Locally analytic vectors and relative Sen's theory

M2 memoire
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Abstract

We introduce the results of Lue Pan [Pan20, Section 3] on a generalisation of [BC16] in one-dimensional geometric setting. Let $\mathrm{Spa}(A, A^+)$ be a one-dimensional smooth affinoid space over $\mathbf{C}_p$ which admits a toric chart, and $\mathrm{Spa}(B, B^+)$ be an affinoid perfectoid pro-étale Galois covering of $\mathrm{Spa}(A, A^+)$ with Galois group a p-adic Lie group G. Then there is a Sen's operator $\theta \in B \otimes_A \mathrm{Lie}(G)$ (unique up to $A^\times$) which annihilates the G-locally analytic vectors of B.

Contents

0 Introduction

Notation and conventions.

- By a hat above the letter, we shall mean the adic completion (say p-adic completion) which will be clear from the context.
- Fix a prime number p. Let $\mathbf{Z}_p$ be the ring of p-adic integers, $\mathbf{Q}_p$ be its fractional field and $\mathbf{C}_p$ the completed algebraic closure of $\mathbf{Q}_p$, all equipped with the p-adic topology and the p-adic valuation v_p.
- For any field k, we denote by $\bar{k}$ its algebraic closure and by $\mathscr{G}_k$ its absolute Galois group.
- For any field k over $\mathbf{Q}_p$, we denote by $\chi_{\text{cyc}} : \mathscr{G}_k \to \mathbf{Z}_p^\times$ the cyclotomic character and by $k^{\text{cyc}} = k(\mu_{p^\infty})$ the cyclotomic extension. Let $\mathbf{Z}_p(1) = \varprojlim_n \mu_{p^n}(\overline{\mathbf{Q}}_p)$. For any $\mathbf{Z}_p$-module V carrying an action of $\mathscr{G}_k$, we write $V(i) = V \otimes \chi_{\text{cyc}}^i$ for the Tate twist by i of V.

0.1. Let k be a complete discretely valued field over $\mathbf{Q}_p$ with perfect residue field. Let $C = \hat{\bar{k}}$ denote the completed algebraic closure of k.

0.2. Tate-Sen's method. Consider $k_\infty := k^{\text{cyc}}$. In [Tat67], Tate calculated the cohomology groups $H^\bullet_{\text{cont}}(\mathscr{G}_k, C(i))$. He proved that any finite extension l_∞ of k_∞ is *almost étale* (the image of the trace map $\text{Tr}_{l_\infty/k_\infty} : \mathcal{O}_{l_\infty} \to \mathcal{O}_{k_\infty}$ between rings of integers contains the maximal ideal of $\mathcal{O}_{k_\infty}$), so that the Galois cohomology $H^\bullet_{\text{cont}}(\mathscr{G}_{k_\infty}, C(i)) = 0$, thus $H^\bullet_{\text{cont}}(\mathscr{G}_k, C(i)) \simeq H^\bullet_{\text{cont}}(\text{Gal}(k_\infty/k), \hat{k}_\infty(i))$. Then he used a *normalised trace map* $\hat{k}_\infty \to k$ to split $\hat{k}_\infty$ into two simpler parts. Finally he proved that $H^0_{\text{cont}}(\mathscr{G}_k, C(i)) = k$, $H^1_{\text{cont}}(\mathscr{G}_k, C(i)) \simeq k$ if $i = 0$ and otherwise zero.

Later in [Sen80], Sen calculated continuous Galois cohomology with values in the non-abelian group $\text{GL}_d(C)$ and proved

$$H^1_{\text{cont}}(\mathscr{G}_k, \text{GL}_d(C)) \simeq H^1_{\text{cont}}(\text{Gal}(k_\infty/k), \text{GL}_d(\hat{k}_\infty)) \simeq \varinjlim_n H^1_{\text{cont}}(\text{Gal}(k_n/k), \text{GL}_d(k_n)),$$

where $k_n = k(\mu_{p^n}) \subset k_\infty$. Sen's method is vaguely starting with a cocycle $\mathscr{G}_k \to \text{GL}_d(C)$ and successively polishing it by coboundaries to arrive at $\text{GL}_d(k_n)$ for some $n \in \mathbf{N}$; there the almost étaleness and Tate's normalised trace maps are essentially used. These isomorphisms amount to saying that we may descend (and decomplete) any semi-linear C-representation V of $\mathscr{G}_k$ to a unique semi-linear k_∞-representation $D_{\text{Sen}}(V)$. Concretely, we have

$$D_{\text{Sen}}(V) = V^{\mathscr{G}_{k_\infty},\, k-\text{finite}}.$$

The infinitesimal action of $\text{Gal}(k_\infty/k)$ (which is a group isomorphic to an open subgroup of $\mathbf{Z}_p^\times$) on $D_{\text{Sen}}(V)$ gives an element $\Theta_{V,\text{Sen}} \in \text{End}_C(V)$, called *Sen's operator*, which encodes a lot of information of the Galois representation V such as generalised Hodge-Tate weights.

We remark that these results extend naturally to general infinitely ramified extensions k_∞ of k cut out by a character $\chi : \mathscr{G}_k \to \mathbf{Z}_p^\times$.

0.3. Generalisations of Sen's theory. (i) Tate-Sen's method consists of an almost étale extension C/k_∞ and an extension k_∞/k cut out by a character $\chi : \mathscr{G}_k \to \mathbf{Z}_p^\times$, which admits well-behaved normalised trace maps. This has been formalised and generalised by Colmez in [Col03, Subsection 3.3] and [BC08, Subsection 3.1]: let G be a profinite group, $\chi : G \to \mathbf{Z}_p^\times$ be a character whose image is infinite and whose kernel is denoted by H. Suppose that we have a system of subrings of a Banach $\mathbf{Z}_p$-algebra $\widetilde{\Lambda}$ equipped with an isometric action of G

$$\begin{array}{ccccc} \widetilde{\Lambda} & \longleftarrow & \widetilde{\Lambda}^{H_1} & \longleftarrow & \widetilde{\Lambda}^{H_2} \\ & & {\scriptstyle R_{H_1,n}} \downarrow\uparrow & & {\scriptstyle R_{H_2,n}} \downarrow\uparrow \\ & & \Lambda_{H_1,n} & \longleftarrow & \Lambda_{H_2,n} \\ & & \uparrow & & \uparrow \\ & & \cdots & \longleftarrow & \cdots \end{array}$$

where $H_1 \subset H_2$ are open subgroups of H, $\widetilde{\Lambda}^{H_i}$ denotes the H_i-invariants, the top row consists of almost étale extensions, and $R_{H_i,n}$ form a compatible system of normalised trace maps for open subgroups H_i of H and n sufficiently large. Then very roughly, Tate-Sen's method contains an almost étale descent from $\widetilde{\Lambda}$-representations of G to $\widetilde{\Lambda}^{H_i}$-representations, followed by a decompletion to $\Lambda_{H_i,n}$-representations.

This formalism has turned out to be useful: for example, given V a $\mathbf{Q}_p$-representation of $\mathscr{G}_k$, one may descend $V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}}$ to some finite free $k_\infty((t))$-representation of $\text{Gal}(k_\infty/k)$, whose infinitesimal action is given by a differential operator ∇ (with respect to the variable t, Fontaine's p-adic analogue of $2\pi i$); this operator is trivial if and only if V is a de Rham representation. One may also use the formalism prove the overconvergence of p-adic representations.

(ii) One unsatisfying point of Sen's method is that, if $G := \mathrm{Gal}(k_\infty/k)$ is a p-adic Lie group of dimension ≥ 2 (no longer being cut out by a character with values in $\mathbf{Z}_p^\times$; for example, k_∞/k be a Lubin-Tate extension in the case where $k \neq \mathbf{Q}_p$), there lack normalised traces, and it is no longer a good idea to take G-smooth vectors of $V^{\mathscr{G}_{k_\infty}}$ at the step of decompletion [Sen80, Proposition 11]. A generalised decompletion theory was proposed by Berger and Colmez in [BC16]. They showed that one should take *G-locally analytic vectors* instead of G-smooth vectors; more precisely, if W is a semi-linear $\hat{k}_\infty$-representation of G, then the natural map $\hat{k}_\infty \otimes_{\hat{k}_\infty^{\mathrm{la}}} W^{\mathrm{la}} \to W$ is an isomorphism (*loc.cit.* Théorème 1.7). Also, they showed that in certain sense, if V is a faithful $\mathbf{Q}_p$-representation of G, then the associated Sen's operator lies in $C \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$ and annihilates the set $\hat{k}_\infty^{\mathrm{la}}$ of locally analytic vectors (*loc.cit.* Théorème 1.9).

0.4. Relative Sen's theory. The Sen's theory is established on the base k, which is a zero-dimensional object from the geometric point of view. It is reasonable to extend the theory to higher dimensional objects, that is to *relative* settings. For example, we may study representations of the (étale) fundamental group of a smooth variety over k. Andreatta and Brinon generalised Tate-Sen's formalism to study them, and their results are in fact closely related to Faltings's *p-adic Simpson correspondence* (cf. [AB08, Remarque 3.15]).

Recently, there has been a paper of Lue Pan [Pan20] in which Berger-Colmez's work on locally analytic vectors was generalised to one-dimensional geometric setting. The local study concerns a one-dimensional smooth affinoid adic space $X = \mathrm{Spa}(A, A^+)$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ admitting a toric chart, together with an affinoid perfectoid pro-étale Galois covering $\widetilde{X} = \mathrm{Spa}(B, B^+)$ of X with Galois group a p-adic Lie group G. He constructed an operator $\theta \in B \otimes_A \mathrm{Lie}(G)$ which annihilates the set $B^{G-\mathrm{la}}$ of G-locally analytic vectors in B; this operator is non-zero if the covering $\widetilde{X} \to X$ is nice (so-called *locally analytic covering*). Comparing with Berger-Colmez's work, we see that the role of $\hat{k}_\infty$ is now played by the affinoid perfectoid space $\widetilde{X}$, which has almost étale property over it by the almost purity theorem; the extension k^{cyc} is here replaced by some $\mathbf{Z}_p$-covering of X pulled back from the canonical covering over the one dimensional torus. So analogously, we may call this θ a *Sen's operator* in this setting. The operator θ has been used in *loc.cit.* to study locally analytic vectors of completed cohomology of perfectoid modular curves. We remark that the condition of X being *one-dimensional* is essentially required for applying the Tate-Sen's formalism of Colmez.

0.5. Rough idea of the construction. Let's keep the notation of the last paragraph. We have a diagram $\widetilde{X} \times_X X_\infty = \mathrm{Spa}(B_\infty, B_\infty^+) \to X_\infty \to X$, where the first arrow allows almost étale descent and the second, which is a $\mathbf{Z}_p(1)$-Galois covering, allows decompletion. By Tate-Sen's formalism, one obtains, for any finite-dimensional continuous $\mathbf{Q}_p$-representation V of G, a linear map

$$\phi_V : \mathrm{Lie}(\mathbf{Z}_p(1)) \to \mathrm{End}_{B_\infty}(B_\infty \otimes_{\mathbf{Q}_p} V)$$

functorial in V and satisfying the Leibniz rule with respect to tensor products of representations. By taking the direct limit over certain system $\{V_i\}_{i\in\mathbf{N}}$ of finite-dimensional sub-$\mathbf{Q}_p$-representations of $\mathscr{C}^{\mathrm{an}}(G, \mathbf{Q}_p)$, one obtains an operator

$$\phi_G : \mathrm{Lie}(\mathbf{Z}_p(1)) \to \mathrm{End}_{B_\infty}(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G, \mathbf{Q}_p)),$$

which is a derivation commuting with the right translation action of G and the action of Γ. Hence it factors through (by abuse of notation)

$$\phi_G : \mathrm{Lie}(\mathbf{Z}_p(1)) \to B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$$

where $\mathrm{Lie}(G)$ acts by the infinitesimal left translation action. Then θ can be just defined as the image of any topological generator of $\mathrm{Lie}(\mathbf{Z}_p(1))$. More canonically, the action ϕ_V is a linear map (so-called *Higgs field*) $B_\infty \otimes_{\mathbf{Q}_p} V \to (B_\infty \otimes_{\mathbf{Q}_p} V) \otimes_A \Omega^1_{A/C}(-1)$, which again suggests a link with p-adic Simpson correspondence; so the operator ϕ_G could be understood as a Higgs field on some big Banach B_∞-module.

0.6. Towards a p-adic Simpson correspondence. (i) In algebraic geometry and differential geometry, the nonabelian Hodge correspondence or Corlette–Simpson correspondence is a correspondence between polystable Higgs bundles of degree 0 and semisimple representations of the fundamental group of a smooth, projective complex algebraic variety, or a compact Kähler manifold. The correspondence is an equivalence of categories (and even bijective real analytic on moduli spaces). The proof is rather indirect and was done via the correspondence from each side to harmonic bundles. These led to the search for a p-adic analogue of such correspondence relating Higgs bundles on a p-adic variety to representations of the fundamental group. Let's just mention a part of the results.

(ii) An important step of p-adic Simpson correspondence was made by Faltings, who constructed, for X a proper scheme over $\mathcal{O}_k$ with toroidal singularities plus some deformation hypothesis, an equivalence (in fact various mutually related equivalences) between the category of small *generalised* representations of the fundamental group $\pi_1(\mathfrak{X}_C)$ and that of small Higgs bundles on $\mathfrak{X}_C$ [Fal05]. The local correspondence is relatively easier. The global construction depends on the choice of a first order lifting of the integral model, which provides a way of gluing local correspondences. Later, Abbes, Gros and Tsuji systematically developed Faltings's method with two new approaches for globalisation [AGT16]. It is natural to ask what kind of Higgs bundles correspond to genuine representations. For example, the semistability was addressed in [LSZ17]. But this question remains widely open in general.

Later, using Scholze's pro-étale site and a period sheaf on it, Liu and Zhu defined a one-way functor from the category of $\mathbf{Q}_p$-local systems on some smooth rigid space X over k to that of nilpotent Higgs bundles on X_C together with a Galois action, which does not rely on integral models [LZ17]. Together with Diao and Lan, they generalised it to the logarithmic case [DLLZ18]. Both these works start from a (rigid analytic) variety over a *discretely valued* field.

Recently in this thesis, in the arena of Liu-Zhu, Wang worked with a liftable formally smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$ and constructed an overconvergent period sheaf to establish a correspondence of Faltings's type [Wan21]. His correspondence depends on the lifting and is compatible with both Liu-Zhu's and Faltings's constructions. Similar comparisons can also be found in [YZ21].

It is also worth mentioning that some progress has been made via the study of vector bundles and C-local systems on diamonds under the v-topology, cf. for example [MW20] and [Heu21]. In the latter paper, Heuer established a correspondence between one-dimensional continuous C-representations of the fundamental group $\pi_1^{\text{ét}}(X,x)$ and pro-finite-étale analytic Higgs bundles (analytic line bundles trivialised by the universal cover together with a Higgs field) of rank one.

0.7. Structure of the memoire. First, we recall some results on adic geometry especially the perfectoid geometry developed by Scholze and Kedlaya-Liu, the Tate-Sen's formalism introduced by Colmez, p-adic Lie groups as well as locally analytic vectors. Then we shall review [Pan20, Section 3] in detail. As it is closely related to p-adic Simpson correspondence, the latter will be the topic of the final section, where Faltings's and Liu-Zhu's constructions are to be quickly reviewed.

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1 Preliminaries

1.1 Continuous group cohomology

1.1.1. Continuous group cohomology. Let G be a topological group. A *topological G-module M* is a topological abelian group M with a G-action in the usual sense such that the corresponding map $G \times M \to M$ is continuous. The morphisms between topological G-modules are continuous G-equivariant group homomorphisms. In this case, Tate introduced the *continuous group cohomology* $H^i_{\text{cont}}(G,M)$ given by the *continuous* cochain complex $C^\bullet_{\text{cont}}(G,M)$, which is the same as the usual one except that all the cochains considered should now be continuous with respect to the topologies on G and M. One recovers the usual group cohomology $H^i(G,M)$ in the case where G is finite or where G is profinite and M has the discrete topology.

Recall that in the discrete case, $H^i(G,-)$, $i \geq 0$ are canonically isomorphic to the derived functors $\text{Ext}^i_{\mathbf{Z}[G]}(\mathbf{Z},-)$ on the abelian category of G-modules. However, in the continuous case, the topological G-modules do not form an abelian category, because the image and coimage do not coincide in general for topological reasons. Thus, it seems that one cannot obtain a derived category from it.

Nevertheless, in [BCC16], they work with certain derived category of pro-discrete G-modules to define group cohomology which verifies some expected good properties and agrees in many cases with Tate's continuous group cohomology.

