ALTERED UNIFORMISATION OF LOG-RIGID SPACES

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Abstract. We prove a rigid-analytic snc version of Hartl's altered local uniformisation theorem [Har03] (cf. [Tem17, Corollary 3.3.2] for arbitrary ground field), which is also the rigid-analytic analogue of [He24, Proposition 5.11].

Contents

Let K be a p-adic local field and $C = \widehat{\overline{K}}$. The results might be generalised to more general ground fields?

One way to generalise the Hyodo-Kato cohomology to log-rigid spaces is to define Hyodo-Kato cohomology for stacks, let us ignore its plausibility.

Another way is to look for a logarithmic version of Hartl's étale-local potentially semistable reduction theorem (generalised by Temkin to arbitray ground field [Tem17]). For this, we will apply the (admissibly) étale altered uniformisation (à la Temkin) of Tongmu He for certain types of log-schemes [He24, Proposition 5.11]:

0.0.1. **Theorem** (Temkin, Tongmu He)**.** *Let K be a complete discrete valuation field. Let $\mathscr{X}$ be an admissible*[1] *$\mathcal{O}_K$-scheme with smooth generic fiber $\mathscr{X}_\eta$ which has a strict normal crossing divisor $D \subset X_\eta$, with complement $X^{\mathrm{triv}} := \mathscr{X}_\eta \backslash D$. Then there exists an admissible étale*

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[1]A scheme over $\operatorname{Spec} \mathcal{O}_K$ is said to be *admissible* if it is flat over $\mathcal{O}_K$ and locally of finite presentation over $\mathcal{O}_K$.

covering[2] $X' \to X \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, *where* K'/K *is a finite field extension, and* X' *is an admissible* $\mathcal{O}_{K'}$*-scheme such that the log-scheme* $(X', M_{X'})$*, whose log-structure is induced from the trivial locus* $X'^{\mathrm{triv}} := X^{\mathrm{triv}} \times_X X'$*, is a regular fs log-scheme, log-smooth and saturated over the regular fs log-scheme* $\operatorname{Spec} \mathcal{O}_{K'}^{\times}$*, whose log-structure is induced by the trivial locus* $\operatorname{Spec} K' \subset \operatorname{Spec} \mathcal{O}_{K'}$.

As a direct consequence of this result, we can show altered local uniformisation theorem in the smooth case with strict normal crossing divisor, cf. Theorem (2.0.3), and in the slightly more general case, cf. Theorem (3.0.3) using éh-local smoothness feature of rigid spaces.

The abbreviations "anlsm", "emb" are not good choices, but it is not meant to push the readers away from the simplicity of these uniformisation results.

Let us point out one potential application of these two results. They are useful towards a global Hyodo-Kato theory for rigid spaces with divisors, just as Temkin's altered local uniformisation is somehow essential towards globalising Hyodo-Kato cohomology in Colmez-Nizioł's works. For example, the results in the smooth case (2.0.3) combined with the étale descent of filtered de Rham cohomology (!) for smooth rigid spaces with strict normal crossing divisors should extend a lot of globalisation constructions of Colmez-Nizioł (about Hyodo-Kato cohomology, local-global compatibility, etc.) into nice log-cases; even better, the results in the slightly more general case (3.0.3) also ought to provide further extension of Colmez-Nizioł's results. Nevertheless, before all these constructions get into work, it seems that one crucial ingredient of local-global compatibility, namely the étale/éh-descent of filtered de Rham cohomology for smooth rigid spaces with strict normal crossing divisor, is missing. We ignore this point for the moment, and hope to come back later in the future.

1. Definitions

1.0.1. A pair $(\mathscr{X}, \mathscr{D})$ of a scheme $\mathscr{X}$ over $\mathcal{O}_K$ and a closed subscheme $\mathscr{D} \subset \mathscr{X}_\eta$ is called *nicely log-smooth over* $\mathcal{O}_L$ for some finite extension L/K if $\mathscr{X}$ is admissible over $\mathcal{O}_L$ with smooth generic fiber $\mathscr{X}_\eta$, and if $\mathscr{D} \subset \mathscr{X}_\eta$ is a strict normal crossing divisor, and if the log-scheme $(\mathscr{X}, M_{\mathscr{X}})$, where $M_{\mathscr{X}} = \mathcal{O}_{\mathscr{X}} \cap j_* \mathcal{O}^{\times}_{\mathscr{X}^{\mathrm{triv}}}$ is the compactifying log-structure induced from the open trivial locus $j : \mathscr{X}^{\mathrm{triv}} := \mathscr{X}_\eta \backslash \mathscr{D} \to \mathscr{X}$, is a regular fs log-scheme, log-smooth and saturated over the regular fs log-scheme $\operatorname{Spec} \mathcal{O}_L^{\times}$.

