_0} R$ with A_0 modelled on R_0), proves (i)–(iii) for A. $\square$

1.6. **Semistable formal case.** Let k be a nonarchimedean field.

1.6.1. Let $\mathfrak{X} = \mathrm{Spf}(A_0)$ be an admissible formal scheme of finite type over k°. Then $(A_0)_\eta := A_0[\frac{1}{\pi}]$ is an affinoid Tate algebra over k. Here are some algebraic properties of $(A_0)_\eta$:

- it is a Noetherian Jacobson ring [Bos05, §3.1, Proposition 3];
- it is also regular if the rigid generic fibre $\mathfrak{X}_\eta$ is smooth over k (see for example [Duc09, Proposition 6.3]).

1.6.2. **Proposition.** *Let A_0 be an admissible k°-algebra and let $\widehat{A_0}$ be its π-adic completion. If A_0 is a strictly semistable k°-algebra, then* $\mathrm{N}^\pi\mathrm{K}(\widehat{A_0}) \simeq 0$ *in* $\mathrm{Pro}(\mathrm{Sp})$.

If moreover B_0 is a flat k°-algebra such that $B_0[\frac{1}{\pi}]$ is regular noetherian[7] *and that $\widehat{B_0} \simeq \widehat{A_0}$, then $\mathrm{N}^\pi\mathrm{K}(B_0) \simeq 0$, too.*

For example, B_0 could be a finite type algebra over a topologically finite type k°-algebra (e.g. a partial algebraisation of $\widehat{A_0}$).

Proof. This follows immediately from the formal-algebraic comparison 1.3.16 and Corollary 1.2.8. ☐

1.6.3. **Lemma.** *Any strictly semistable formal scheme $\mathfrak{X}$ is Zariski-locally of the form $\mathrm{Spf}(\widehat{A_0})$ for some strictly semistable k°-algebra A_0.*

Proof. We may assume that there is an étale morphism $f\colon \mathfrak{X} = \mathrm{Spf}(A_0) \to \mathrm{Spf}(k^\circ\langle t_0, \dots, t_d\rangle/(t_0 \dots t_r - \pi))$ of formal schemes, as in Definition 1.1.1. By topological invariance of the small étale site, f is uniquely determined by its reduction mod π, which we denote by $f_0\colon \mathrm{Spec}(A/\pi) \to \mathrm{Spec}(k^\circ/\pi[t_0, \dots, t_d]/(t_0 \dots t_r - \pi))$. This lifts to a (not necessarily unique) étale map $f'\colon A_0 \to \mathrm{Spec}(k^\circ/\pi[t_0, \dots, t_d]/(t_0 \dots t_r - \pi))$ (see 1.5.2). By the uniqueness of formal étale liftings, we have $\widehat{f'} \simeq f$. ☐

1.6.4. Consider the following condition for an affinoid k-algebra A:

($\dagger_A^{(\mathrm{V})}$) There exists a ring of definition A_0 of A, and a proper morphism of schemes $p\colon X \to \mathrm{Spec}(A_0)$ such that p is an isomorphism over $\mathrm{Spec}(A)$, that X is flat over k° and that its formal π-adic completion $\mathfrak{X}$ is a strictly semistable (or even quasi-log-smooth?) formal scheme over k°.

Beware that k is not necessarily discretely valued, so X is non-Noetherian and even not regular (in the non-Noetherian sense), contrary to its counterpart in the dagger condition in Kerz–Saito–Tamme's work[KST19]. We use the superscript "(V)" to indicate that it is our suggesstion of a generalisation of its friend in *loc. cit.* into the case of affinoid algebras "of type (V)".

Here are some basic properties of the rings involved:

(i) The scheme $X_\eta \xrightarrow{\simeq} \mathrm{Spec}(A)$ is Noetherian [Bos05, §3.1, Proposition 3]. It is also regular. Indeed, by [Hub96, (1.9.5), Proposition 1.9.6], using properness of p, the natural morphism $\mathfrak{X}_\eta \to \mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} X$[8] of adic spaces (actually, adic spaces) is an isomorphism. Since p is an isomorphism over the generic fibre, we see that $\mathfrak{X}_\eta \xrightarrow{\simeq} \mathrm{Spa}(A, A^\circ)$. Then, since $\mathfrak{X}_\eta$ is smooth over k, so is $\mathrm{Spa}(A, A^\circ)$, hence A is also regular by 1.6.1. (This is needed for applying Proposition 1.6.2.)

(ii) For any affine open $\mathrm{Spf}(B) \subset \mathfrak{X}$, by semistability B_η is a smooth affinoid over k, hence it is Noetherian regular by 1.6.1.