1.1.2 - Proposition ([Jan88, (2.1) and (2.2)]). *Let G be a profinite group and $M = \varprojlim_{n\in\mathbf{N}} M_n$ be the (topological) inverse limit of a Mittag-Leffler system of discrete G-modules $(M_n)_{n\in\mathbf{N}}$. Then for all $i \geq 0$, there is a natural short* *exact sequence*

$$0 \to R^1 \varprojlim_n H^{i-1}_{\mathrm{cont}}(G, M_n) \to H^i_{\mathrm{cont}}(G, M) \to \varprojlim_n H^i_{\mathrm{cont}}(G, M_n) \to 0.$$[1]

1.1.3. Lyndon-Hochschild-Serre spectral sequence. Let G be a topological group and M be a topological G-module. Let H be a closed normal subgroup of G. For discrete group cohomology, we have a spectral sequence

$$E_2^{rs} = H^r(G/H, H^s(H, M)) \Rightarrow H^{r+s}(G, M).$$

But for continuous group cohomology, such a spectral sequence is missing in general. Nevertheless, Boggi and Cook proved it to be true if G is profinite and M is a pro-discrete G-module [BCC16, Theorem 7.13]. In fact, they dealt with certain derived category of the (well-behaving) category of the pro-discrete G-modules to define group cohomology. We must point out that their definition of group cohomology (*loc.cit*, p. 1302) coincides with Tate's continuous group cohomology in the case mentioned above (see *loc.cit*, line 7–9 from the bottom, p. 1309 for discussion), so that we can cite their results in terms of Tate's continuous group cohomology.

As a degenerate case of the spectral sequence, we have the following statement which can be verified by hand:

> *If $H^i_{\mathrm{cont}}(H, M) = 0$, $i > 0$ and if the restriction maps $C^i_{\mathrm{cont}}(G, M) \to C^i_{\mathrm{cont}}(H, M)$, $i \geq 0$ are surjective, then the inflation maps $H^i_{\mathrm{cont}}(G/H, M^H) \to H^i_{\mathrm{cont}}(G, M)$, $i \geq 0$ are bijective.*

The second condition is verified for example when H is a direct factor of G.

Let's prove this statement. The surjectivity is immediate from assumption, and it is clearly enough to show the injectivity for $i \geq 2$. Consider $f \in C^i_{\mathrm{cont}}(G/H, M^H)$ such that $\partial f = 0$ and that its inflation to $C^i_{\mathrm{cont}}(G, M)$ is of the form ∂F for some $F \in C^{i-1}_{\mathrm{cont}}(G, M)$. Since $\partial F|_{H^i} = 0$, there exists by assumption some $F' \in C^{i-2}_{\mathrm{cont}}(G, M)$ such that $F|_{H^{i-1}} = \partial F'|_{H^{i-1}}$. Then $F - \partial F'$ factors through $(G/H)^{i-1}$ and $\partial(F - \partial F') = f$.

1.1.4. Induced long exact sequence. Let G be a topological group and $0 \to M' \to M \to M'' \to 0$ be an exact sequence of topological G-modules. In general, we do not have as usual an induced long exact sequence of continuous cohomology groups; but we do have it if the short sequence

$$0 \to C^i_{\mathrm{cont}}(G, M') \to C^i_{\mathrm{cont}}(G, M) \to C^i_{\mathrm{cont}}(G, M'') \to 0$$

(which is already exact on the left) is exact for all i. This last condition is verified if M', M, M'' are $\mathbf{Q}_p$-Banach spaces; indeed, any continuous surjective map of $\mathbf{Q}_p$-adic Banach spaces admits a continuous splitting [Ber, Corollary 11.6], so we have exactness on the right; being the kernel of the continuous morphism $M \to M''$, the image of M' in M is closed, thus is isomorphic to M' by the open mapping theorem, hence we obtain the exactness in the middle.

1.2 Almost étale extensions

1.2.1. Set-up. Let V be a ring together with a sequence of principal ideals $\mathfrak{m}_\alpha \subset V$ parametrised by positive elements $\alpha \in \Lambda^+$ of some subgroup $\Lambda \subset \mathbf{Q}$ dense in $\mathbf{R}$ (so $1 \in \Lambda^+$). Denote by π a generator of $\mathfrak{m}_1$, and choose once for all a generator π^α of $\mathfrak{m}_\alpha$, $\alpha \in \Lambda^+$. Assume furthermore that $\pi^\alpha \pi^\beta = \text{unit} \cdot \pi^{\alpha+\beta}$, and that π^α is not a zero-divisor. Denote $\mathfrak{m} = \bigcup_\alpha \mathfrak{m}_\alpha$.

Observe that $\mathfrak{m}$ is a flat V-module, being the filtered colimit of free V-modules $\mathfrak{m}_\alpha$ (since π^α is not a zero-divisor). In particular, the multiplication map $\mathfrak{m} \otimes_V \mathfrak{m} \to \mathfrak{m}$ is an isomorphism.

The ideal $\mathfrak{m}$ may be thought of as an "approximating unit ideal". The basic idea of almost mathematics is to not distinguish this from the unit ideal V, which could be beneficial for integral problems.

1.2.2 - Example. (o) The very special case $\mathfrak{m} = V$. In this case, most of the following notions and results will recover the usual ones in commutative algebra.

(i) Let V be the ring of integers of an infinitely ramified algebraic extension K of $\mathbf{Q}_p$ and Λ be the p-adic valuation group of $K^\times$, and $\mathfrak{m}_\alpha\{$elements of valuation $\geq \alpha\}$ for $\alpha \in \Lambda$.

(ii) More generally, we shall apply the results of this section to $V = R^+$ for some perfectoid Huber pair (R, R^+), and $\mathfrak{m}_{l/p^n} = \varpi^{l/p^n}$, $l, n \in \mathbf{N}$, where $\varpi \in R^+$ is a pseudo-uniformiser with compatible p-power roots, see subsection 2.1.

[1] This exact sequence agrees with the one induced by spectral sequence (if the latter exists) by vanishing of $R^i \varprojlim_{n\in\mathbf{N}}$ for $i \geq 2$.

1.2.3. Almost modules. Let $\Sigma \subset V - \mathbf{Mod}$ be the full subcategory of modules annihilated by $\mathfrak{m}$; it is closed taking submodules, quotients and extensions. The *category of almost V-modules* (or *V^a-modules*), denoted by $V^a - \mathbf{Mod}$, is then defined as the quotient category $V - \mathbf{Mod}/\Sigma$ in the sense of [Gab62, Chapitre III, §1]. For any V-module M, we denote by M^a the associate V^a-module. The category $V^a - \mathbf{Mod}$ is naturally an abelian tensor category [GR03, 2.2]; the tensor product is given by

$$M^a \otimes_{V^a} N^a := (M \otimes_V N)^a.$$

A morphism of V-modules is said to be an *almost isomorphism* (*resp. almost injective*, *almost surjective*) if the induced morphism of V^a-modules is an isomorphism (*resp.* a monomorphism, epimorphism). Two V-modules M, N are said to be *almost isomorphic* if they become isomorphic in $V^a - \mathbf{Mod}$, and we write $M^a \simeq N^a$ or $M \approx N$.

What do the morphisms in $V^a - \mathbf{Mod}$ look like? For any V-modules M and N, we have [GR03, 2.2.2]

$$\mathrm{Hom}_{V^a}(M^a, N^a) \simeq \mathrm{Hom}_V(\mathfrak{m} \otimes_V M, N).$$

Hence $\mathrm{Hom}_{V^a}(M^a, N^a)$ acquires a natural structure of V-module, and has trivial $\mathfrak{m}$-torsion.

For any V-module M, we define $(M^a)_* := \mathrm{Hom}_{V^a}(V^a, M^a)$ the *module of almost elements of M^a*. By the above isomorphism, we have $(M^a)_* \simeq \mathrm{Hom}_V(\mathfrak{m}, M)$, and it is naturally a V-module with trivial $\mathfrak{m}$-torsion. For simplicity, we shall write M_* in place of $(M^a)_*$. We have a natural morphism $M \to M_*$, $m \mapsto (x \mapsto xm)$, which is easily seen to be an almost isomorphism. Also, for any V-modules M, N, we have by adjunction

$$\mathrm{Hom}_{V^a}(M^a, N^a) \simeq \mathrm{Hom}_V(M, N_*).$$

1.2.4. Almost modules over a V-algbera. One can also work with the subcategory $A^a - \mathbf{Mod}$ of A^a-modules for any V-algebra A [2]. Le M, N be A-modules, the abelian group $\mathrm{Hom}_{A^a}(M^a, N^a)$ has a natural structure of A_*-module; more explicitly, we have natural isomorphisms

$$\mathrm{Hom}_{A^a}(M^a, N^a) \simeq \mathrm{Hom}_A(M, N_*) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_V(\mathfrak{m}, N)) \simeq \mathrm{Hom}_A(M \otimes_V \mathfrak{m}, N),$$

Hence we may define the internal Hom functor on $A^a - \mathbf{Mod}$ by

$$\mathrm{alHom}_{A^a}(M^a, N^a) := \mathrm{Hom}_{A^a}(M^a, N^a)^a.$$

Also, we define the functor of tensor product by

$$M^a \otimes_{A^a} N^a := \left(M_* \otimes_{A_*} N_*\right)^a.$$

The functors $\mathrm{alHom}_{A^a}(-,-)$ and $- \otimes_{A^a} -$ enjoy the same adjoint property as in the category $A - \mathbf{Mod}$.

1.2.5 - Definition. Let A be a V-algebra. An A-module M is said to be

(i) *almost flat* [3] (*resp. almost faithfully flat*) if the functor $M^a \otimes_{A^a} -$ from $A^a - \mathbf{Mod}$ to itself is exact (*resp.* exact and faithful);

(ii) *almost projective* if the functor $\mathrm{alHom}_{A^a}(M^a, -)$ is exact;

(iii) *almost finitely generated* (*resp. almost finitely presented*) if for each $\alpha \in \Lambda^+$, there exists a finitely generated (*resp.* finitely presented) A-module M_α and a morphism $f_\alpha : M_\alpha \to M$ of A-modules whose kernel and cokernel are annihilated by π^α;

(iii') *uniformly almost finitely generated* if M is almost finitely generated and if furthermore there exists some $n \in N$ independent of α such that each M_α can be generated by n elements.

(iv) A morphism $A \to B$ of V-algebras is called *almost unramified* if there exists some almost element $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu_*(e) = 1$ and e annihilates $(\ker \mu)_*$, where $\mu : B \otimes_A B \to B$ denotes the multiplication map.

1.2.6 - Remark. (i) An A-module M is almost flat and almost finitely presented if and only if it is almost projective and almost finitely generated [GR03, Proposition 2.4.18].

(ii) The definition (iii) (*resp.* (iv)) agrees with the general one, cf. [GR03, Corollary 2.3.13 and its proof (*resp.* Proposition 3.1.4)].

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[2] Naturally, one should consider a V^a-algebra A^a. But such A^a must come from some V-algebra, for example, A_* with the natural structure of V-algebra. Similarly, any A^a-module M^a comes from some A-module, for example, M_*.

[3] Most references such as [GR03] and [Sch12] use *flat* instead of *almost flat*.

1.2.7 - Definition. A morphism of $A \to B$ of V-algebras is said to be:

(i) *almost étale* if it is almost unramified and B is an almost flat A-module;

(ii) *almost finite étale* if it is almost étale and B is an almost finitely presented A-module;

(iii) an *almost étale G-Galois covering* for some finite group G of automorphisms of B over A, if the morphisms $A \to B^G$ and $B \otimes_A B \to \prod_{g \in G} B$, $x \otimes y \mapsto (xg(y))_{g\in G}$ are almost isomorphisms, and if B is a uniformly almost finitely generated A-module.

We need to justify our terminology of (iii).

1.2.8 - Lemma. *Any almost étale G-Galois covering $\phi : A \to B$ is almost finite étale and almost faithfully flat.*

Proof. Indeed, it suffices to prove that: (1) ϕ is almost unramified, and as an A-module, (2) B is almost finitely generated and almost projective (1.2.6, i), (3) B is almost faithfully flat.

(1) The almost isomorphism $B\otimes_A B \to \prod_{g\in G} B$ provides an almost element $e \in (B\otimes_A B)_* = \mathrm{Hom}_V(\mathfrak{m}, B\otimes_A B)$ corresponding to $(\delta_g)_{g\in G} \in \prod_{g\in G} B$ (here $\delta_g = 1$ if $g = \mathbf{1}_G$, and $\delta_g = 0$ otherwise), which verifies the conditions of being almost unramified.

(2) We define a trace map

$$\mathrm{Tr}_{B/A} = \sum_{g\in G} g : B \to B.$$

Thanks to the almost isomorphism $A \to B^G$, the almost morphism $\mathrm{Tr}^a_{B/A}$ factors through A^a. Thus, for any $\alpha \in \Lambda^+$, we get a genuine morphism

$$\pi^\alpha \,\mathrm{Tr}_{B/A} : B \to A,\ b \mapsto \sum_{g\in G} \pi^\alpha g(b).$$

For each $\alpha \in \Lambda^+$, write

$$e(\pi^\alpha) = \sum x_i \otimes y_i \in B \otimes B,$$

where the sum is finite. The construction of e implies that we have in B

$$\sum x_i g(y_i) = \pi^\alpha \delta_g.$$

We check that the inclusion of finitely generated sub-A-modules $B_\alpha = \sum A x_i \subset B$ has $\pi^{2\alpha}$-torsion cokernel: for $b \in B$, we have

$$\pi^{2\alpha} b = \sum_{g\in G} \pi^\alpha g(b) \sum x_i g(y_i) = \sum \pi^\alpha \,\mathrm{Tr}_{B/A}(by_i)x_i \in B_\alpha.$$

Therefore, B is an almost finitely generated A-module (although it is contained in another assumption).

Now, let's prove that, for any epimorphism $M^a \to N^a$ of A^a-modules, $\alpha \in \Lambda^+$ and any morphism $\psi \in \mathrm{Hom}_{A^a}(B^a, N^a)$, we can lift $\pi^{3\alpha}\psi$ to some $\widetilde{\psi} \in \mathrm{Hom}_{A^a}(B^a, M^a)$. By $M^a \simeq (M_*)^a$ and the adjunction $\mathrm{Hom}_{A^a}((M_*)^a, N^a) \simeq \mathrm{Hom}_A(M_*, N_*)$, we may assume that $M^a \to N^a$ is induced by $\phi : M_* \to N_*$, a morphism of A-modules; samely, we may view ψ as a morphism $B \to N_*$ of A-modules. Now, let's use the notation x_i, y_i as above. By almost surjectivity, we can choose a lift $\widetilde{x_i} \in M$ for each $\pi^\alpha\psi(x_i)$, and define

$$\widetilde{\psi} : B \to M_*,\ b \mapsto \sum \pi^\alpha \,\mathrm{Tr}_{B/A}(by_i)\widetilde{x_i}.$$

Then we check that for $b \in B$,

$$\begin{aligned} \phi \circ \widetilde{\psi}(b) &= \sum \pi^\alpha \,\mathrm{Tr}_{B/A}(by_i)\phi(\widetilde{x_i}) \\ &= \sum \pi^\alpha \,\mathrm{Tr}_{B/A}(by_i)\pi^\alpha\psi(x_i) \\ &= \psi\left(\sum \pi^\alpha \,\mathrm{Tr}_{B/A}(by_i)\pi^\alpha x_i\right) \quad \text{by } A\text{-linearity} \\ &= \psi(\pi^{3\alpha} b). \end{aligned}$$

Therefore, $\widetilde{\psi}$ lifts $\pi^{3\alpha}\psi$.

Up to now, we have shown that $A \to B$ is almost finite étale.