[2]Admissible étale coverings in the sense of [He24, Definition 5.4] are equivalent to the η-étale $\mathcal{O}_K$-admissible topology in the sense of [Tem17, 2.2.3], see *op. cit.* Remark 2.2.4.

A pair $(\mathfrak{X}, D)$ of a formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ and a closed adic subspace $D \subset \mathfrak{X}_\eta$ is called *algebraically nicely log-smooth over* $\mathcal{O}_L$ if it is the p-adic formal completion of a pair of schemes $(\mathscr{X}, \mathscr{D})$ nicely log-smooth over $\mathcal{O}_L$; the p-adic completion procedure is as follows: $\mathfrak{X} := \mathscr{X}_p^{\wedge}$[3] and $D := \mathscr{D}^{\mathrm{an}} \times_{(\mathscr{X}_\eta)^{\mathrm{an}}} \mathfrak{X}_\eta$, where $\mathfrak{X}_\eta \to (\mathscr{X}_\eta)^{\mathrm{an}}$ is the canonical open embedding.

1.0.2. **Lemma.** *For any pair $(\mathscr{X}, \mathcal{D})$ nicely log-smooth over $\mathcal{O}_L$, the corresponding log-scheme $(\mathscr{X}, M_{\mathscr{X}})$ has* Cartier type reduction*, i.e. its base change to* $\operatorname{Spec} \mathcal{O}^0_{F_L}$*, denoted by $(\mathscr{X}^0_1, M_{\mathscr{X}^0_1})$, is of Cartier type over* $\operatorname{Spec} \mathcal{O}^0_{F_L}$ [HK94, 2.12].

Proof. According to a result of Kato, a morphism of fs log-schemes is of Cartier type if and only if it is saturated. Since $(\mathscr{X}, M_{\mathscr{X}})$ is saturated over $\mathcal{O}_L^\times$ by definition, so is its base change along $\operatorname{Spec} \mathcal{O}^0_{F_L} \hookrightarrow \operatorname{Spec} \mathcal{O}_L^\times$. Therefore, $(\mathscr{X}^0_1, M_{\mathscr{X}^0_1})$ is saturated over $\operatorname{Spec} \mathcal{O}^0_{F_L}$, hence it is of Cartier type over $\operatorname{Spec} \mathcal{O}^0_{F_L}$ by the first sentence. ☐

1.0.3**.** For any algebraically nicely log-smooth pair $(\mathfrak{X}, D)$, we can associated with it a log-rigid space (X, M_X) log-smooth over $\operatorname{Spa}(K, \mathcal{O}_K)$, where $X := \mathfrak{X}_\eta$ and $M_X := M^{\mathrm{an}}_{\mathscr{X}_\eta}$ is an (étale) sheaf of monoids induced from the log-structure $M_{\mathscr{X}_\eta} := M_{\mathscr{X}}|_{\mathscr{X}_\eta}$ on $\mathscr{X}_\eta$ (for example locally taking a chart of it). Denote the open trivial locus $j : X^{\mathrm{triv}} := X \backslash D \subset X$. We claim that $M_X \simeq \mathcal{O}_X \cap j_* \mathcal{O}^\times_{X^{\mathrm{triv}}}$, which is the compactifying log-structure induced by the open trivial locus. Indeed, since we have an open immersion $X \subset (\mathscr{X}_\eta)^{\mathrm{an}}$ such that $D = X \cap \mathscr{D}^{\mathrm{an}}$, the claim follows from the following lemma:

1.0.4. **Lemma.** *Let $\mathscr{D} \subset \mathscr{X}$ be a closed immersion of a strict normal crossing divisor into a smooth scheme over K, and let $M_{\mathscr{X}}$ be the compactifying log-structure induced by the open complement embedding $j : \mathscr{X} \backslash \mathscr{D} \subset \mathscr{X}$. Then the induced analytified log-structure $M^{\mathrm{an}}_{\mathscr{X}}$ on $\mathscr{X}^{\mathrm{an}}$ agrees with the compactifying log-structure $M_{\mathscr{X}^{\mathrm{an}}}$ induced by the by the open immersion $j^{\mathrm{an}} : \mathscr{X}^{\mathrm{an}} \backslash \mathscr{D}^{\mathrm{an}} \subset \mathscr{X}^{\mathrm{an}}$.*

Proof. Indeed, we have $j^{\mathrm{an}*} M^{\mathrm{an}}_{\mathscr{X}} = \mathcal{O}_{\mathscr{X}^{\mathrm{an,triv}}} \cap \mathcal{O}^\times_{\mathscr{X}^{\mathrm{an,triv}}}$, whence by adjunction a map

$$M^{\mathrm{an}}_{\mathscr{X}} \to \mathcal{O}_{\mathscr{X}^{\mathrm{an}}} \cap j^{\mathrm{an}}_* \mathcal{O}^\times_{\mathscr{X}^{\mathrm{an,triv}}} =: M_{\mathscr{X}^{\mathrm{an}}}. \tag{1.0.4.1}$$

That this morphism is an isomorphism is an étale local property, since both sides commute with pullback along étale morphisms. Now, the question being étale local, we may assume

[3] Let $(\mathscr{X}_n, M_{\mathscr{X}_n})$ be the reduction mod p^n of $(\mathscr{X}, M_{\mathscr{X}})$, then $\mathscr{X}^\wedge_p := \varinjlim \mathscr{X}_n$. It is moreover equipped with the induced mod p^n log-structure $M_{\mathscr{X}_n}$.

$\mathscr{X} = \operatorname{Spec} A$ with an étale morphism

$$\varphi : K[T_1, \ldots, T_d] \to A$$

such that $\mathscr{D} = V(\prod_{i=1}^r T_i)$, for some $0 \le r \le d$ (if $r = 0$, then $\mathscr{D} = \emptyset$). Again, by étale locality, we may assume φ to be identity, in which case (1.0.4.1) is easily seen to be an isomorphism [DLLZ19, Example 2.3.17]. □

1.0.5. *Remark* ((strict) normal crossing divisor). (i) Any strict normal crossing divisor $\mathscr{D} \subset \mathscr{X}$ in a smooth scheme $\mathscr{X}$ over K (a perfect field) admits an equivalent local description: there is an étale morphism

$$\varphi : \mathscr{X} \to K[T_1, \ldots, T_d]$$

such that $\mathscr{D} = V(\prod_{i=1}^r \varphi^* T_i)$ for some $0 \le r \le d$.

(ii) A closed subspace $D \subset X$ in a smooth rigid space X over K is called a *(strict) normal crossing divisor* if locally for the analytic topology (or equivalently for the étale topology) on X, up to a finite separable base field extension L/K, it is of the form $S \times V(\prod_{i=1}^r T_i) \subset S \times \mathbf{B}_L^r$ with S smooth connected affinoid rigid space over K [DLLZ19, Example 2.3.17]. Hence we will not distinguish them.

(iii) As a result, the rigid-analytification of a (strict or not) normal crossing divisor on a smooth K-scheme is a normal crossing divisor.

1.0.6. **Lemma.** *Let $(\mathfrak{X}, D)$ be an algebraically nicely log-smooth pair over $\mathcal{O}_L$, with chosen algebraic model $(\mathscr{X}, \mathscr{D})$. The p-adic formal log-scheme $(\mathfrak{X}, M_{\mathfrak{X}})$ defined as the system of $(\mathscr{X}/p^n, M_{\mathscr{X}/p^n})$ is independent of the choice of $(\mathscr{X}, M_{\mathscr{X}})$.*