Here is our analogue of [KST19, Corollary 4.7, Lemma 5.13].

1.6.5. **Definition.** For any topological k°-algebra R with π-adic topology, define the pro-spectra

$$\hat{\mathrm{N}}^\pi\mathrm{K}(R) \coloneqq \mathrm{fib}(\text{“}\lim_{T\mapsto \pi T}\text{”}\,\mathrm{K}(R\langle T\rangle) \to \mathrm{K}(R)),$$

[7] This noetherian regularity is important in our argument.

[8] The fibre product, as an adic space, always exists [Hub94, Proposition 3.8]. Instead of looking at X which are proper over $\mathrm{Spec}(A_0)$, one may attempt to compute $Y \coloneqq \mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A)$: writing $A = A_0[X]/(X\pi - 1)$, then Y is the increasing union (gluing) of the rigid-analytic spectra of affinoid Tate k-algebras $A\langle \pi^k X\rangle/(X\pi - 1), k \geq 1$. But these are canonically isomorphic to A, so that $\mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A) \xrightarrow{\simeq} \mathrm{Spa}(A, A^\circ)$. More generally, $\mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} X_\eta \xrightarrow{\simeq} \mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} X$ for any X locally of finite type over $\mathrm{Spec}(A_0)$.

where $R\langle T\rangle$ is the π-adic completion of $R[T]$.

1.6.6. **Proposition** ([KST19, Corollary 4.7, Lemma 5.13 (ii)]). *Let A be an affinoid k-algebra satisfying $(\dagger_A^{(\mathrm{V})})$ together with a choice of ring of definition A_0. Then we have weak equivalences of pro-spectra:*

(i) $\mathrm{N}^\pi\mathrm{K}(A_0 \text{ on } \pi) \simeq 0$ *and* $\mathrm{N}^\pi\mathrm{K}(A_0) \simeq 0$,

(ii) $\hat{\mathrm{N}}^\pi\mathrm{K}(A_0 \text{ on } \pi) \simeq 0$.

Proof. We copy the proof of [KST19, Lemma 5.13 (ii)].

Let $p\colon X \to \mathrm{Spec}(A_0)$ be as in $(\dagger_A^{(\mathrm{V})})$: X is flat over k°, its π-adic completion $\mathfrak{X}$ is a strictly semistable formal scheme over k°. (One can view X as a partial algebraisation of $\mathfrak{X}$.)

Notice that (ii) follows from (i). Indeed, by excision [TT07, Theorem 7.1], we have $\mathrm{N}^\pi\mathrm{K}(A_0 \text{ on } \pi) \xrightarrow{\simeq} \hat{\mathrm{N}}^\pi\mathrm{K}(A_0 \text{ on } \pi)$.

Now it remains to prove (i). First, by excision [TT07, Theorem 7.1], we get a commutative diagram of (pro-)spectra with exact rows:

$$\begin{tikzcd}
\mathrm{K}(\mathfrak{X} \text{ on } \pi) \arrow[r] & \mathrm{K}(\mathfrak{X}) \arrow[r] & \mathrm{K}(\mathfrak{X}_\eta) \\
\mathrm{K}(X \text{ on } \pi) \arrow[r] \arrow[u, "\simeq"] & \mathrm{K}(X) \arrow[r] \arrow[u] & \mathrm{K}(X_\eta) \arrow[u] \\
\mathrm{K}(A_0 \text{ on } \pi) \arrow[r] \arrow[u] & \mathrm{K}(A_0) \arrow[r] \arrow[u] & \mathrm{K}(A) \arrow[u, "\simeq"]
\end{tikzcd}$$

and its $\mathrm{N}^\pi\mathrm{K}(-)$ version. Now, observe the following:

- By the generalised pro-cdh descent [KST26, Theorem A][9] instead of pro-cdh descent for noetherian schemes [KST17, Theorem A], we have a *weak* equivalence[10] of pro-spectra:
$$\mathrm{N}^\pi\mathrm{K}(X) \xrightarrow{\simeq} \mathrm{N}^\pi\mathrm{K}(A_0).$$

This point corresponds exactly to $\mathrm{K}(-)$ satisfying the pro-descent condition (P1), cf. [KST19, Lemma 4.5]. However, the verification of the "regular vanishing" condition (P2) is unclear for X, so that the direct analogue of [KST19, Proposition 4.6, Corollary 4.7] does not immediately follows. That is why we need our treatment as follows in order to see that X behaves like K-regular schemes. Now let us continue with the observations:

- By smoothness of $\mathfrak{X}_\eta$ (from semistabiliy of $\mathfrak{X}$), we have an equivalence
$$\mathrm{N}^\pi\mathrm{K}(\mathfrak{X}_\eta) \simeq 0.$$
- By semistability of $\mathfrak{X}$, using (the first part of) Lemma 1.6.3 and Proposition 1.6.2 (and suitable descent as is needed to define $\mathrm{N}^\pi\mathrm{K}(\mathfrak{X})$, which I did not define actually...), we see that there is an equivalence
$$\mathrm{N}^\pi\mathrm{K}(\mathfrak{X}) \simeq 0.$$

[9] Using the topologically universal adhesiveness of (k°, π) [FK18, Chapter 0, Corollary 9.2.7], one can show that the admissible blow-up p, which is *a priori* only of finite type, is almost of finite presentation. This stronger finiteness assumption is imposed in [KST26, Theorem A]. Thanks George Tamme for pointing this out.

[10] Due to pro-cdh descent, the whole result can only be weak equivalence, i.e. become an equivalence in $\mathrm{Pro}(\mathrm{Sp}^+)$, rather than an equivalence in $\mathrm{Pro}(\mathrm{Sp})$. This is the only place where *weak* equivalences get involved in the statement.

Therefore, we obtain a *weak* equivalence

$$\mathrm{N}^\pi\mathrm{K}(A_0 \text{ on } \pi) \simeq \mathrm{N}^\pi\mathrm{K}(X \text{ on } \pi) \simeq 0.$$

Now everything in the $\mathrm{N}^\pi\mathrm{K}$-version of the above diagram vanishes, except yet $\mathrm{N}^\pi\mathrm{K}(X) \overset{\simeq}{\rightrightarrows} \mathrm{N}^\pi\mathrm{K}(A_0)$[11]. Since $X_\eta \overset{\simeq}{\rightrightarrows} \mathrm{Spec}(A)$ is Noetherian and regular by 1.6.4 (i), we may apply (the second part of) Proposition 1.6.2 to conclude that there is a *weak* equivalence

$$\mathrm{N}^\pi\mathrm{K}(A_0) \simeq \mathrm{N}^\pi\mathrm{K}(X) \simeq 0.$$

This proves the second vanishing of (i), hence completes the proof. □

2. Kerz–Saito–Tamme's work

2.1. **Recollections of basics.** Let us record more facts about K-theory of Tate rings, verbatim as in [KST19].

2.1.1. **Proposition** ([KST19, Lemma 5.13 (i)])**.** *Let A_0 be a topological algebra with π-adic topology. Then*

$$\hat{\mathrm{N}}^\pi\mathrm{K}^{\mathrm{cont}}(A_0) \coloneqq \mathrm{fib}(\underset{T\mapsto \pi T}{\text{“}\lim\text{”}}\, \mathrm{K}^{\mathrm{cont}}(A_0\langle T\rangle) \to \mathrm{K}^{\mathrm{cont}}(A_0))$$

is contractible in $\mathrm{Pro}(\mathrm{Sp})$.

Proof. It can be checked on each level: the pro-system $\big(\mathrm{K}(A_0\langle T\rangle/\pi^j)\big)_{T\mapsto\pi T}$ is equivalent to the constant A_0/π^j for any $j \in \mathbf{N}$. □

We claim that the whole works of Kerz–Saito–Tamme [KST19] and [KST23] remain true for our new dagger condition $(\dagger_A^{(\mathrm{V})})$ as long as we replace [KST19, Corollary 4.7, Lemma 5.13 (i)–(ii)] with our Proposition 2.1.1 and Proposition 1.6.6. We expect to add a more detailed account of logic thread very soon!

This slight generalisation is intended, for example, for justifying that the étale representability of analytic K-theory in the rigid analytic motivic $\mathbf{A}^1$-homotopy category [DYZ26, Remark 4.1 (3), Remark 5.2] still hold for nonarchimedean field k which is not necessarily discretely valued[12].

[11] This seems strange because in the proof of [KST19, Lemma 5.13] we deduce the vanishing of $\mathrm{N}^\pi\mathrm{K}(A_0 \text{ on } \pi)$ from the vanishing of $\mathrm{N}^\pi\mathrm{K}(A_0)$, or equivalently, from the vanishing of $\mathrm{N}^\pi\mathrm{K}(X)$; but here this vanishing is not immediate because we have to show first that $X_\eta \overset{\simeq}{\rightrightarrows} \mathrm{Spec}(A)$ is regular Noetherian!

[12] In [DYZ26, Remark 5.2], we should have imposed the discrete valuation since it relies on [DYZ26, Remark 4.1 (3)]. However, as can be seen from this note, the assertion of [DYZ26, Remark 5.2] remains correct as stated.

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