(3) For faithful flatness, since B is already an almost flat A-module, it suffices to prove that, if $I \subset A$ is an ideal such that $IB \approx B$, then $I \approx A$. Now let's use the assumption that B is a uniformly almost finitely generated A-module, so that we may apply the following almost version of Nakayama's lemma: for any $\alpha \in \Lambda^+$, there exists $r_\alpha \in I$ such that $\pi^\alpha + r_\alpha$ annihilates B. In particular, since $A \to B$ is almost injective, $\pi^\alpha(\pi^\alpha + r_\alpha)$ annihilates A. Hence $\pi^{2\alpha} A \subset I$. □

1.2.9 - Lemma. *Let $A \to B$ be an almost étale G-Galois covering of V-algebras for some finite group G. If C be an A-algebra, then $C \to B \otimes_A C$ is an almost étale G-Galois covering.*

Proof. By almost faithfully flat descent $A \to B$, we are reduced to the case $B \simeq \prod_{g\in G} A$, which is clear. □

1.2.10 - Lemma. *Let $A \to B$ be an almost étale G-Galois covering of V-algebras for some finite group G. If H is a subgroup of G, then $B^H \to B$ is an almost étale H-Galois covering. If furthermore H is a normal subgroup, then $A \to B^H$ is an almost étale G/H-Galois covering.*

Proof. It is enough to prove that

$$B \otimes_{B^H} B \to \prod_{h\in H} B, \quad B^H \otimes_A B^H \to \prod_{g\in G/H} B^H$$

are almost isomorphisms under their respective conditions. This is rather formal: the first one follows by using the canonical idempotent $e \in (B \otimes_A B)_*$, and the second is immediate from the almost faithfully flat descent for the base change $A \to B$. □

1.2.11 - Lemma. *Let $A \to B$ be an almost étale G-Galois covering of V-algebras for some finite group G. Then the trace map $\mathrm{Tr}_{B/A} : B^a \to A^a$ is an epimorphism of A^a-modules.*

Proof. By almost faithfully flat descent $A \to B$, one reduces to the case where $B^a = \prod_{g\in G} A^a$. □

Recall that given a topological ring B equipped with a continuous action of a topological group G, a *(topological) G-semi-linear B-module* is a B-module M with a topological G-module structure such that $g(bm) = g(b)g(m)$ for all $b \in B,\ m \in M$.

1.2.12 - Corollary. *Let $A \to B$ be an almost étale G-Galois covering of V-algebras for some finite group G. For any G-semi-linear B-module M, we have $H^i(G, M) \approx 0,\ i > 0$.*

Proof. It follows from the almost surjectivity of $\mathrm{Tr}_{B/A}$ (1.2.11) and the fact that if B is any A-algebra with an A-linear G-action, then $H^i(G, M)$ is annihilated by all elements of the form $\sum_{g\in G} g(b),\ b \in B$ for any $i > 0$. □

1.2.13 - Corollary. *Let A be a V-algebra. Let $(B_j)_{j\in J}$ be a filtered direct system of almost étale G_j-Galois covering over A, where $(G_j)_{j\in J}$ is a compatible cofiltered inverse system of finite groups. Let $B = \varinjlim_j B_j$ and $G = \varprojlim_j G_j$.*

(i) If M is a discrete G-semi-linear B-module, then $H^i(G, M) \approx 0,\ i > 0$.

(ii) Let $M = \varprojlim_n M_n$ with the inverse limit topology, where $(M_n, \phi_{n+1,n})_{n\in\mathbf{N}}$ is a Mittag-Leffler inverse system of discrete G-semi-linear B-modules. Then we have $H^i(G, M) \approx 0,\ i > 0$.

Proof. (i) For each j, applying the previous corollary to the almost étale G/H_j-Galois covering $B_j^{G/H_j} \to B_j$, where we denote $H_j = \ker(G \to G_j)$, we get $H^i(G_j/H_j, M^{H_j}) \approx 0,\ i > 0$. Taking limit over j, we obtain $H^i(G, M) \simeq \varinjlim_j H^i(G/H_j, M^{H_j}) \approx 0,\ i > 0$.

(ii) We use the exact sequence $R^1\varprojlim_n H^{i-1}(G, M_n) \to H^i(G, M) \to \varprojlim_n H^i(G, M_n)$ (1.1.2). Applying (i), it suffices then to prove $R^1\varprojlim_n (M_n)^G \approx 0$. First, consider the G-equivariant short exact sequence of discrete B-modules $0 \to \ker\phi_{n+1,n} \to M_{n+1} \to M_n \to 0$. We deduce by (i) that $(M_{n+1})^G \to (M_n)^G$ is almost surjective, so $\mathfrak{m}(M_{n+1})^G \to \mathfrak{m}(M_n)^G$ is surjective. Now consider the exact sequence

$$R^1\varprojlim_n \mathfrak{m}(M_n)^G \to R^1\varprojlim_n (M_n)^G \to R^1\varprojlim_n \left((M_n)^G/\mathfrak{m}(M_n)^G\right).$$

The left part vanishes since $\left(\mathfrak{m}(M_n)^G\right)_{n\in\mathbf{N}}$ is Mittag-Leffler, and the right part is annihilated by $\mathfrak{m}$ since $R^1\varprojlim_i$ is functorial and each quotient $(M_n)^G/\mathfrak{m}(M_n)^G$ is annihilated by $\mathfrak{m}$. We conclude that $R^1\varprojlim_n (M_n)^G \approx 0$. □

2 Adic spaces and perfectoid spaces

The rings will be commutative and unital.

2.1 Huber rings

2.1.1 - Definition. A *Huber ring* is a topological ring R which admits an open subring $R_0 \subset R$ and a finitely generated ideal $I \subset R_0$ such that the subspace topology on R_0 is the I-adic topology. In that case, we say that R_0 is a *ring of definition*, I is an *ideal of definition* and (R_0, I) is a *couple of definition*.

2.1.2. Let R be a Huber ring. A subset $S \subset R$ is called *bounded* if for each open neighbourhood $0 \in U \subset R$, there exists an open neighbourhood $0 \in V \subset R$ such that $SV \subset U$. An element $f \in R$ is called *power bounded* if $f^{\mathbf{N}} = \{f^n : n \in \mathbf{N}\}$ is bounded. An element $f \in R$ is called *topologically nilpotent* if $f^n \to 0$ as $n \to \infty$. We denote by R° the subset of power bounded elements of R and by $R^{\circ\circ}$ the subset of topologically nilpotent elements of R.

2.1.3 - Definition. (i) Let R be Huber ring.

- R is called *uniform* if R° is bounded.
- A subring $R^+ \subset R$ is called a *subring of integral elements* if it is open and integrally closed in R and $R^+ \subset R^\circ$. Any such pair (R, R^+) is called a *Huber pair.*

(ii) A *Tate ring* is a Huber ring R which admits a topologically nilpotent unit $\varpi \in R$. Any such element ϖ is called a *pseudo-uniformiser* of R. A *Tate-Huber pair* is a Huber pair (R, R^+) with R a Tate ring.

2.1.4 - Remark. Let (R, R^+) be a Tate-Huber pair. Then R is uniform if and only if R^+ is a ring of definition. Indeed, let ϖ be a pseudo-uniformiser. We have $\varpi R^\circ \subset R^+ \subset R^\circ$, so R° is bounded if and only if R^+ is bounded. A subring of R is a ring of definition if and only if it is open and bounded. Since R^+ is already open, R^+ is bounded if and only if R^+ is a ring of definition.

2.1.5. Completion. Let (R, R^+) be a Huber pair. We have $R^{\circ\circ} \subset R^+ \subset R^\circ$. The subset $R^\circ \subset R$ is an integrally closed subring. Let (R_0, I) be a couple of definition of R. Then $R^{\circ\circ}$ contains I since the elements of the latter are topologically nilpotent. We define

$$\widehat{R} = \varprojlim_n R/I^n, \quad \widehat{R^+} = \varprojlim_n R^+/I^n, \quad \widehat{R_0} = \varprojlim_n R_0/I^n.$$

Then $(\widehat{R}, \widehat{R^+})$ is a Huber pair with couple of definition $(\widehat{R_0}, I\widehat{R_0})$. We call $(\widehat{R}, \widehat{R^+})$ the *completion* of the Huber pair (R, R^+). We say that (R, R^+) is *complete* if it is naturally isomorphic to its completion. These definitions are independent of the choice of the couple of definition.

2.1.6 - Example. The pair $(\mathbf{Q}, \mathbf{Z})$ with couple of definition $(\mathbf{Z}, p\mathbf{Z})$ is a Tate-Huber pair. It has a pseudo-uniformiser $p \in \mathbf{Q}$. Its completion is $(\mathbf{Q}_p, \mathbf{Z}_p)$.

2.1.7 - Definition (Fontaine). A Tate ring R is called *perfectoid* if it is complete, uniform, and there exists a pseudo-uniformiser $\varpi \in R$ such that $p \in \varpi^p R^\circ$ in R° and the Frobenius map

$$\Phi : R^\circ/\varpi \to R^\circ/\varpi^p, \quad x \mapsto x^p$$

is an isomorphism. Such ϖ is said to be a *perfectoid pseudo-uniformiser of* R°. A Huber pair (R, R^+) with R a perfectoid Tate ring is called *perfectoid.*

2.1.8 - Remark ([Sch17, Remark 3.2]). The other conditions being verified, the last condition that $\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism is equivalent to $\Phi : R^\circ/p \to R^\circ/p$ being surjective.

2.1.9. Perfectoid Tate rings as Banach spaces. Given a perfectoid Huber pair (R, R^+) with a pseudo-uniformiser ϖ as in the definition. There exists a unit $u \in R^+$ such that $u\varpi$ admits a system of compatible p-power roots in R^+ [BMS18, Lemma 3.9]; therefore, up to multiplying ϖ by a unit, we may assume that ϖ admits a system of compatible p-power roots $\{\varpi^{1/p^n}\}_{n \in \mathbf{N}}$.

We can equip R with a norm (so called *spectral norm*)

$$\|x\|_R := \inf\left\{e^\alpha : \alpha \in \mathbf{Z}[\tfrac{1}{p}],\ \varpi^\alpha x \in R^+\right\},$$

for which R becomes a Banach space. Equivalently, we may consider the valuation

$$v_R(x) := \sup\left\{\alpha \in \mathbf{Z}[\tfrac{1}{p}] : \varpi^{-\alpha} x \in R^+\right\} = -\log \|x\|.$$

These norm and valuation are uniquely defined up to normalisation and are independent of the choice of $R^+ \subset R^\circ$. Clearly, R° is the unit ball of this Banach structure.

2.1.10 - Definition. A *perfectoid field* is a perfectoid Tate ring which is a non-archimedean field[4].

Recall here that *a non-archimedean field* is a topological field k whose topololgy is induced by a nontrivial valuation of rank one. In particular, k admits a norm $|\cdot| : k \mapsto \mathbf{R}_{\geq 0}$.

2.1.11 - Proposition ([Sch17, Proposition 3.8]). *Let K be a topological field. Then K is a perfectoid field if and only if K is a complete non-archimedean field whose topology is induced by a valuation of rank one $|\cdot| : K \to \mathbf{R}_{\geq 0}$ such that:*

- *The valuation group $|K^\times| \subset \mathbf{R}$ is not discrete;*
- $|p| < 1$*;*
- *The Frobenius $\Phi : K^\circ/p \to K^\circ/p$ is surjective.*

2.1.12 - Example. (i) Let R be a Tate ring of characteristic p. Then R is perfectoid if and only if R is complete and is a perfect ring [Sch17, Proposition 3.5].

(ii) The p-adic completion of the cyclotomic extension $\mathbf{Q}_p(\mu_{p^\infty})$ is a perfectoid field. In fact, for any finite field extension k of $\mathbf{Q}_p$, the p-adic completion of $k_\infty = k(\mu_{p^\infty})$ is perfectoid.

(iii) Let K be a perfectoid field of characteristic 0. The p-adic completion of $\bigcup_{n\in\mathbf{N}} K\langle T_1^{\pm 1/p^n}, \ldots, T_d^{\pm 1/p^n}\rangle$, denoted by $K\langle T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\rangle$, is a perfectoid K-algebra.

2.1.13. Tilting. The *tilt* of a ring A is the ring $A^\flat := \varprojlim_{(\cdot)^p} A/p$, which is a perfect $\mathbf{F}_p$-algebra. Now let R be a perfectoid Tate ring. We define its *tilt* to be

$$R^\flat := R^{\circ\flat}[\frac{1}{\varpi^\flat}],$$

where ϖ is any pseudo-uniformiser of R admitting a compatible system of p-power roots, and $\varpi^\flat := (\varpi^{1/p^n})_n \in R^{\circ\flat}$. Then $R^\flat$ admits a natural structure of topological $\mathbf{F}_p$-algebra which makes it a perfect Tate ring; we have $R^{\flat\circ} = R^{\circ\flat}$, and $\varpi^\flat$ is a pseudo-uniformiser of $R^\flat$ ([SW20, Lemma 6.2.2]).

2.1.14 - Theorem (Tilting correspondence, [SW20, Theorem 6.2.7]). *Let R be a perfectoid Tate ring with tilt $R^\flat$. Then tilting $S \mapsto S^\flat$ induces an equivalence of categories between perfectoid R-algberas and perfectoid $R^\flat$-algebras.*

2.2 Adic spaces

2.2.1. Recall that a *valuation* on some ring R is a multiplicative map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is some totally ordered abelian group (written multiplicatively), such that $|0| = 0$, $|1| = 1$ and $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in R$. We denote by $\Gamma_{|\cdot|} \subset \Gamma$ the subgroup generated by $|x|$, $x \in R$ such that $|x| \neq 0$. If R is a topological ring, then a valuation $|\cdot|$ on R is said to be *continuous* if the subsets $\{x \in R : |x| < \gamma\}$, $\gamma \in \Gamma$ are all open. Two valuations $|\cdot|$ and $|\cdot|'$ are said to be *equivalent*, denoted by $|\cdot| \simeq |\cdot|'$, if there is an isomorphism of totally ordered groups $\alpha : \Gamma_{|\cdot|} \to \Gamma_{|\cdot|'}$ such that $|\cdot|' = \alpha \circ |\cdot|$.

The *support* of a valuation $|\cdot|$ on R is $\operatorname{supp}|\cdot| := \{f \in R : |f| = 0\}$. This is a prime ideal of R by triangular inequality and the multiplicativity of $|\cdot|_x$, and it depends only on the equivalence class of the valuation.

2.2.2. Adic spectrum. Let (R, R^+) be Huber pair. We define the associated *adic spectrum*

$$\operatorname{Spa}(R, R^+) := \{|\cdot| : R \to \Gamma \cup \{0\} \text{ continuous valuation} : \forall f \in R^+, |f| \leq 1\}/\simeq .$$

For any $x \in \operatorname{Spa}(R, R^+)$, we write $|\cdot|_x$ or $f \mapsto f(x)$ for the corresponding valuation on R. We equip $\operatorname{Spa}(R, R^+)$ with the topology generated by *rational subsets*

$$\operatorname{Spa}(R, R^+)\left(\frac{f_1, \ldots, f_n}{g}\right) := \{x \in X : \forall i,\ |f_i|_x \leq |g|_x \neq 0\},$$

where $f_1, \ldots, f_n \in R$ generate an open ideal and $g \in R$. This topology is the coarsest topology on $\operatorname{Spa}(R, R^+)$ for which the subsets $\{x \in \operatorname{Spa}(R, R^+) : |f|_x \leq |g|_x \neq 0\}$ are open, for all $f, g \in R$ [Hub93, Theorem 3.5, (ii)] (there Huber put this coarsest topology on $\operatorname{Spa}(R, R^+)$ and showed that the rational subsets form a basis).

[4] It is a non-trivial fact that a perfectoid Tate ring which is a field is a perfectoid field [Ked18, Theorem 4.2].

Next, we define the *structural presheaves* $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on $X = \mathrm{Spa}(R,R^+)$ by defining them on the basis of rational open subsets: let $U = \mathrm{Spa}(R,R^+)\left(\frac{f_1,\dots,f_n}{g}\right)$ be a rational open subset and (R_0,I) be a couple of definition of R, then we define

$$(\mathcal{O}_X(U),\mathcal{O}_X^+(U)) = \left(R\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle, R\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle^+\right)$$

to be the completion of the Huber pair $(R[\frac{1}{g}], R^+[\frac{f_1}{g},\dots,\frac{f_n}{g}]^{\text{int. closure}})$ with respect to the couple of definition $(R_0[\frac{f_1}{g},\dots,\frac{f_n}{g}], I[\frac{f_1}{g},\dots,\frac{f_n}{g}])$. This is a Huber pair and depends only on U itself but not on the choice of $f_1,\dots,f_n,g$ representing U nor on the choice of couple of definition.

For every $x \in X$, let $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$ be the stalks of $\mathcal{O}_X$ and $\mathcal{O}_X^+$ at x. Let $|\cdot|_x$ represent x. By definition of the structural presheaves, $|\cdot|_x$ extends uniquely to a continuous valuation on $\mathcal{O}_X(U)$ for rational subsets U containing x. Passing to the direct limit over such U, we obtain a valuation (by abuse of notation) $|\cdot|_x$ on $\mathcal{O}_{X,x}$, whose equivalence class is uniquely determined by x.