Proof. There is a natural "open immersion" $j : \mathfrak{X}_\eta \to \mathfrak{X}$ identifying $\mathscr{D}_p^\wedge$ with the Zariski closure of $j(D)$; more precisely, let $j : (\mathfrak{X}_\eta)_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ be the morphism of sites with induced morphism of topoi $(j^*, j_*) : (\mathfrak{X}_\eta)_{\text{ét}}^\sim \to \mathfrak{X}_{\text{ét}}^\sim$, then the open complement $\mathfrak{X} \backslash \mathscr{D}_p^\wedge$ is the largest open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ such that $(j_* h_D)(\mathfrak{U}) = 0$. Since $M_{\mathscr{X}/p^n}$ is uniquely determined by the open trivial locus $\mathfrak{X} \backslash \mathscr{D}_p^\wedge$, we are done. □

1.0.7. As a corollary, any algebraically nicely log-smooth $(\mathfrak{X}, D)$ determines a unique canonical p-adic formal log-scheme $(\mathfrak{X}, M_{\mathfrak{X}})$, which is log-smooth over $\operatorname{Spf} \mathcal{O}_L^\times$. Conversely, any such $(\mathfrak{X}, M_{\mathfrak{X}})$ determines $(\mathfrak{X}, D)$ by taking D to be the reduced closed complement of the rigid generic fiber of the open trivial locus of $\mathfrak{X}$.

2. Altered étale local uniformisation

2.0.1. **Definition.** Let $\mathcal{M}_K^{\log,\mathrm{anlsm}}$ denote the full subcategory of the category of p-adic formal log-schemes over $\mathcal{O}_K$ consisting of formal schemes $(\mathfrak{X}, M_{\mathfrak{X}})$ that are associated with an algebraically nicely log-smooth pair $(\mathfrak{X}, D)$ over $\mathcal{O}_L$ for a finite extension L/K.

Similarly we define $\mathcal{M}_C^{\log,\mathrm{anlsm},b}$.

Recall that the formal schemes here have Cartier type reduction 1.0.2.

2.0.2. **Definition.** Let $\mathcal{R}\mathrm{ig}_K^{\log,\mathrm{snc}}$ be the full subcategory of *snc embedded log-rigid spaces* of the category of log-rigid spaces; such a space is a log-rigid space (X, M_X) whose log-structure M_X is induced by some (necessarily dense) Zariski-open subspace $U \subset X$, such that the reduced closed subspace $D := X\backslash U$ is a normal crossing divisor.

The morphisms of $f : (X', M_{X'}) \to (X, M_X)$ are étale morphisms $f : X' \to X$ of rigid spaces with $D' = f^{-1}(D)$.

Fiber product exists: $(X', M_{X'}) \times_{(X,M_X)} (X'', M_{X''})$ is $X' \times_X X''$ together with open trivial locus $U' \times_U U''$.

We equip it with the étale topology, where coverings are étale coverings of the underlying rigid space.

Similarly we define $\mathcal{R}\mathrm{ig}_C^{\log,\mathrm{snc}}$.

2.0.3. **Theorem** (Altered local uniformisation for smooth schemes with a strict normal crossing divisor)**.** *Let $F_\eta : \mathcal{M}_K^{\log,\mathrm{anlsm}} \to \mathcal{R}\mathrm{ig}_K^{\log,\mathrm{snc}}$ (*resp. *$F_\eta : \mathcal{M}_C^{\log,\mathrm{anlsm},b} \to \mathcal{R}\mathrm{ig}_C^{\log,\mathrm{snc}}$) denote the rigid generic fiber functor taking $(\mathfrak{X}, M_{\mathfrak{X}})$ to (X, M_X) over* $\mathrm{Spa}(L, \mathcal{O}_L)$ *which is viewed as over* $\mathrm{Spa}(K, \mathcal{O}_K)$ *(*resp. *to (X, M_X) over* $\mathrm{Spa}(C, \mathcal{O}_C)$*).*

(i) The pair $(\mathcal{M}_K^{\log,\mathrm{anlsm}}, F_\eta)$ forms a base for $\mathcal{R}\mathrm{ig}_{K,\mathrm{\acute{e}h}}^{\log,\mathrm{snc}}$.

(ii) The pair $(\mathcal{M}_C^{\log,\mathrm{anlsm},b}, F_\eta)$ forms a base for $\mathcal{R}\mathrm{ig}_{C,\mathrm{\acute{e}h}}^{\log,\mathrm{snc}}$.