2.2.3 - Remark. If (R,R^+) is a Tate-Huber pair, then thanks to the existence of pseudo-uniformiser, the rational subsets are

$$\mathrm{Spa}(R,R^+)\left(\frac{f_1,\dots,f_n}{g}\right) = \{x \in X : \forall i,\ |f_i|_x \le |g|_x\},$$

where $f_1,\dots,f_n \in R$ generate R as an ideal and $g \in R$.

2.2.4 - Proposition ([Hub94, Proposition 1.6 (i), (ii)]). *Let (R,R^+) be a Huber pair and $x \in X = \mathrm{Spa}(R,R^+)$. The stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal* $\mathrm{supp}\,|\cdot|_x = \{f \in \mathcal{O}_{X,x} : |f| = 0\}$. *The stalk $\mathcal{O}_{X,x}^+$ is equal to $\{f \in \mathcal{O}_{X,x} : |f| \le 1\}$ and is a local ring with maximal ideal $\{f \in \mathcal{O}_{X,x} : |f| < 1\}$.*

2.2.5 - Proposition ([Hub93, Proposition 3.9]). *Let (R,R^+) be Huber pair. Then the natural map* $\mathrm{Spa}(\widehat{R},\widehat{R^+}) \to \mathrm{Spa}(R,R^+)$ *is a bijection identifying rational subsets and structure presheaves.*

2.2.6 - Theorem ([Hub93, Lemma 3.3, Theorem 3.5]). *Let (R,R^+) be a Huber pair.*

(i) The underlying topological space of $\mathrm{Spa}(R,R^+)$ *is spectral.*
(ii) Any rational subset is quasi-compact.
(iii) $\mathrm{Spa}(R,R^+) = \emptyset$ *if and only if the closure of $\{0\}$ in R is the whole ring R.*
(iv) we have $R^+ = \{f \in R : |f|_x \le 1,\ \forall x \in \mathrm{Spa}(R,R^+)\}$; hence $R^{\circ\circ} = \{f \in R : |f|_x < 1,\ \forall x \in \mathrm{Spa}(R,R^+)\}$.

The (iv) shows that we can recover R^+ from the topological space $X = \mathrm{Spa}(R,R^+)$, the structural presheaf $\mathcal{O}_X$ and the valuations on its stalks.

2.2.7 - Definition. (i) Let $\mathscr{V}$ denote the category of triples $(X,\mathcal{O}_X,(|\cdot|_x)_{x\in X})$ where $(X,\mathcal{O}_X)$ is a topologically ringed space, and for every $x \in X$, $|\cdot|_x$ is an equivalence class of valuations of the stalk $\mathcal{O}_{X,x}$. We denote also by $|X|$ the underlying topological space. The morphisms $(X,\mathcal{O}_X,(|\cdot|_x)_{x\in X}) \to (Y,\mathcal{O}_Y,(|\cdot|_y)_{y\in Y})$ are morphisms of topologically ringed spaces compatible with valuations of stalks.

(ii) An *adic space* is an object of $\mathscr{V}$ locally an affinoid adic space, i.e. an object of $\mathscr{V}$ which is isomorphic to an adic spectrum.

(iii) A *perfectoid space* is an adic space which is locally an affinoid perfectoid space, i.e. an object of $\mathscr{V}$ which is isomorphic to the adic spectrum of a perfectoid Huber pair[5].

(iv) An adic space is called *analytic* if it is locally the adic spectrum of a Tate-Huber pair.

We say that a Huber pair (R,R^+) is *sheafy* if the presheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(R,R^+)$ is a sheaf; in this case, one verifies that $\mathcal{O}_X^+$ is also a sheaf. Not all spectra $\mathrm{Spa}(R,R^+)$ are sheafy [BV18, 4.1]. However, there is no such issue of sheafyness in most practical cases.

2.2.8 - Theorem ([Hub94, Theorem 2.2], [BV18, Theorem 7]). *Let (R,R^+) be a Huber pair. Suppose that we are in each of the following cases:*

(i) R is Tate and strongly Noetherian, i.e. the algebra of convergent series $R\langle X_1,\dots,X_n\rangle := \{\sum a_\nu X^\nu \in \widehat{R}[[X_1,\dots,X_n]] : a_\nu \to 0\}$ is Noetherian for all $n \in \mathbf{N}$);
(ii) R has a Noetherian ring of definition;

[5] If (R,R^+) is a sheafy Tate-Huber pair such that $\mathrm{Spa}(R,R^+)$ is a perfectoid space, it is not clear whether R has to be perfectoid.

(iii) (R,R^+) *is Tate and stably uniform, i.e. every rational subset* $U \subset \mathrm{Spa}(R,R^+)$ *has the property that* $\mathcal{O}_X(U)$ *is uniform.*

Then (R,R^+) *is sheafy, and, if we denote by* X *the associated affinoid adic space, we have* $H^i(U,\mathcal{O}_X) = 0$ *for all rational subsets* $U \subset X$ *and* $i \geq 1$.

2.2.9 - Definition. We call a Huber pair (R,R^+) *Noetherian* if R verifies the condition (i). An adic space is called *locally Noetherian* if it is locally the adic spectrum of a Noetherian Huber pair.

2.2.10 - Example. (i) Any algebra topologically of finite type over a non-archimedean field k is Tate and strongly Noetherian. This is the case of rigid analytic varieties.

(ii) Any perfectoid Huber pair (R,R^+) verifies the condition (iii) of the theorem: for every rational subset $U \subset \mathrm{Spa}(R,R^+)$, the topological ring $\mathcal{O}_X(U)$ is still perfectoid by [Sch12, Theorem 6.3], thus uniform. But a perfectoid Tate ring often fails to be Noetherian, so does not verify the conditions (i) or (ii) of the theorem.

2.2.11 - Proposition ([Hub94, Proposition 2.1]). *Let* (R,R^+) *be a Huber pair. Then functor* $(R,R^+) \to \mathrm{Spa}(R,R^+)$ *from sheafy complete Huber pairs to adic spaces is fully faithful.*

2.2.12 - Proposition ([SW20, Lemma 5.1.2, Proposition 5.1.5, Remark 5.1.6]). *Let* $(B,B^+) \leftarrow (A,A^+) \to (C,C^+)$ *be a diagram of of Huber pairs. Suppose that* A *is a Tate ring. Then there is a diagram* $B_0 \leftarrow A_0 \to C_0$ *of rings of definition with* A_0 *containing a pseudo-uniformiser* ϖ. *Let* $D = B \otimes_A C$, D_0 *be the image of* $B_0 \otimes_{A_0} C_0$ *in* D, *and* D^+ *be the integral closure of* $B^+ \otimes_{A^+} C^+$ *in* D. *We equip* D *with the topology given by the* ϖ*-adic on* D_0. *Then* D *is a Huber ring with* $(D_0, \varpi D_0)$ *a couple of definition, and* (D,D^+) *is the pushout of the diagram in the category of Huber pairs.*

If furthermore A, B, C *are assumed to be complete, then the completion of* (D,D^+) *is the pushout of the diagram in the category of complete Huber pairs.*

2.2.13 - Remark. The fibre product does not exist in general in the category of adic spaces. The pushout complete Tate-Huber pairs might not be sheafy even if (A,A^+), (B,B^+) and (C,C^+) are; however, it is sheafy for example when (A,A^+) satisfies one of the conditions in (2.2.8) and the morphisms are *finite étale* (see the next subsection).

2.2.14 - Proposition. *For any diagram* $Y \to X \leftarrow Z$ *of perfectoid spaces, the fibre product* $Y \times_X Z$ *in the category of adic spaces is a perfectoid space.*

Proof. We may also assume $X = \mathrm{Spa}(A,A^+)$, $Y = \mathrm{Spa}(B,B^+)$, $Z = \mathrm{Spa}(C,C^+)$ are affinoid perfectoid. Following [Sch12, Proposition 6.18] (complemented by [He21, Proposition 5.28]), we deal only with the situation where everything is over some affinoid perfectoid field (K,K^+) with a perfectoid pseudo-uniformiser $\varpi \in K^+$[6]. It is enough to prove the following stronger statement:

The topological ring $B^\circ \hat{\otimes}_{A^\circ} C^\circ$ *is almost integral perfectoid over* K°.

Here, we say that a K°-algebra R^+ is *almost integral perfectoid* if it is ϖ-adically complete, almost flat over K° and if the Frobenius $R^+/\varpi \approx R^+/\varpi^p$ is an almost isomorphism; in this case, $R = R^+[\frac{1}{\varpi}]$ is a perfectoid Tate ring with $R^\circ \simeq (R^+)_*$.

Assume first that $\mathrm{char}\, K = p$. Then the ϖ-adic completed tensor product $B^\circ \hat{\otimes}_{A^\circ} C^\circ$ is almost integral perfectoid, because it is almost flat over K° and the Frobenius is bijective. We need to see the almost flatness. Note first that any $\varpi^\flat$-torsion element $x \in B^\circ \hat{\otimes}_{A^\circ} C^\circ$ is almost zero: if $\varpi x = 0$, then by perfectness (2.1.12, i) $\varpi^{1/p} x^{1/p} = 0$, so $\varpi^{1/p} x = 0$; iterating this gives the result. Next, we observe that

A K°*-module* M *is almost flat if and only if* M *has almost zero* ϖ*-torsion.*

The "only if" part is clear by considering the injective maps $K^\circ \to K^\circ$, $x \mapsto \varpi^{1/p^n} x$. Conversely, assuming the ϖ-torsion of M to be almost zero, it is enough to show that $\mathrm{Tor}_i^{K^\circ}(M, K^\circ/I) \approx 0$, $i > 0$ for any for any ideal $I \subset K^\circ$. Roughly, since K has a nontrivial valuation of rank 1 defining its topology and is densely valued, we are easily reduced to the case where $I = (\varpi^{l/p^n})$ for some $l,n \in \mathbf{N}^*$, which is clear by assumption. More precisely, if $I = (x)$ is a principal ideal, then since there exists $n \in \mathbf{N}$ such that $\varpi^n \in I$, the x-torsion of M is almost zero by assumption, so that $\mathrm{Tor}_\bullet^{K^{\circ a}}(M^a, K/I) \approx 0$. In general, for any $l,n \in \mathbf{N}^*$, there exists $x \in K^\circ$ such that $I \subset (x)$ with cokernel annihilated by ϖ^{l/p^n}. Then, considering the exact sequence $0 \to (x)/I \to K^\circ/I \to K^\circ/x \to 0$, we find that $\mathrm{Tor}_i^{K^\circ}(M, K^\circ/I)$ is almost killed by ϖ^{l/p^n} for all $i > 0$.

[6]To deal with the general case, one may still use the tilting correspondence, but based on the Witt vector construction, see [KL19, Theorem 3.3.13] or [Ked17, Lemma 2.8.7].

Now we deal with the case where char $K = 0$[7]. According to the previous case and by tilting correspondence (2.1.14), we get a perfectoid Tate ring D such that D° fits into the commutative diagram

$$\begin{array}{ccc} B^\circ & \longrightarrow & D^\circ \\ \uparrow & & \uparrow \\ A^\circ & \longrightarrow & C^\circ \end{array}$$

and such that $D^{\flat\circ} \approx B^{\flat\circ}\hat{\otimes}_{A^{\flat\circ}} C^{\flat\circ}$. We obtain, by ϖ-adic completeness of D°, a natural map $\iota : B^\circ\hat{\otimes}_{A^\circ} C^\circ \to D^\circ$; its reduction $B^\circ\hat{\otimes}_{A^\circ} C^\circ/\varpi \to D^\circ/\varpi$ is an almost isomorphism. Then we deduce that ι itself is an almost isomorphism by the following observation:

> *Let $M \to N$ be a morphism of A-modules. If N has almost zero ϖ-torsion and $M/\varpi \to N/\varpi$ is an almost isomorphism, then the maps $M/\varpi^n \to N/\varpi^n$ are almost isomorphisms for all $n \geq 1$.*

In fact, this follows from the almost snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} & & M/\varpi^n & \xrightarrow{\cdot\varpi} & M/\varpi^{n+1} & \longrightarrow & M/\varpi f & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N/\varpi^n & \xrightarrow{\cdot\varpi} & N/\varpi^{n+1} & \longrightarrow & N/\varpi & \longrightarrow & 0 \end{array}$$

with exact rows, and by induction on n. □

2.2.15 - Corollary. *Let $(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$ be a diagram of perfectoid Huber pairs with pushout (D, D^+). Then the canonical map $B^+\hat{\otimes}_{A^+} C^+ \to D^+$ is an almost isomorphism.*

Proof. When everything is over an affinoid perfectoid field (K, K^+), this follows from the proof above. For the general case, see [KL19, Theorem 3.3.13] or [Ked17, Lemma 2.8.7]. □

2.3 Étale topology

2.3.1 - Definition. (i) A morphism $(R, R^+) \to (S, S^+)$ of Tate-Huber pairs is called *finite étale* if S is a finite étale R-algebra and carries the canonical topology (see below), and if S^+ is the integral closure of R^+ in S.

(ii) A morphism $f : X \to Y$ of analytic adic spaces is called *finite étale* if there is a cover of Y by open affinoids $V \subset Y$ such that $U = f^{-1}(V)$ is affinoid, and the associated map $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite étale.

(iii) A morphism $f : X \to Y$ of analytic adic spaces is called *étale* if locally on open subspaces $U \subset X$ and $V \subset Y$, $f|_U$ factors as $U \to W \to V$ where the first is an open immersion and the second finite étale.

For locally Noetherian analytic adic spaces, the notion of étaleness here coincides with that in [Hub96] (cf. *loc.cit.* Lemma 2.2.8).

2.3.2 - Remark ([GR03, 5.4.11]). If A is a ring with $t \in A$ a non-zero-divisor, then any finitely generated $A[\frac{1}{t}]$-module M carries a canonical topology, which induces the t-adic topology on any finitely generated A-submodule of M. This topology is well-defined as different finitely generated A-submodules of M are "equivalent" under the scaling by t. Any morphism of finitely generated $A[\frac{1}{t}]$-modules is continuous for this topology.

If furthermore A is complete and M is projective, then M is complete, being a direct summand of a finitely generated free $A[\frac{1}{t}]$-module.

2.3.3 - Remark. Let $f : \mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ be induced by a finite étale morphism $(R, R^+) \to (S, S^+)$ of complete Tate-Huber pairs. Then the map f is surjective if and only if the associated morphism of schemes $f_0 : \mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is surjective.

Indeed, first assume that f is finite étale and surjective, then the morphism $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is finite étale. Any closed point of $\mathrm{Spec}\, R$ lies in the image of the support map $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}\, R$, $x \mapsto \mathrm{supp}\, |\cdot|_x$ [Hub94, Lemma 1.4]. By surjectivity of f and the compatibility between f, f_0 and support maps, the image

[7] At this stage, one might prefer tilting to get a pushout (D, D^+) of $(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$ such that $B^{\flat\circ}\hat{\otimes}_{A^{\flat\circ}} C^{\flat\circ} \approx D^{\flat\circ}$ in the category of perfectoid pairs, but it is *a priori* not obvious to see whether this is also the pushout in the category of *completed Huber pairs*. To argue directly by tilting, especially by deformation theory, the main difficulty is to show the almost flatness of the K°-module $B^\circ\hat{\otimes}_{A^\circ} C^\circ$.

of f_0 contains all the closed points of $\operatorname{Spec} R$. Since the étale morphism f_0 is open, we conclude the "only if" part. Conversely, suppose f_0 is surjective, then for any valuation $x \in \operatorname{Spa}(R, R^+)$, there is a prime ideal $\mathfrak{q}$ lying over the support $\mathfrak{p}$ of x. Any valuation on $k(\mathfrak{p})$ extends to a valuation on $k(\mathfrak{q})$ [Bou06a, Chapitre VI, §2, n°4, Proposition 4], inducing a valuation y on S, which is continuous since S is a finite R-module. Since S^+ is the integral closure of R^+ in S and $|\cdot|_x \leq 1$ on R^+, we see that $|\cdot|_y \leq 1$ on S^+, hence $y \in \operatorname{Spa}(S, S^+)$ is mapped to x.