Proof. Let us prove (i). The proof of (ii) will be similar. Let $(X, M_X) \in \mathcal{R}\mathrm{ig}_K^{\log,\mathrm{snc}}$ with associated normal crossing divisor D. The question being étale local on X, we may assume that

$$X = S \times \mathbf{B}_L^r$$

with S smooth connected affinoid rigid space over K, and $D = S \times V(\prod_{i=1}^r T_i)$ for some $0 \leq r \leq d$. Applying Elkik's algebraisation theorem [Tem17, Theorem 3.13], we may assume $S = \mathscr{S}_\eta^{\mathrm{rig}}$ with $\mathscr{S}$ being an η-smooth admissible $\mathcal{O}_K$-scheme. Let

$$\mathscr{X} := \mathscr{S} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathcal{O}_L[T_1, \ldots, T_d], \quad \mathscr{D} := \mathscr{S} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathcal{O}_L[T_1, \ldots, T_d].$$

Then $\mathscr{X}_\eta$ is smooth with strict normal crossing divisor $\mathscr{D}_\eta$, thus satisfying the condition of the algebraic altered uniformisation theorem 0.0.1. Applying this theorem, we obtain a finite extension K'/K and an admissibly étale covering $\varphi : \mathscr{X}' \to \mathscr{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ by an admissible $\mathcal{O}_{K'}$-scheme $\mathscr{X}'$ such that the pair

$$(\mathscr{X}', \mathscr{D}') := (\mathscr{X}', (\varphi^{-1}(\mathscr{D}) \amalg \mathscr{X}'_s)_{\mathrm{red}})$$

is nicely log-smooth over $\mathcal{O}_L$. As the rigid generic fiber of an admissibly étale covering, $\varphi_\eta^{\mathrm{rig}} : (X', D') \to (X, D)$ is an étale covering by [Tem17, Lemma 3.2.2]. So we are done. □

3. A naive éh-topology

We expect a good variant of éh-topology for descent of sheaves of differential forms $\Omega^\bullet_{(X,M_X)/K}$.

3.0.1. **Definition.** Let $\mathcal{R}\mathrm{ig}_K^{\log,\mathrm{emb},+}$ be the full subcategory of *embedded log-rigid spaces* of the category of log-rigid spaces; such a space is a log-rigid space (X, M_X) whose log-structure M_X is induced by some nowhere dense[4] Zariski-open subspace $U \subset X$[5], such that $U = X^{\mathrm{triv}}$ is equal to the trivial locus[6]. We denote by $Z = X \backslash U \subset X$ the reduced closed rigid subspace supported on the closed complement of U[7]. We call (X, Z) the *associated embedding pair*.

[4] Éh-locally on X (with blowup centers contained in Z), the pair (X, Z) could have X smooth with Z in either of the two cases[Guo23, Corollary 2.4.12]:

- $Z \subset X$ is a strict normal crossing divisor;
- $Z = X$, which we call the *degenerate case*.

The degenrate case is quite awful, since this allows $M_X = \mathcal{O}_X$ as monoidal sheaves. The problems could come from éh-descent of differential forms. For this reason, we should not allow the degenerate case (on any irreducible components), whence our assumption "nowhere density" on irreducible components.

[5] On irreducible components of X, there are two cases: either U is dense, or $U = \emptyset$. We include the latter case for descent arguments, but it is not really the most interesting objects that we consider. When $U =$, M_X is induced by the morphism of monoid sheaves $\mathcal{O}_X \to \mathcal{O}_X$

[6] If we start from an open U, then the trivial locus could be strictly larger than U due to Hartogs extension phenomenon. By [Lüt93, Proposition 2.13 (a)], if X is normal and $Z = X \backslash U$ has codimension ≥ 2, then $\mathcal{O}_X(X) \xrightarrow{\simeq} \mathcal{O}_X(U)$; in particular, $j_* \mathcal{O}_U^\times = \mathcal{O}_X^\times$, so $M_X := \mathcal{O}_X \cap j_* \mathcal{O}_U^\times = \mathcal{O}_X^\times$. (Without normality, we only have $\mathscr{R}(X) \xrightarrow{\simeq} \mathscr{R}(U)$.)

[7] Again, as above, on irreducible components of X, we could have $Z = X$.

The morphisms of $f : (X', M_{X'}) \to (X, M_X)$ are morphisms $f : X' \to X$ of rigid spaces $Z' = f^{-1}(Z)$[8].