2.3.4 - Lemma. *Let $Y \to X$ and $Z \to X$ be finite étale (*resp. *étale) morphisms of locally Noetherian analytic*[8] *adic spaces. Then the fibre product $Y \times_X Z$ exists in the category of adic spaces. Moreover, the map of underlying topological spaces $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ is surjective.*

Proof. For a diagram $(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$ of finite étale morphisms of Tate-Huber pairs with (A, A^+) complete and Noetherian, the pair (D, D^+), where $D = B \otimes_A C$ and D^+ the integral closure of $B^+ \otimes_{A^+} C^+$ in D, is the pushout in the category of complete Huber pairs. It is sheafy because D being a finite A-module, (D, D^+) is also Noetherian. The étale case follows immediately and the general case follows by glueing. The surjectivity is by [Hub94, Lemma 3.9, (i)]. □

2.3.5 - Lemma. *If $Y \to X$ is a finite étale morphisms of locally Noetherian analytic adic spaces, then for any open affinoid $U = \operatorname{Spa}(A, A^+) \subset X$ with (A, A^+) complete, Tate and Noetherian, its preimage V is of the form $\operatorname{Spa}(B, B^+)$ with (B, B^+) finite étale over (A, A^+).*

Proof. This is [Hub96, Example 1.6.6 (ii)]. □

2.3.6. Now we turn to the étale theory over perfectoid spaces. Before employing the terminology of almost mathematics, let's clarify based on which $(V, \mathfrak{m})$ we will be talking about "almost". Suppose we have morphisms $(A, A^+) \to (B, B^+) \to (C, C^+)$ of perfectoid Huber pairs. Let $\varpi_A \in A^+$ (*resp.* $\varpi_B \in B^+$) be a pseudo-uniformiser with compatible p-power roots. Then we may take $(V, \mathfrak{m})$ to be $(A^+, \bigcup_{n \in \mathbf{N}} \varpi_A^{1/p^n})$ or $(B^+, \bigcup_{n \in \mathbf{N}} \varpi_B^{1/p^n})$. The associated almost world does not depend on the choice of pseudo-uniformiser. So we may take ϖ_B to be the image of ϖ_A in B, and thus do not distinguish between the category $B^{+a} - \mathbf{Mod}$ and the full subcategory of $A^{+a} - \mathbf{Mod}$ consisting of all B^{+a}-modules. Also, since $\varpi_A^{1/p^n} A^\circ \subset A^+$ for all $n \in \mathbf{N}$, we have $A^{+a} - \mathbf{Mod} \simeq A^{\circ a} - \mathbf{Mod}$.

2.3.7 - Theorem (Almost purity, [Sch12, Theorem 5.25, Proposition 7.6, Theorem 7.9]). *Let R be a perfectoid Tate ring.*

(i) For any finite étale R-algebra S, S equipped with the canonical topology is perfectoid, S° is almost finite étale over R°, and S° is a uniformly almost finitely generated R°-module. This induces an equivalence of categories

$$\begin{array}{ccc} \{\text{finite étale } R\text{-algebras}\} & \to & \{\text{almost finite étale } R^\circ\text{-algebras}\} \\ S & \mapsto & (S^\circ)^a, \end{array}$$

and an inverse is given by inverting ϖ.

(ii) Let R^+ be a subring of integral elements of R. The functor

$$\begin{array}{ccc} \{\text{finite étale } R\text{-algebras}\} & \to & \{\text{Adic spaces finite étale over } \operatorname{Spa}(R, R^+)\} \\ S & \mapsto & \operatorname{Spa}(S, S^+) \end{array}$$

is an equivalence of categories, where S^+ denotes the integral closure of R^+ in S.

2.3.8 - Corollary. *Let $Y \to X$ be a finite étale (*resp. *étale) morphism of analytic adic spaces. Let Z be a perfectoid space over X. Then the fibre product $Y \times_X Z$ exists and is a perfectoid space, and the projection $Y \times_X Z \to Z$ is finite étale (*resp. *étale). Moreover, the map of underlying topological spaces $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ is surjective.*

Proof. Assume first that $Y \to X$ is induced by a finite étale morphism $(A, A^+) \to (B, B^+)$ of Tate-Huber pairs and that $Z = \operatorname{Spa}(C, C^+)$ with C perfectoid. Then $D = B \otimes_A C$ is a finite étale C-algebra, hence D is perfectoid by the almost purity theorem and (D, D^+), where D^+ is the integral closure of $B^+ \otimes_{A^+} C^+$ in D, is perfectoid and is the pushout of the diagram $(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$ in the category of complete Huber pairs. The étale case and the global case follow immediately. The last sentence follows faithfully from the proof of [Hub94, Lemma 3.9, (i)]. □

[8] The condition of analyticity is in fact unnecessary (as is for the next lemma, etc.), cf. [Hub96, Proposition 1.2.2, Corollary 1.2.3]. However, we focus only on analytic cases.

2.3.9 - Proposition. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of adic spaces, where Z is either locally Noetherian analytic or perfectoid.*

*(i) If f and g are finite étale (*resp. *étale), then so is $g \circ f$.*

*(ii) Let $Y' \to Y$ be an étale morphism. If f is finite étale (*resp. *étale), then so is $X \times_Y Y' \to Y'$.*

*(iii) If g and $g \circ f$ are finite étale (*resp. *étale), then so is f.*

Proof. According to (2.3.4) or (2.3.8), the fibre product in (ii) exists and is locally described by the algebraic tensor product. Then (i) and (ii) follow by definition and (2.3.2).

The finite étale part of (iii) follows also from definition. Let's prove the étale part of (iii). If g is étale, by working locally we may assume that g factors as $Y \to \overline{Y} \to Z$ where the first is an open immersion and the second is finite étale. The diagonal $\Delta_{Y/Z} : Y \to Y \times_Z Y$ is the base change of $\Delta_{\overline{Y}/Z} : \overline{Y} \to \overline{Y} \times_Z \overline{Y}$ by $Y \times_Z Y \to \overline{Y} \times_Z \overline{Y}$. By (i), (ii) and the finite étale part of (iii), $\Delta_{\overline{Y}/Z}$ is finite étale, so $\Delta_{Y/Z}$ is also finite étale. Further base change of $\Delta_{Y/Z}$ by $X \times_Z Y \to Y \times_Z Y$ shows that the graph $X \to X \times_Z Y$ is finite étale. On the other hand, if $g \circ f : X \to Z$ is étale, then its base change by g, i.e. $X \times_Z Y \to Y$, is étale. Therefore, the composition $f : X \to X \times_Z Y \to Y$ is étale. □

This allows us to define the *étale site* $X_{ét}$ of a locally Noetherian analytic space or a perfectoid space X as the full subcategory of adic spaces over X consisting of all étale objects over X, whose coverings are jointly surjective morphisms. We denote by $\mathcal{O}_{X_{ét}}$ the presheaf $U \mapsto \mathcal{O}_U(U)$ on $X_{ét}$.

2.3.10 - Proposition ([Sch12, Proposition 7.13]). *Let X be an affinoid perfectoid space. For any étale covering $\{U_i\}_i$ of X, the sequence*

$$0 \to \mathcal{O}_X^+(X) \to \prod_i \mathcal{O}_{U_i}^+(U_i) \to \prod_{i,j} \mathcal{O}_{U_i \times_X U_j}^+(U_i \times_X U_j) \to \cdots$$

is almost exact. In particular, $H^i(X_{ét}, \mathcal{O}_{X_{ét}}^+) \approx 0$ for $i > 0$, and the presheaf $\mathcal{O}_{Y_{ét}}$ is a sheaf for any perfectoid space Y.

2.3.11. Étale Galois coverings. (i) Let $\varphi : (R, R^+) \to (S, S^+)$ be a morphism of complete Tate-Huber pair and G a finite group of automorphisms of (S, S^+) over (R, R^+). We say that φ is an *étale G-Galois covering* if $R \to S$ is an étale G-Galois covering, that is

$$R \simeq S^G, \qquad S \otimes_R S \simeq \textstyle\prod_{g \in G} S,\ x \otimes y \mapsto (xg(y))_g.$$

In this case, S is faithfully flat over R by classical commutative algebra. Note that this definition concerns only the ring morphisms $R \to S$. If $H \subset G$ is a subgroup, then (S^H, S^{+H}) is a Huber pair finite étale over (R, R^+), and $(S^H, S^{+H}) \to (S, S^+)$ is an étale H-Galois covering. If moreover H is normal, then $(R, R^+) \to (S^H, S^{+H})$ is en étale G/H-Galois covering.

(ii) Let (R, R^+) be a complete Tate-Huber pair either Noetherian or perfectoid and $f : Y \to X = \mathrm{Spa}(R, R^+)$ be a finite étale morphism. According to (2.3.5) and (2.3.7, ii), $Y = \mathrm{Spa}(S, S^+)$ and f is induced by a finite étale morphism $f^\sharp : (R, R^+) \to (S, S^+)$. We say that f is *an étale G-Galois covering* if $f^\sharp$ is so. In this case, for any subgroup $H \subset G$, we denote $Y/H := \mathrm{Spa}(S^H, S^{+H})$. It is an affinoid adic space finite étale over X, and $h : Y \to Y/H$ is an étale G-Galois covering. If moreover H is normal, then $Y/H \to X$ is an étale G/H-Galois covering.

Let's justify the notation Y/H. According to [Han16, Theorem 3.1], there is a natural homeomorphism $|Y|/H \to |Y/H|$. Since the adic spectrum Y/H is already sheafy in our situation and $\mathcal{O}_{Y/H} = (h_* \mathcal{O}_Y)^G$ (it is enough to check on global sections [Ked17, Theorem 1.4.2]), we deduce that Y/H is the categorical quotient of Y by the action of finite group H in the category $\mathscr{V}$.

(iii) If Y and Y' are respectively étale G-Galois and G'-Galois coverings of X, and if $Y \to Y'$ is surjective morphism over X compatible with the group actions via a morphism of groups $G \to G'$, then we have natural isomorphisms $G' \simeq G/H$ and $Y/H \simeq Y'$ where H is kernel of the map $G \to G'$.

Indeed, we are easily reduced to the statement concerning étale Galois covering of rings $R \to S$ and $R \to S'$ together with a morphism of R-algebras $S \to S'$, which is finite étale and faithfully flat by (2.3.3). The statement then follows immediately from definition and faithfully flat descent.

2.3.12 - Lemma. *Let $(R, R^+) \to (S, S^+)$ be an étale G-Galois covering of perfectoid Huber pairs and $\varpi \in R^+$ be a pseudo-uniformiser.*

(i) S° is an almost étale G-Galois covering of R°.

(ii) $R^+/\varpi \to (S^+/\varpi)^G$ is injective and almost surjective.

Proof. (i) We have $R \simeq S^G$ and $\prod_{g\in G} S \simeq S \otimes_R S$. By almost purity, the R°-module S° is uniformly almost finitely generated. Also, we have $S^{\circ a} \otimes_{R^{\circ a}} S^{\circ a} \simeq (S \otimes_R S)^{\circ a}$ by (2.2.15). Consequently, we have $R^+ = (S^+)^G$ hence $R^{\circ a} \simeq (S^{\circ a})^G$, and also $S^{\circ a} \otimes_{R^{\circ a}} S^{\circ a} \simeq \left(\prod_{g\in G} S\right)^{\circ a} \simeq \prod_{g\in G} S^{\circ a}$.

(ii) Recall that ϖ is a unit in R. If $x \in R^+ \cap \varpi S^+$, then $\varpi^{-1}x \in R \cap S^+ = R^+$, so that $x \in \varpi R^+$. Also, (i) implies that $R^+/\varpi \to S^+/\varpi$ is an almost étale G-Galois covering (1.2.9), so $R^+/\varpi \to (S^+/\varpi)^G$ is almost surjective. □

2.4 Pro-étale topology

2.4.1. Recall that the category pro $-\mathscr{C}$ of pro-objects of some category $\mathscr{C}$ may be defined as the category whose objects are functors $F : I \to \mathscr{C}$ from small cofiltered index categories I, and whose morphisms are given by $\mathrm{Hom}(F,G) = \varprojlim_j \varinjlim_i \mathrm{Hom}(F(i), G(j))$. We also present F as $\varprojlim_i F(i)$.

Specialised to our case, we consider the category pro $-X_{\text{ét}}$.

2.4.2 - Definition ([Sch13a, Definition 3.9]). Let X be a locally Noetherian analytic adic space.

(i) A morphism $U \to V$ of objects of pro $-X_{\text{ét}}$ is called ***étale*** (*resp.* ***finite étale***), if it is induced by an étale (*resp.* finite étale) morphism $U_0 \to V_0$ of objects in $X_{\text{ét}}$, i.e. $U = U_0 \times_{V_0} V$ via some morphism $V \to V_0$.

(ii) A morphism $U \to V$ of objects of pro $-X_{\text{ét}}$ is called ***pro-étale***, if it can be written as $U = \varprojlim_i U_i$, a cofiltered inverse limit of objects $U_i \in$ pro $-X_{\text{ét}}$ which are étale over V, such that $U_i \to U_j$ is finite étale and surjective for large $i > j$. Such a presentation $U = \varprojlim_i U_i \to V$ is called a ***pro-étale presentation***.

(iii) The ***pro-étale site*** $X_{\text{proét}}$ has as the underlying category the full subcategory of pro $-X_{\text{ét}}$ of pro-étale objects over X, and a covering is given by a family of morphisms $\{U_i \to U\}_{i\in I}$ such that $|U| = \bigcup_i |U_i|$. Here, to any $V = \varprojlim_i V_i \in$ pro $-X_{\text{ét}}$, we associate the underlying topological space $|V| := \varprojlim_i |V_i|$.

The pro-étale site $X_{\text{proét}}$ is indeed a site [Sch13a, Lemma 3.10]. The Noetherian hypothesis guarantees in fact that objects of $X_{\text{ét}}$ locally have only finitely many connected components, which is essentially used to verify the existence of equalisers in $X_{\text{proét}}$; this is needed for verifying that the site $X_{\text{proét}}$ is algebraic [Sch13a, Lemma 3.12], which makes topological arguments available.

There is a natural projection $\nu : X_{\text{proét}} \to X_{\text{ét}}$. For X lying over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, we define the structural sheaves $\mathcal{O}_X^+ = \nu^*\mathcal{O}_{X_{\text{ét}}}^+$, $\widehat{\mathcal{O}}_X^+ = \varprojlim_n \mathcal{O}_X^+/p^n$, and also $\mathcal{O}_X = \mathcal{O}_X^+[\frac{1}{p}]$, $\widehat{\mathcal{O}}_X = \widehat{\mathcal{O}}_X^+[\frac{1}{p}]$.

2.4.3 - Definition. Let X be a locally Noetherian adic space over an affinoid field (k, k^+) with a pseudo-uniformiser ϖ. An object $U \in X_{\text{proét}}$ is ***affinoid perfectoid*** if U has a pro-étale presentation $U = \varprojlim_i U_i \to X$ by affinoids $U_i = \mathrm{Spa}(R_i, R_i^+)$, such that, denoting by R^+ the ϖ-adic completion of $\varinjlim_i R_i$ and denoting $R = R^+[\frac{1}{\varpi}]$, the Huber pair (R, R^+) is perfectoid. We denote then $\hat{U} := \mathrm{Spa}(R, R^+)$.

One checks that the definition of $\hat{U}$ is independent of the pro-étale presentation $U = \varprojlim_i U_i$.

2.4.4 - Example. Let (K, K^+) be an affinoid perfectoid field of characteristic 0. For $n \in \mathbf{N} \cup \{\infty\}$, denote

$$\mathbf{T}_n^d = \mathrm{Spa}(R_n, R_n^+) := \mathrm{Spa}(K\langle T_1^{\pm 1/p^n}, \ldots, T_d^{\pm 1/p^n}\rangle, K^+\langle T_1^{\pm 1/p^n}, \ldots, T_d^{\pm 1/p^n}\rangle),$$

We denote also $\mathbf{T}^d := \mathbf{T}_0^d$, which is a standard torus over (K, K^+). Then $\mathbf{T}_n^d \to \mathbf{T}_m^d$ is finite étale surjective for all $n > m$, the object $U := \varprojlim_n \mathbf{T}_n^d \in \mathbf{T}^d_{\text{proét}}$ is affinoid perfectoid with associated affinoid perfectoid space $\hat{U} = \mathbf{T}_\infty^d$.

2.4.5 - Proposition. *Let X be a locally Noetherian adic space over an affinoid perfectoid field (K, K^+) of characteristic 0. If $U \in X_{\text{proét}}$ is affinoid perfectoid, then we have $\widehat{\mathcal{O}}_X(\hat{U}) = R$, $\widehat{\mathcal{O}}_X^+(\hat{U}) = R^+$ and $H^i(X_{\text{proét}}/U, \widehat{\mathcal{O}}_X^+) \approx 0$ for $i > 0$.*

This allows descent in the computation of cohomology of finite locally free sheaves on $X_{\text{proét}}$.