Fiber product exists: $(X', M_{X'}) \times_{(X,M_X)} (X'', M_{X''})$ is $X' \times_X X''$ together with open trivial locus $U' \times_U U''$[9].

We equip it with the *naive éh-topology* where coverings are those whose trivial loci form an éh-covering of rigid spaces in the sense of [Guo23, §2].

Similarly we define $\mathcal{R}\mathrm{ig}_C^{\log,\mathrm{emb},+}$.

3.0.2. *Remark.* (i) Let X be a normal rigid space and $D \subset X$ an effective Cartier divisor. Then (X, M_X) with trivial locus $X^{\mathrm{triv}} = U = X \backslash D$ is fs over $\mathrm{Spa}(K, \mathcal{O}_K)$ [DLLZ19, Example 2.3.16]. So $(X, M_X) \in \mathcal{R}\mathrm{ig}_K^{\log,\mathrm{emb},+}$.

(ii) Whenever X is smooth (regular), any such closed subspace $Z =: D$ is an effective Cartier divisor. Indeed, we may assume that D does not have codimension ≥ 2 components by footnote (6) since X is normal; then D is a Weil divisor, which comes from a Cartier divisor since X is regular [Cai24, Proposition 4.14].

3.0.3. **Theorem** (Altered local uniformisation for general embedded pairs)**.** *Let* $F_\eta : \mathcal{M}_K^{\log,\mathrm{anlsm}} \to \mathcal{R}\mathrm{ig}_K^{\log,\mathrm{emb},+}$ *(*resp. $F_\eta : \mathcal{M}_C^{\log,\mathrm{anlsm},b} \to \mathcal{R}\mathrm{ig}_C^{\log,\mathrm{emb},+}$*) denote the rigid generic fiber functor taking* $(\mathfrak{X}, M_{\mathfrak{X}})$ *to* (X, M_X) *over* $\mathrm{Spa}(L, \mathcal{O}_L)$ *which is viewed as over* $\mathrm{Spa}(K, \mathcal{O}_K)$ *(*resp. *to* (X, M_X) *over* $\mathrm{Spa}(C, \mathcal{O}_C)$*).*

(i) The pair $(\mathcal{M}_K^{\log,\mathrm{anlsm}}, F_\eta) \amalg (\{(X, X) \mid X \text{ smooth}\}, \mathrm{id})$ *forms a base for* $\mathcal{R}\mathrm{ig}_{K,\text{éh}}^{\log,\mathrm{emb},+}$.

(ii) The pair $(\mathcal{M}_C^{\log,\mathrm{anlsm},b}, F_\eta) \amalg (\{(X, X) \mid X \text{ smooth}\}, \mathrm{id})$ *forms a base for* $\mathcal{R}\mathrm{ig}_{C,\text{éh}}^{\log,\mathrm{emb},+}$.

Proof. Let us prove (i). The proof of (ii) will be similar. Let $(X, M_X) \in \mathcal{R}\mathrm{ig}_K^{\log,\mathrm{emb},+}$ with associated reduced closed subspace Z. The question being éh-local on X, we may assume X to be affinoid smooth over K [Guo23, Corollary 2.4.8].

Applying Elkik's algebraisation theorem [Tem17, Theorem 3.13] we may assume that $X = \mathscr{X}_\eta^{\mathrm{rig}}$ with $\mathscr{X} := \mathrm{Spec}\, A$ is admissible and η-smooth over $\mathrm{Spec}\, \mathcal{O}_K$. Then Z is the rigid generic fiber of certain closed subscheme $\mathscr{Z} \subset \mathscr{X}$.

By Hironaka's resolution of singulaties applied to $\mathscr{X}_\eta$ over K, there exists a finite sequence of $\mathscr{Z}_\eta$-supported blowups of schemes over K

$$\mathcal{X}' = \mathcal{X}_n \to \cdots \to \mathcal{X}_0 = \mathscr{X}_\eta \tag{3.0.3.1}$$

[8] One can more generally consider the morphisms such that $Z' \supset f^{-1}(Z)$, or equivalently $U' \subset f^{-1}(U)$; I do not know any h-descent results here

[9] This might not be the fiber product in the category of log-rigid spaces.

with smooth centers $\mathcal{C}_j \subset \mathcal{X}_j$ such that $\mathcal{X}'$ is smooth and $\mathcal{Z}' := \mathscr{Z}_\eta \times_{\mathscr{X}_\eta} \mathcal{X}' \subset \mathcal{X}'$ is a strict normal crossing divisor.