2.4.6 - Lemma. *Let (R_i, R_i^+), $i \in I$ be a direct system of Noetherian Huber pairs over an affinoid perfectoid field (K, K^+) of characteristic 0, for some small filtered index I, such that the p-adic completion (R, R^+) of the direct limit of (R_i, R_i^+) is perfectoid. Denote $\hat{U} = \mathrm{Spa}(R, R^+)$.*

Fix an index $0 \in I$. Let $V_0 = \mathrm{Spa}(S_0, S_0^+) \to U_0$ be an étale map which can be written as a composition of rational subsets and finite étale maps. For $i \geq 0$, write $V_i = V_0 \times_{U_0} U_i = \mathrm{Spa}(S_i, S_i^+)$.

(i) The completion (S,S^+) of the direct limit of (S_i,S_i^+) is perfectoid over (K,K^+). Moreover, for any $i \geq 0$, we have $\mathrm{Spa}(S,S^+) = V_i \times_{U_i} \hat{U}$ *in the category of adic spaces. In particular, if $U \in X_{pro\acute{e}t}$ is affinoid perfectoid, then $V = V_0 \times_{U_0} U = \varprojlim_i V_i \in X_{pro\acute{e}t}$ is also affinoid perfectoid.*

Let $A_i^+ = (S_i^+ \otimes_{R_i^+} R)^{\mathrm{tf},\wedge}$ be the p-adic completion of the p-torsion free quotient of $S_i^+ \otimes_{R_i^+} R^+$.

(ii) We have $S = R_i^+[\frac{1}{p}]$, for any $i \geq 0$. More precisely, the natural maps $A_i^+ \to S^+$, $i \geq 0$ is injective, their cokernels are respectively annihilated by some p^{ϵ_i}, with $\epsilon_i \to 0$.

Proof. This is essentially [Sch13a, Lemma 4.5] with relaxed conditions on the system $\{(R_i,R_i^+)\}_i$. The transition maps were assumed to be eventually finite étale coverings, which was in fact not needed in the proof. □

2.4.7 - Definition. Let (R,R^+) be a complete Tate-Huber pair either Noetherian or perfectoid, $\varpi \in R$ a pseudo-uniformiser and $X = \mathrm{Spa}(R,R^+)$. Let G be a (profinite) topological group. An object $U \in \mathrm{pro}-X_{\acute{e}t}$ is said to be an *affinoid perfectoid pro-étale G-Galois covering* of X, if U has a presentation $U = \varprojlim_i U_i$ by affinoids $U_i = \mathrm{Spa}(S_i,S_i^+)$, such that:

- the transition map $U_i \to U_j$ is finite étale and surjective for any $i > j$;
- $U_i \to X$ is an étale G_i-Galois covering for all i, and $G \simeq \varprojlim_i G_i$ as topological groups;
- the pair (S,S^+) is a perfectoid Huber pair, where S^+ denotes the p-adic completion of $\varinjlim_i S_i$ and $S = S^+[\frac{1}{\varpi}]$.

An adic space of the form $\hat{U} = \mathrm{Spa}(S,S^+)$ is then called an *affinoid perfectoid pro-étale G-Galois covering of X*.

The transition maps $G_i \to G_j$ are surjective by (2.3.11, iii), so the maps $G \to G_i$ are also surjective.

2.4.8 - Example. Let (K,K^+) be an affinoid perfectoid field over $\mathbf{Q}_p(\mu_{p^\infty})$. Then $\varprojlim_n \mathbf{T}_n^d$ considered in (2.4.4) is affinoid perfectoid pro-étale Galois over $\mathbf{T}^d$ with Galois group $\Gamma \simeq \mathbf{Z}_p(1)^d$. If $\zeta = (\zeta_{p^n})_n \in \mathbf{Z}_p(1)$ is a $\mathbf{Z}_p$-basis, then the canonical isomorphism writes

$$\Gamma = \bigoplus_i \mathbf{Z}_p\gamma_i \simeq \mathbf{Z}_p(1)^d, \quad \gamma_i \mapsto (0,\ldots,\underset{i\text{-th}}{\zeta},\ldots,0),$$

where the action of γ_i is given by $\gamma_i(T_j^{1/p^n}) = \zeta_{p^n}^{\delta_{ij}} T_j^{1/p^n}$.

2.4.9 - Lemma. *Let (R,R^+) be a perfectoid Huber pair with pseudo-uniformiser ϖ. Let* $\mathrm{Spa}(S,S^+)$ *be an affinoid perfectoid pro-étale G-Galois covering of* $\mathrm{Spa}(R,R^+)$.

(i) We have $R^+/\varpi \approx (S^+/\varpi)^G$, $R^+ \approx S^{+G}$ and $S^G = R$.

(ii) If M is a flat R^+-module with the trivial G-action, then $(S^+\hat{\otimes}_{R^+}\widehat{M}[\frac{1}{\varpi}])^G \simeq \widehat{M}[\frac{1}{\varpi}]$, where $\widehat{M}$ is the ϖ-adic completion of M.

Proof. Let ϖ be a perfectoid pseudo-uniformiser of R.

(i) Let $\{S_i\}_i$ be as (2.4.7). According to the previous remark, we may assume that for $i > j$, the transition map $G_i \to G_j$ is surjective and $S_j^+ \to S_i^+$ is an almost étale G_i/G_j-covering.

We have

$$R^+/\varpi \to (\varinjlim_j (S_j^+/\varpi))^G \simeq (\varinjlim_j S_j^+/\varpi)^G = (S^+/\varpi)^G.$$

Let's show that the first arrow is injective and almost surjective. On the one hand, for any $g \in G$, $x \in S_j^+$ such that $[gx] = [x] \in \varinjlim_j S_j^+/\varpi$ (the bracket meaning the equivalence classe), then $[gx] = [x] \in S_{i_g}^+/\varpi$ for some index $i_g > j$. Since the action of G on $\varinjlim_j (S_j^+/\varpi)$ is smooth and J is cofiltered, one may find $i > j$ such that $[gx] = [x] \in S_i^+/\varpi$ for all $g \in G$. The almost surjectivity then follows from that of $R^+/\varpi \approx (S_i^+/\varpi)^{G_i}$. On the other hand, since each map $R^+/\varpi \to S_j^+/\varpi$ is injective (2.3.12), the direct limit is also injective.

Now, this implies that the natural map $\varphi : R^+ \to S^{+G}$ is injective and almost surjective. Indeed, on the one hand, we have $\ker\varphi \subset \varpi R^+$ for $n \geq 0$. But ϖ being a non-zero-divisor in S^+, we have furthermore $\ker\varphi \subset \varpi \ker\varphi$. A simple induction and the ϖ-adic completeness of R^+ imply $\ker\varphi = 0$. For almost surjectivity, we have $\varpi^{1/p^n} S^{+G} \subset R^+ + \varpi S^+$ for all $n \geq 0$. Since $\varpi \in S^+$ is a non-zero-divisor, we have furthermore $\varpi^{1/p^n} S^{+G} \subset R^+ + \varpi S^{+G}$ for all $n \geq 0$. An induction shows that for any $y \in S^{+G}$ and $n \geq 1$, one can find $x_m \in R^+$ such that

$$\varpi^{1/p^n} y = \sum_{m\in\mathbf{N}} \varpi^{m(1-1/p^n)} x_m.$$

The almost surjectivity then follows from the ϖ-adic completeness of R^+.

(ii) The flatness implies that $\widehat{M}$ and $S^+\hat{\otimes}_{R^+}\widehat{M}$ are ϖ-torsion free, that $M/\varpi \to (S^+ \otimes_{R^+} M/\varpi)^G$ is an almost isomorphism, and that this lifts to an almost isomorphism $\widehat{M} \to (S^+\hat{\otimes}_{R^+}\widehat{M})^G$. Inverting ϖ, we obtain the desired isomorphism. □

2.4.10 - Lemma. *Let (R, R^+) be a perfectoid Huber pair and* $\mathrm{Spa}(S, S^+)$ *be an affinoid perfectoid pro-étale G-Galois covering of* $\mathrm{Spa}(R, R^+)$. *Let H be a compact open subgroup of G. Then $U/H := \varprojlim_i U_i/H_i$ is represented by the adic space* $\mathrm{Spa}(S^H, (S^H)^+)$, *where $(S^H)^+$ denotes the integral closure of R in S^H. The object $U \in (U/H)_{\text{proét}}$ is an affinoid perfectoid pro-étale H-Galois covering. The adic space U/H is finite étale over X, and is an étale G/H-Galois covering of X if H is normal in G.*

In particular, without loss of generality, we may insert U/H (seen as an adic space) into the index system of the presentation $U = \varprojlim_i U_i$ if needed.

Proof. We may assume that $G_j \to G_i$ is surjective for all $j > i$. Denote $H_i := \mathrm{Im}(G \to G_i)$ for any index i. By (2.3.11), $U_i = \mathrm{Spa}(R_i, R_i^+)$ is H_i-Galois over $\mathrm{Spa}(R_i^{H_i}, R_i^{+H_i})$. The latter is finite étale over X and is an étale G_i/H_i-Galois covering of X if H is normal in G. On the other hand, $(R_i^{H_i}, R_i^{+H_i})$ is independent of the index i; indeed, since for $j > i$, $H_j \to H_i$ is surjective with kernel $K_{ij} = \ker(G_j \to G_i)$ and $R_j^{K_{ij}} = R_i$. Then let $(R', R'^+) := (R_i^{H_i}, R_i^{+H_i})$. Its adic spectrum represents U/H. Clearly by definition, $U = \varprojlim_i U_i$ is an affinoid perfectoid pro-étale H-covering of U/H. □

2.4.11 - Lemma. *Let (A, A^+) be a Noetherian Tate-Huber pair and $X = \mathrm{Spa}(A, A^+)$. If $U \in X_{\text{proét}}$ is affinoid perfectoid and $V \in X_{\text{proét}}$ is an affinoid perfectoid pro-étale G-Galois covering of X, then*

(i) $W = V \times U \in X_{\text{proét}}$ is affinoid perfectoid;

(ii) $V \times \hat{U}$ is affinoid perfectoid pro-étale G-Galois over $\hat{U}$ with associated affinoid perfectoid space $\hat{W}$;

(iii) we have $\hat{W} = \hat{U} \times_X \hat{V}$ in the category of adic spaces;

(iv) if furthermore V is affinoid perfectoid pro-étale H-Galois over X, then W is affinoid perfectoid pro-étale $G \times H$-Galois over X.

Proof. We follow the proof of [Sch13a, Lemma 4.6]. We have a factorisation $V = \varprojlim_j V_j \to V_0 \to X$ with $V_j \to V_0$ finite étale and surjective, $V_0 \to X$ finite étale (thus small) and V_i étale G_i-Galois over X. Write $U = \varprojlim_i U_i$. We write $W = \varprojlim_{i,j} U_i \times_X V_j = \varprojlim_{i,j} \mathrm{Spa}(S_{ij}, S_{ij}^+)$ and denote $W_j := V_j \times U$.

By (2.4.6), $\hat{W}_0 = V_0 \times_X \hat{U}$ is affinoid perfectoid and finite étale G_0-Galois over $\hat{U}$; similarly, $\hat{W}_j = V_j \times_X \hat{U} = \mathrm{Spa}(S_j, S_j^+)$ is affinoid perfectoid, finite étale and surjective over $\hat{W}_0$ and is étale G_j-Galois over $\hat{U}$; and (S_j, S_j^+) is the completion of the direct limit over i of (S_{ij}, S_{ij}^+). It follows that the completion of the direct limit over i, j of (S_{ij}, S_{ij}^+) is the completion of the direct limit over j of (S_j, S_j^+); but the completion of a direct limit of perfectoid Huber pairs is again perfectoid (use for example (2.1.4) to show the uniformity of the direct limit, since each S_j^+ is a ring of definition). This proves (i) and (ii).

Then (iv) follows immediately, since in that case, $U_i \times_X V_j$ will be étale $G_i \times H_j$-Galois over X.

For (iii), write $\hat{W} = \mathrm{Spa}(S, S^+)$, $\hat{U} = \mathrm{Spa}(R, R^+)$ and $\hat{V} = \mathrm{Spa}(R', R'^+)$. We check that (S, S^+), being the completion of the direct limit over i, j of (S_{ij}, S_{ij}^+), is also the pushout of $(R, R^+) \leftarrow (A, A^+) \to (R', R'^+)$ in the category of complete Huber pairs, cf. (2.2.12). □

We will need the following lemma when studying the dependence of Sen's operator on toric charts.

2.4.12 - Lemma ([Sch13b, Lemma 3.24]). *Let C be a complete algebraic closed extension of $\mathbf{Q}_p$. Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid adic space which is small, that is there is a toric chart*

$$f : X \to \mathbf{T}^d = \mathrm{Spa}(C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, C^\circ\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$$

which is a composition of finite étale maps and rational subsets. Consider the exact sequence

$$0 \to \hat{\mathbf{Z}}_p(1) \to \varprojlim_{(\cdot)^p} \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0$$

on $X_{proét}$, where $\hat{\mathbf{Z}}_p(1) = \varprojlim_{(\cdot)^p} \mu_{p^n}$. It induces a boundary map $\delta_0 : \nu_* \mathcal{O}_X^\times \to R^1\nu_* \hat{\mathbf{Z}}_p(1)$. Then there is a unique $\mathcal{O}_{X_{ét}}$-linear map $\delta : \Omega^1_{X_{ét}} \to R^1\nu_*\hat{\mathcal{O}}_X(1)$ such that the diagram

$$\begin{array}{ccc} A^\times = H^0(X_{proét}, \mathcal{O}_X^\times) & \xrightarrow{\delta_0} & H^1(X_{proét}, \hat{\mathbf{Z}}_p(1)) \\ \downarrow \mathrm{dlog} & & \downarrow \\ \Omega^1_{A/\mathbf{C}} = H^0(X_{proét}, \Omega^1_X) & \xrightarrow{\delta} & H^1(X_{proét}, \widehat{\mathcal{O}}_X(1)) \end{array}$$

commutes; this map is an isomorphism.

Let's fix a toric chart $f : X \to \mathbf{T}^d$ and describe the map δ in the diagram. Let $X_\infty = \mathrm{Spa}(A_\infty, A_\infty^+) := X \times_{\mathbf{T}^d} \mathbf{T}^d_\infty$; this is an affinoid perfectoid pro-étale Galois covering of X with Galois group $\Gamma^d \simeq \mathbf{Z}_p^d(1)$. We have a natural commutative diagram

$$\begin{array}{ccccc} H^1_{\mathrm{cont}}(\Gamma^d, \varprojlim_n \mu_{p^n}(C)) & \longrightarrow & \check{H}^1(X_\infty/X, \hat{\mathbf{Z}}_p(1)) & \longrightarrow & H^1(X_{\mathrm{proét}}, \hat{\mathbf{Z}}_p(1)) \\ \downarrow & & & & \downarrow \\ H^1_{\mathrm{cont}}(\Gamma^d, R_\infty(1)) & \xrightarrow{\simeq} & \check{H}^1(X_\infty/X, \widehat{\mathcal{O}}_X(1)) & \xrightarrow{\simeq} & H^1(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X(1)) \end{array}$$

By abuse of notation, we denote the image $f^*T_i \in A_\infty$ still by T_i for each $i = 1, \ldots, d$. By construction of δ_0, the image $\delta_0(T_i)$ is represented by the cocycle $g \mapsto \{g(T_i^{1/p^n})/T_i^{1/p^n}\}_n$. Then we may define $\delta(\mathrm{dlog} T_i) \in H^1(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X(1))$ to be the cocycle defined by the same expression.

3 Tate-Sen's formalism

We present the Tate-Sen's formalism of Colmez in the following form. Our references are [Col03] and [BC08].

3.1 Tate-Sen's conditions

3.1.1. Set-up. Let G be a profinite group and $\chi : G \to \mathbf{Z}_p^\times$ a continuous character with infinite (thus open) image. Let $H = \ker \chi$. For each open subgroup G_0 of G, let $H_0 = G_0 \cap H$, denote $\widetilde{\Gamma}_{H_0} = N_G(H_0)/H_0$ and $C_{H_0} = \mathrm{Cent}(\widetilde{\Gamma}_{H_0})$. If G_0 is normal in G, then $\widetilde{\Gamma}_{H_0} = G/H_0$.

Let $\widetilde{\Lambda}$ be a commutative unital $\mathbf{Z}_p$-algebra equipped with a valuation $v : \widetilde{\Lambda} \to \mathbf{R} \cup \{+\infty\}$ such that:

- $v(x) = +\infty$ if and only if $x = 0$;
- $v(xy) \geq v(x) + v(y)$;
- $v(x + y) \geq \inf\{v(x), v(y)\}$;
- $v(p) > 0$ and $v(px) = v(p) + v(x)$ for $x \in \widetilde{\Lambda}$.

This induces the valuation topology on $\widetilde{\Lambda}$, for which we assume $\widetilde{\Lambda}$ to be complete. We suppose in addition that G acts continuously and isometrically on $\widetilde{\Lambda}$, i.e. $v(g(x)) = v(x)$ for all $g \in G$ and $x \in \widetilde{\Lambda}$.

3.1.2 - Remark. Let $d \geq 1$ be an integer. We equip $\mathrm{Mat}_d(\widetilde{\Lambda})$ with the valuation (by abuse of notation) $v(A) = \min_{i,j} v(a_{ij})$ where a_{ij} are the entries of $A \in \mathrm{Mat}_d(\widetilde{\Lambda})$. Clearly, we have that

- v verifies the four conditions listed above but with $x, y \in \mathrm{Mat}_d(\widetilde{\Lambda})$;
- $\mathrm{Mat}_d(\widetilde{\Lambda})$ is a $\mathbf{Z}_p$-algebra complete for the valuation topology, and on which G acts continuously and isometrically;
- if $A \in \mathrm{Mat}_d(\widetilde{\Lambda})$ satisfies $v(A - 1) > 0$, then $v(A) = 0$ and A is invertible with inverse $\sum_{i \geq 0}(1 - A)^i$ by completeness of $\widetilde{\Lambda}$ with respect to the valuation topology.

3.1.3 - Definition. The *Tate-Sen's conditions* are the following:

(TS1) There exists $c_1 > 0$ such that, for any open subgroups $H_1 \subset H_0$ of H, there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ satisfying $v(\alpha) > -c_1$ and $\sum_{\tau \in H_0/H_1} \tau(\alpha) = 1$.

(TS2) There exist $c_2 > 0$, and for each open subgroup H_0 of H, an integer $n(H_0) \in \mathbf{N}$, an increasing sequence $(\Lambda_{H_0,n})$ of closed sub-$\mathbf{Z}_p$-algebras of $\widetilde{\Lambda}^{H_0}$ together with $\mathbf{Z}_p$-linear maps $R_{H_0,n} : \widetilde{\Lambda}^H \to \Lambda_{H_0,n}$, such that:

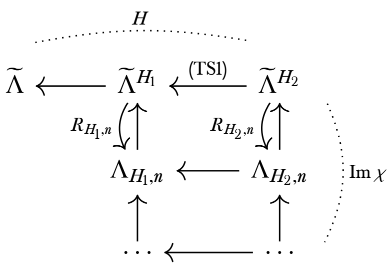
  a) If $H_1 \subset H_2$, then $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and $R_{H_1,n} = R_{H_2,n}$ on $\widetilde{\Lambda}^{H_2}$;
  b) $R_{H_0,n}$ is $\Lambda_{H_0,n}$-linear and $R_{H_0,n}(x) = x$ for $x \in \Lambda_{H_0,n}$;
  c) $g(\Lambda_{H_0,n}) = \Lambda_{gH_0g^{-1},n}$ and $g(R_{H_0,n}(x)) = R_{gH_0g^{-1},n}(gx)$ for $g \in G$ and $x \in \widetilde{\Lambda}$;
  d) If $n \geq n(H_0)$ and $x \in \widetilde{\Lambda}^{H_0}$, then $v(R_{H_0,n}(x)) \geq v(x) - c_2$;
  e) $\lim_{n\to+\infty} R_{H_0,n}(x) = x$ for $x \in \widetilde{\Lambda}^{H_0}$.

(TS3) There exist $c_3 > 0$, and for each open subgroup G_0 of G, an integer $n(G_0)$, such that:

  a) $\chi(C_{H_0}) \supset 1 + p^{n(G_0)}\mathbf{Z}_p$ where $H_0 = G_0 \cap H$;
  b) For $\gamma \in \widetilde{\Gamma}_{H_0}$, if $n \geq \max\{n(G_0), v_p(\chi(\gamma) - 1)\}$, then $\gamma - 1$ is invertible on $X_{H_0,n} := \ker R_{H_0,n} \subset \widetilde{\Lambda}^{H_0}$;
  c) $v((\gamma - 1)^{-1}(x)) \geq v(x) - c_3$ for $x \in X_{H_0,n}$.

Roughly, up to constants c_1, c_2, c_3 which we hope to be small, the condition (TS1) is an "étaleness" condition up to some constant c_1, (TS2) means the existence of compatible Tate's normalised traces $R_{H_0,n}$, and (TS3) indicates the invertibility of $\gamma - 1$ on $\ker R_{H_0,n}$ and asks the operator norm of $(\gamma - 1)^{-1}$ to be not much greater than 1. Here is a commutative diagram of some of these objects:

(3.1.3.1)



3.1.4 - Remark. It follows from the definition that the Tate-Sen's conditions are still verified if we shrink G to some open subgroup of it containing $H = \ker\chi$.

3.1.5 - Proposition ([BC08, Proposition 3.2.6]). *Suppose that (TS1)-(TS3) are verified. Let $G \to \mathrm{GL}_d(\widetilde{\Lambda})$, $\sigma \mapsto U_\sigma$ be a continuous cocycle. Let G_0 be an open normal subgroup of G such that $U_\sigma - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(U_\sigma - 1) > c_1 + 2c_2 + 2c_3$ for all $\sigma \in G_0$. Then there exists $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ such that:*

- *$M - 1 \in 1 + p^k\mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M-1) > c_2 + c_3$,*
- *the cocycle $\sigma \mapsto M^{-1}U_\sigma\sigma(M)$ is trivial on $H_0 := G_0 \cap H$ and takes values in $\mathrm{GL}_d(\Lambda_{H_0,n(G_0)})$.*

By superscript "+" on $\widetilde{\Lambda}$ (*resp.* $\Lambda_{H_0,n}$), we mean the corresponding unit ball, i.e. the subset of elements with non-negative valuation.

3.1.6 - Theorem ([BC08, Proposition 3.3.1]). *Suppose that (TS1)-(TS3) are verified. Let T be a $\mathbf{Z}_p$-representation of G free of rank d. Let k be an integer such that $v(p^k) > c_1 + 2c_2 + 2c_3$. Let G_0 be an open normal subgroup of G acting trivially on T/p^kT and write $H_0 = G_0 \cap H$. Then for each $n \geq n(G_0)$, $\widetilde{\Lambda}^+ \otimes_{\mathbf{Z}_p} T$ contains a unique sub-$\Lambda^+_{H_0,n}$-module $D^+_{H_0,n}(T)$ free of rank d satisfying:*

(i) $D^+_{H_0,n}(T)$ is fixed by H_0 and stable under G;

(ii) The natural G_0-equivariant map $\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_0,n}} D^+_{H_0,n}(T) \to \widetilde{\Lambda}^+ \otimes_{\mathbf{Z}_p} T$ is an isomorphism;

(iii) $D^+_{H_0,n}(T)$ has a $\Lambda^+_{H_0,n}$-basis which is c_3-fixed *(i.e. such that for all $\gamma \in G_0/H_0$, the corresponding matrix W_γ satisfies $v(W_\gamma - 1) > c_3$).*

Proof. By choosing a $\mathbf{Z}_p$-basis of T, the action of G is represented by a continuous group homomorphism $U : G \to \mathrm{GL}_d(\mathbf{Z}_p)$ seen as a continuous cocycle $G \to \mathrm{GL}_d(\widetilde{\Lambda}^+)$. The hypotheses on G_0 and k show that the conditions of the previous proposition are satisfied. Therefore we obtained a transition matrix $M \in \mathrm{GL}_d(\widetilde{\Lambda}^+)$ with properties which indicate the existence of a desired sub-$\Lambda^+_{H_0,n}$-module $D^+_{H_0,n}(T)$.

Suppose that there is another such submodule. Fix $\gamma \in G$ satisfying $n(\gamma) = n$ and let $U_\gamma, U'_\gamma \in \mathrm{GL}_d(\Lambda^+_{H_0,n})$ be the matrices of the action of γ on each submodule under respective $\Lambda^+_{H_0,n}$-basis verifying the condition (iii). Let $B \in \mathrm{GL}_d(\widetilde{\Lambda}^+)$ be the transition matrix between two bases. So we obtain $U'_\gamma = B^{-1}U_\gamma\gamma(B)$, that is

$$\gamma(B) = U_\gamma^{-1}BU'_\gamma.$$

By condition (iii), it follows from the next lemma that $B \in \mathrm{GL}_d(\Lambda^+_{H_0,n})$, which shows the uniqueness. □

3.1.7 - Lemma. *Let H_0 be an open subgroup of H, $n \geq n(H_0)$, $\gamma \in \widetilde{\Gamma}_{H_0}$ satisfying $n \geq v_p(\chi(\gamma) - 1)$. Let $B \in \mathrm{GL}_d(\widetilde{\Lambda}^{H_0})$. If there exists $V_1, V_2 \in \mathrm{GL}_d(\Lambda_{H_0,n})$ satisfying $v(V_1 - 1) > c_3$ and $v(V_2 - 1) > c_3$ such that $\gamma(B) = V_1BV_2$, then $B \in \mathrm{GL}_d(\Lambda_{H_0,n})$.*

Proof. Writing $B = R_{H_0,n}(B) + C$, it suffices to show $C = 0$. By $\Lambda_{H_0,n}$-linearity of trace maps, we have $\gamma(C) = V_1CV_2$, so $\gamma(C) - C = (V_1 - 1)CV_2 + C(V_2 - 1)$, so that

$$v((\gamma - 1)(C)) \geq v(C) + \min\{v(V_1 - 1), v(V_2 - 1)\} \geq v(C) + v_3.$$

If C were non-zero, then the last inequality would be strict since $v(C) \neq 0$, contradicting (TS3). □

The proof of (3.1.5) generalises that in [Sen80] and splits into two steps: almost étale descent and decompletion.

3.2 Descent

3.2.1 - Lemma. *Suppose that (TS1) is verified. Let $a > c_1$, $k \in \mathbf{N}$ and H_0 an open subgroup of H. Suppose that $U : H_0 \to \mathrm{GL}_d(\widetilde{\Lambda})$ is a continuous cocycle satisfying $U_\tau - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(U_\tau - 1) \geq a$ for all $\tau \in H_0$. Then there exists a matrix $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ satisfying $M - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M-1) \geq a - c_1$ such that the cocycle $\tau \mapsto M^{-1}U_\tau\tau(M)$ verifies for all $\tau \in H_0$*

$$v(M^{-1}U_\tau\tau(M) - 1) \geq a + 1.$$

Proof. By continuity of U, let H_1 be an open subgroup of H_0 such that $v(U_\tau - 1) \geq a + 1 + c_1$ for all $\tau \in H_1$. By (TS1), let $\alpha \in \widetilde{\Lambda}^{H_1}$ be such that $v(\alpha) > -c_1$ and $\sum_{\tau \in H_0/H_1} \tau(\alpha) = 1$. If Q is a set of representatives of H_0/H_1, we write

$$M_Q = \sum_{\sigma \in Q} \sigma(\alpha) U_\sigma.$$

Then $M_Q - 1 \in p^k \mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M_Q - 1) \geq a - c_1$; in particular $v(M_Q - 1) > 0$, hence M_Q is has valuation 0 and is invertible. On the other hand, for $\tau \in H_0$, we have

$$U_\tau \tau(M_Q) = \sum_{\sigma \in Q} \tau\sigma(\alpha) U_\tau \tau(U_\sigma) = \sum_{\sigma \in Q} \tau\sigma(\alpha) U_{\tau\sigma} = M_{\tau Q},$$

so that

$$M_Q^{-1} U_\tau \tau(M_Q) = 1 + M_Q^{-1}(M_{\tau Q} - M_Q).$$

For each $\sigma \in Q$, there are $\sigma' \in Q$, $\tau' \in H_1$ such that $\tau\sigma = \sigma'\tau'$; these σ' form a permutation of Q. Since also α is fixed by H_1, we have

$$M_{\tau Q} - M_Q = \sum_{\sigma \in Q} (\sigma'\tau'(\alpha) U_{\sigma'\tau'} - \sigma' U_{\sigma'}) = \sum_{\sigma' \in Q} \sigma'(\alpha) U_{\sigma'} \sigma(U_{\tau'} - 1),$$

so $v(M_{\tau Q} - M_Q) \geq -c_1 + (a + 1 + c_1) = a + 1$. Therefore we can choose $M = M_Q$. ◻

3.2.2 - Corollary. *Under the conditions of (3.2.1), there exists $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ satisfying $M - 1 \in p^k \mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M - 1) \geq a - c_1$ such that for all $\tau \in H_0$,*

$$M^{-1} U_\tau \tau(M) = 1.$$

Proof. By induction using the lemma, we obtain matrices M_i, $i \in \mathbf{N}$ satisfying $M_i - 1 \in p^k \mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M_i - 1) \geq a - c_1 + i$ such that the cocycle $\tau \mapsto U_{n,\tau} = (\prod_{i=0}^n M_i)^{-1} U_\tau \tau(\prod_{i=0}^n M_i)$ verifies $v(U_{n,\tau} - 1) \geq a + n + 1$ for all $\tau \in H_0$. Then the infinite product $\prod_{i \in \mathbf{N}} M_i$ converges to an element $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ with the desired properties. ◻

3.3 Decompletion

3.3.1 - Lemma. *Suppose that (TS2) and (TS3) are verified. Let $\delta > 0$ and $b \in \mathbf{R}$ satisfying $b \geq 2c_2 + 2c_3 + \delta$. Let H_0 be an open subgroup of H, $n \geq n(H_0)$, $\gamma \in \widetilde{\Gamma}_{H_0}$ satisfying $n \geq v_p(\chi(\gamma) - 1)$. If $a \in \mathbf{R}$ satisfies $c_2 + c_3 + \delta \leq a \leq b - c_2$, and $U = 1 + p^k U_1 + p^k U_2 \in \mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$ with*

$$U_1 \in \mathrm{Mat}_d(\Lambda_{H_0,n}), \quad v(p^k U_1) \geq a$$
$$U_2 \in \mathrm{Mat}_d(\widetilde{\Lambda}^{H_0}), \quad v(p^k U_2) \geq b,$$

then there exists $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ satisfying $M - 1 \in p^k \mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$ and $v(M - 1) \geq b - c_2 - c_3$ such that $M^{-1} U \gamma(M) = 1 + p^k V_1 + p^k V_2$ with

$$V_1 \in \mathrm{Mat}_d(\Lambda_{H_0,n}), \qquad v(p^k V_1) \geq a$$
$$V_2 \in \mathrm{Mat}_d(\widetilde{\Lambda}^{H_0}), \quad v(p^k V_2) \geq b + \delta.$$

Proof. According to (TS2) and (TS3), we can write as $U_2 = R_{H_0,n}(U_2) + (\gamma - 1)(W)$ with $v(R_{H_0,n}(p^k U_2)) \geq b - c_2 \geq a$ and $v(p^k W) \geq b - c_2 - c_3$. We claim that we can take $M = 1 - p^k W$. Indeed, we have

$$M^{-1} U \gamma(M) = (\textstyle\sum_{i \geq 0} p^{ik} W^i)(1 + p^k U_1 + p^k U_2)(1 - p^k \gamma(W)) = 1 + p^k U_1 + p^k U_2 - p^k (\gamma - 1)(W) + p^{2k} V$$

for some $V \in \mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$. Putting $V_1 = U_1 + U_2 - (\gamma - 1)(W) = U_1 + R_{H_0,n}(U_2) \in \mathrm{Mat}_d(\Lambda_{H_0,n})$, we have

$$v(p^k V_1) \geq \min\left\{v(p^k U_1), v(p^k U_2) - c_2\right\} \geq a.$$

On the other hand, by developing the product series and ultrametric inequality, we obtain

$$v(p^{2k} V) = \min\left\{2v(p^k W), v(p^k W) + v(p^k U_1), v(p^k W) + v(p^k U_2)\right\} \geq 2(b - c_2 - c_3) \geq b + \delta.$$

So $V_2 = p^k V$ together with V_1 verifies the desired properties. ◻

3.3.2 - Corollary. *Under the conditions of (3.3.1), if* $U \in \mathrm{Mat}_d(\widetilde{\Lambda})$ *satisfies* $U-1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$ *and* $v(U-1) \geq b$. *Then there exists* $M \in \mathrm{GL}_d(\widetilde{\Lambda}^{H_0})$ *satisfying* $M-1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$ *and* $v(M-1) \geq b - c_2 - c_3$ *such that*

$$M^{-1}U\gamma(M) \in 1 + p^k\mathrm{Mat}_d(\Lambda_{H_0,n}).$$

Proof. Applying the lemma to $a = b - c_2$ and by induction, we obtain matrices M_i, $i \in \mathbf{N}$ satisfying $M_i - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$ and $v(M_i - 1) \geq b + i\delta - c_2 - c_3$ such that $(\prod_{i=1}^n M_i)^{-1}U\gamma(\prod_{i=1}^n M_i) = 1 + p^kV_{n,1} + p^kV_{n,2}$ with

$$V_{n,1} \in \mathrm{Mat}_d(\Lambda_{H_0,n}), \qquad v(p^kV_{n,1}) \geq a$$
$$V_{n,2} \in \mathrm{Mat}_d(\widetilde{\Lambda}^{H_0}), \quad v(p^kV_{n,2}) \geq b + (n+1)\delta.$$

Then the infinite product $\prod_{i\in\mathbf{N}} M_i$ converges to an element $M \in \mathrm{GL}_d(\widetilde{\Lambda}^{H_0})$. Then $M^{-1}U\gamma(M) = 1 + p^kV_1$ where $V_1 = \lim_n V_{n,1} \in \Lambda_{H_0,n}$ because $\Lambda_{H_0,n}$ is closed in $\widetilde{\Lambda}^{H_0}$. The other properties are easily checked. □

Proof of (3.1.5). By (3.2.2), there exists $M_1 \in \mathrm{GL}_d(\widetilde{\Lambda})$ satisfying $M_1 - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M_1-1) > 2c_2 + 2c_3$ such that the cocycle $\tau \mapsto U'_\tau = M_1^{-1}U_\tau\tau(M_1)$ is trivial on H_0. Hence U' is inflated from a cocycle $\widetilde{\Gamma}_{H_0} = G/H_0 \to \mathrm{GL}_d(\Lambda^{H_0})$.