If $\mathcal{X}_j$ has a admissible $\mathcal{O}_K$-model $\mathscr{X}_j$, then taking the scheme theoretic closure of $\mathcal{C}_j$ in $\mathscr{X}_j$, we obtain a closed subscheme $\mathscr{C}_j := \overline{\mathcal{C}_j} \subset \mathscr{X}_j$. Then the admissible blowup $\mathscr{X}_{j+1} := \mathrm{Bl}^{\mathrm{adm}}_{\mathscr{C}_j} \mathscr{X}_j \to \mathscr{X}_j$ agrees on generic fibers with the given blowup $\mathcal{X}_{j+1} \to \mathcal{X}_j$. Repeating this process, we obtain a finite sequence of admissible blowups of admissible schemes over $\mathcal{O}_K$

$$\mathscr{X}' = \mathscr{X}_n \to \cdots \to \mathscr{X}_0 = \mathscr{X} \tag{3.0.3.2}$$

that agrees with (3.0.3.1) on generic fibers. Letting $\mathscr{Z}' := \mathscr{Z} \times_{\mathscr{X}} \mathscr{X}'$, then $(\mathscr{X}', \mathscr{Z}'_\eta) \in \mathcal{M}_K^{\mathrm{log,anlsm}}$, and $(X', Z') := (\mathscr{X}'^{\mathrm{rig}}_\eta, \mathscr{Z}'^{\mathrm{rig}}_\eta) \to (\mathscr{X}^{\mathrm{rig}}_\eta, \mathscr{Z}^{\mathrm{rig}}_\eta) = (X, Z)$ is the composite of a finite sequence of blow ups with smooth centers $C_j := (\mathscr{C}_j)^{\mathrm{rig}}_\eta$ such that $Z' \subset X'$ is a normal crossing divisor. Then we obtain an éh-covering

$$(X', Z') \to (X, Z)$$

in $\mathcal{R}\mathrm{ig}^{\mathrm{log,emb},+}_{K,\mathrm{\acute{e}h}}$. It remains to show that *étale locally*, (X', Z') lies in the image of $(\mathcal{M}_K^{\mathrm{log,anlsm}}, F_\eta)$.

Changing notation, we may assume that (X, Z) is the rigid generic fiber of $(\mathscr{X}, \mathscr{D})$ with $\mathscr{X}$ admissible $\mathcal{O}_K$-scheme and $\mathscr{Z} \subset \mathscr{X}$ a closed subscheme such that $\mathscr{Z}_\eta \subset \mathscr{X}$ is a strict normal crossing divisor. Now we can conclude as in the proof of (2.0.3). □

References

[Cai24] Yulin Cai. Intersection theory on non-archimedean analytic spaces, 2024.

[DLLZ19] Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu. Logarithmic adic spaces: some foundational results, 2019.

[Guo23] Haoyang Guo. Hodge-Tate decomposition for non-smooth spaces. *J. Eur. Math. Soc. (JEMS)*, 25(4):1553–1625, 2023.

[Har03] Urs T. Hartl. Semi-stable models for rigid-analytic spaces. *manuscripta mathematica*, 110:365–380, 2003.

[He24] Tongmu He. Perfectoidness via Sen Theory and Applications to Shimura Varieties, 2024.

[HK94] Osamu Hyodo and Kazuya Kato. Exposé V : Semi-stable reduction and crystalline cohomology with logarithmic poles. In Jean-Marc Fontaine, editor, *Périodes p-adiques - Séminaire de Bures, 1988*, number 223 in Astérisque, pages 221–268. Société mathématique de France, 1994. talk:5.

[Lüt93] Werner Lütkebohmert. Riemann's existence problem for a p-adic field. *Inventiones mathematicae*, 111(1):309–330, 1993.

[Tem17] Michael Temkin. Altered local uniformization of berkovich spaces. *Israel Journal of Mathematics*, 221(2):585–603, 2017.

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