Let $\gamma \in C_{H_0} = \mathrm{Cent}(G/H_0)$ such that $v_p(\chi(\gamma)-1) = n(G_0)$; in particular, $\gamma \in G_0$ and $U'_\gamma - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda}^H)$ with $v(U'_\gamma - 1) > 2c_2 + 2c_3$. By (3.3.2), there exists $M_2 \in \mathrm{GL}_d(\widetilde{\Lambda}^{H_0})$ satisfying $M_2 - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda}^{H_0})$ and $v(M_2-1) > c_2 + c_3$ such that $M_2^{-1}U'_\gamma\gamma(M_2) \in \mathrm{GL}_d(\Lambda_{H_0,n(G_0)})$.

Let $M = M_1M_2$ and V denote the cocycle $G \to \mathrm{GL}_d(\Lambda_{H_0.n(G_0)})\tau \mapsto M^{-1}U_\tau\tau(M)$. We have $M - 1 \in p^k\mathrm{Mat}_d(\widetilde{\Lambda})$ and $v(M-1) > c_2 + c_3$, so

$$v(V_\gamma - 1) \geq \min\left\{v(U_\gamma - 1), v(M-1)\right\} > c_2 + c_3 > c_3.$$

Since γ lies in the center of G/H_0, for any $\tau \in G$, we have $\tau\gamma = \gamma\tau$ modulo H_0, so the cocycle relation reads

$$V_\tau\tau(V_\gamma) = V_{\tau\gamma} = V_{\gamma\tau} = V_\gamma\gamma(V_\tau)$$

$$\gamma(V_\tau) = V_\gamma^{-1}V_\tau\tau(V_\gamma).$$

Then we deduce by (3.1.7) that $V_\tau \in \mathrm{GL}_d(\Lambda_{H_0,n(G_0)})$ for any $\tau \in G$. □

4 Locally analytic vectors

4.1 p-adic Lie groups and locally analytic functions

4.1.1. A $\mathbf{Q}_p$*-Banach space* is a topological $\mathbf{Q}_p$-vector space B whose topology comes from an ultrametric norm $\|\cdot\|_B : B \to \mathbf{R}$ for which it is complete. The morphisms between $\mathbf{Q}_p$-Banach spaces are the continuous $\mathbf{Q}_p$-linear maps; a morphism is called *strict* if its image is closed.

For two $\mathbf{Q}_p$-Banach spaces V, W with respective unit balls V°, W°, we define the (p-adically) completed tensor product $V\hat{\otimes}_{\mathbf{Q}_p}W := \left(V^\circ \otimes_{\mathbf{Z}_p} W^\circ/p^n\right)[\frac{1}{p}]$. Replacing the norms on V and W by equivalent ones does not affect the definition.

Any surjective morphism of $\mathbf{Q}_p$-Banach spaces admits a continuous splitting, see for example [Ber, Corollary 11.6]. Hence the completed tensor product with a $\mathbf{Q}_p$-Banach space preserves the exactness of short exact sequences of $\mathbf{Q}_p$-Banach spaces.

4.1.2. Locally analytic functions and manifolds. Let $d \geq 0$ be an integer. Let $U \subset \mathbf{Q}_p^d$ be an open subset containing 0 and $(B, \|\cdot\|_B)$ be a $\mathbf{Q}_p$-Banach space.

A function $f : U \to B$ is called *analytic* if it can be expressed by a convergent power series $\sum_{\alpha\in\mathbf{N}^d} b_\alpha x^\alpha$ in d variables with coefficients $b_\alpha \in B$. Denote by $\mathscr{C}^{\mathrm{an}}(U,B)$ the set of all analytic functions $U \to B$, which is naturally a $\mathbf{Q}_p$-vector space. If furthermore U is compact, then $\mathscr{C}^{\mathrm{an}}(U,B)$ is a $\mathbf{Q}_p$-Banach space with respect to the norm $\|f\|_U := \sup_{\alpha\in\mathbf{N}^d, x\in U}\|b_\alpha x^\alpha\|_B$ (the coefficients b_α for analytic function is uniquely determined by the Identity Theorem [Sch11, Corollary 5.8]). There is a canonical isomorphism $\mathscr{C}^{\mathrm{an}}(U,\mathbf{Q}_p)\hat{\otimes}_{\mathbf{Q}_p}B \simeq \mathscr{C}^{\mathrm{an}}(U,B)$. Any morphism of $\mathbf{Q}_p$-Banach spaces $B \to B'$ induces a map $\mathscr{C}^{\mathrm{an}}(U,B) \to \mathscr{C}^{\mathrm{an}}(U,B')$.

A function $f : U \to B$ is called *locally analytic* if for any $x \in U$, there exists an open polydisc $B(x,\epsilon) \subset U$ such that $f(x+\cdot)$ is analytic on $B(0,\epsilon)$.

A *locally analytic manifold over* $\mathbf{Q}_p$ of dimension d is a Hausdorff topological space locally modelled on $\mathbf{Z}_p^d \subset \mathbf{Q}_p^d$ with locally analytic transition maps. The *locally analytic maps* between such manifolds are defined using charts; the locally analytic functions on such manifolds are locally analytic maps to $\mathbf{Q}_p$.

4.1.3 - Example. Let $(B, \|\cdot\|_B)$ be a $\mathbf{Q}_p$-Banach space with unit ball B°. By definition of analytic functions, for all $n \geq 0$, there is a natural isometric identification

$$\mathscr{C}^{\mathrm{an}}(p^n \mathbf{Z}_p^d, B) \simeq B \left\langle \frac{T_1}{p^n}, \dots, \frac{T_d}{p^n} \right\rangle$$

with $\|\sum_\alpha b_\alpha T^\alpha\|_{p^n \mathbf{Z}_p^d} = \sup_\alpha \left\| p^{n|\alpha|} b_\alpha \right\|_B$ on the first space and the Gauss norm on the latter. This norm is identified with $\|\cdot\|_\infty$ on the set $p^n \mathscr{O}_{\overline{\mathbf{Q}}_p}^d$, thus in particular stays invariant under analytic automorphism of $p^n \mathbf{Z}_p^d$. If furthermore B is a $\mathbf{Q}_p$-algebra such that $\left\|xy\right\|_B \leq \|x\|_B \left\|y\right\|_B$ for any $x, y \in B$ (for example, if $B = \mathbf{Q}_p$), then $\mathscr{C}^{\mathrm{an}}(p^n \mathbf{Z}_p^d, B)$ is an algebra such that $\left\|fg\right\|_{p^n \mathbf{Z}_p^d} \leq \left\|f\right\|_{p^n \mathbf{Z}_p^d} \left\|g\right\|_{p^n \mathbf{Z}_p^d}$.

4.1.4. p-adic Lie groups. Parallel to real Lie groups, we define a *p-adic Lie group* as a locally analytic manifold G over $\mathbf{Q}_p$ equipped with a group law such the multiplication map $G \times G \to G$ is locally analytic. Then the map $G \to G$ taking inverse is automatically locally analytic [Sch11, Proposition 13.6]. Many facts about real Lie groups stay true for p-adic Lie groups. But a great distinction is that, unlike Lie groups having no-small-subgroup property, any p-adic Lie group admits a nice filtration of compact open subgroups, in the following sense.

4.1.5 - Theorem (Lazard, [Sch11, Theorem 27.1]). *Let G be a p-adic Lie group of dimension d. Then there exists a compact open subgroup $G_0 \subset G$ of rank d equipped with an integral valued, saturated p-valuation ω defining the topology of G_0.*

4.1.6. Let's explain the terminology in the theorem. Let G_0 be a compact Hausdorff topological group.

(i) A *p-valuation on* G_0 (supposed to be compact Hausdorff) is a function $\omega : G_0 - \{e_{G_0}\} \to \mathbf{R}_{>0}$ with convention $\omega(e_{G_0}) = +\infty$ such that for all $g, h \in G_0$,

(a) $\omega(g) > \frac{1}{p-1}$,
(b) $\omega(g^{-1}h) \geq \min\{\omega(g), \omega(h)\}$,
(c) $\omega(ghg^{-1}h^{-1}) \geq \omega(g) + \omega(h)$, and
(d) $\omega(g^p) = \omega(g) + 1$.

From this, we obtain subgroups

$$(G_0)_\nu := \{g \in G_0 : \omega(g) \geq \nu\} \quad \text{and} \quad (G_0)_{\nu+} := \{g \in G_0 : \omega(g) > \nu\}, \quad \nu \in \mathbf{R}_{>0}$$

which are normal subgroups as a consequence of (b) and (c). Let

$$\mathrm{gr}(G_0) := \bigoplus_{\nu > 0} (G_0)_\nu / (G_0)_{\nu+}.$$

We have a natural map $\sigma : G_0 \to \mathrm{gr}(G_0)$, $g \mapsto g(G_0)_{\omega(g)+}$. Raising to the p-th power defines an operator $P : \mathrm{gr}(G_0) \to \mathrm{gr}(G_0)$, $\sigma(g) \to \sigma(g^p)$. From (d), it is not hard to see that $\mathrm{gr}(G_0)$ is a torsion free (thus free) $\mathbf{F}_p[P]$-module, so that we may define the rank

$$\mathrm{rank}(G_0, \omega) := \mathrm{rank}_{\mathbf{F}_p[P]} \mathrm{gr}(G_0)$$

if $\mathrm{gr}(G_0)$ is a finitely generated $\mathbf{F}_p[P]$-module.

In addition, there exists a unique Hausdorff topological group structure on G_0 for which the G_ν form a fundamental system of open neighbourhoods of e_{G_0}; this topology is called the *topology defined by* ω. In the presence of a p-valuation defining its topology, the compact group G_0 is necessarily a pro-p-group.

(ii) Suppose that (G_0, ω) has finite rank and that ω defines the topology on G_0. For any $g_1, \dots, g_d \in G_0$, we have a well-defined continuous map

$$c : \mathbf{Z}_p^d \to G_0, \ (x_1, \dots, x_d) \mapsto g_1^{x_1} \cdots g_d^{x_d} \tag{4.1.6.1}$$

thanks to ω being a p-valuation defining the topology. The above c is bijective (thus a homeomorphism by compactness) and, $\omega(c(x_1, \dots, x_d)) = \min_i \{\omega(g_i) + v_p(x_i)\}$ for any $x_1, \dots, x_d \in \mathbf{Z}_p$ if and only if the classes $\sigma(g_1), \dots, \sigma(g_d)$ form a $\mathbf{F}_p[P]$-basis [Sch11, Proposition 26.5], in which case $(g_1, \dots, g_d)$ is called an *ordered basis of* (G_0, ω). Any (G_0, ω) of finite rank has an ordered basis [Sch11, Proposition 26.6] (the point is to show that $\mathrm{gr}(G_0)$ has a $\mathbf{F}_p[P]$-basis of *homogeneous* elements).

(iii) It is unclear whether $G_0^{p^n} \subset G_{(n+\frac{p}{p-1})+} = \{g \in G_0 : \omega(g) > n + \frac{p}{p-1}\}$ is a subgroup. We call (G_0, ω) *saturated* if the above inclusion is an equality for any $n \geq 0$. In this case, $G_0^{p^n}$ is a subgroup of G_0 for all $n \geq 0$; moreover, for any ordered basis $(g_1, \ldots, g_d)$ of (G_0, ω) and the map $c : \mathbf{Z}_p^d \to G_0$ defined as above, we have $c^{-1}(G_0^{p^n}) = p^n \mathbf{Z}_p^d$.

(iv) Note that if (G_0, ω) is of finite rank and saturated, and if ω defines the topology on G_0, then any $g \in G_0 - G_0^p$ can be extended to an ordered basis of (G_0, ω). Indeed, since $\omega(g) \leq \frac{p}{p-1}$ by saturation property, the $\mathbf{F}_p[P]$-coefficients of $\sigma(g)$ under any basis $\sigma(g_1), \ldots, \sigma(g_d)$ belong to $\mathbf{F}_p$ and one of them does not vanish. So we still have a $\mathbf{F}_p[P]$-basis if we replace some $\sigma(g_i)$ with $\sigma(g)$.

4.1.7 - Example. (i) A trivial example is $G \simeq \mathbf{Z}_p$ with $G_0 = G$ and $\omega = v_p$, which is an integral valued, saturated p-valuation defining the topology.

(ii) Let E be a finite extension of $\mathbf{Q}_p$ and consider the p-adic Lie group $G = \mathrm{GL}_n(E)$, which has dimension $[E : \mathbf{Q}_p] n^2$. Let

$$G_0 = \{g \in \mathrm{GL}_n(E) : v_p(g-1) > \frac{1}{p-1}\},$$

which is a compact open subgroup of G. Then $\omega(g) = v_p(g-1)$ is a p-valuation on G_0 which is saturated and defines the topology of G_0, but ω is not integral valued unless E is unramified.

4.1.8. Lie algebra. For a locally analytic manifold G, we can define the tangent space TG using 1-jets just as for real analytic manifolds; vector fields $\xi \in \Gamma(G, TG)$ correspond bijectively to derivations on the space of locally analytic functions $\mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p)$ via $\xi \mapsto D_\xi$.

Suppose furthermore G to be a p-adic Lie group. Just as for real Lie groups, there are bijections between tangent vectors at $e = e_G \in G$, right invariant vector fields on G and right equivariant derivations on $\mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p)$, as follows

$$\begin{array}{ccccc} T_e G & \longleftarrow & \Gamma(G, TG)^G & \longrightarrow & \mathrm{Der}(\mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p))^G \\ \xi(e) & \longleftarrow\!\shortmid & \xi & \longmapsto & D_\xi \end{array}$$

where $\Gamma(G, TG)^G$ denotes the space of vector fields invariant under *right* translation action of G, and $\mathrm{Der}(\mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p))^G$ denotes the space of $\mathbf{Q}_p$-linear derivations on $\mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p)$ that commute with the *right* translation action of G [Sch11, Corollary 13.11, Proposition 9.16].

We denote $\mathrm{Lie}(G) = T_e G$ with the Lie bracket transported from $\mathrm{Der}(\mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p))$. More explicitly, let (G_0, ω) be as in (4.1.5) and $(g_1, \ldots, g_d)$ an ordered basis of it. We take the chart (G_0, c) given by (4.1.6.1). Then $\mathrm{Lie}(G) = \bigoplus_{i=1}^d \mathbf{Q}_p \frac{\partial}{\partial x_i}$, and for any $f \in \mathscr{C}^{\mathrm{la}}(G, \mathbf{Q}_p)$,

$$(D_{\frac{\partial}{\partial x_i}} f)(e) = \left.\frac{\partial}{\partial x_i}\right|_{x_i=0} f(g^{x_i}).$$

If $G_0 \subset G$ is an open subgroup, then we have a natural identification $\mathrm{Lie}(G_0) = \mathrm{Lie}(G)$.

4.1.9 - Proposition. *Any p-adic Lie group G has an open subgroup G_0 which admits a faithful finite-dimensional continuous $\mathbf{Q}_p$-representation V.*

Proof. By Ado's theorem [Bou07, Chapitre I, §7, n°3, Théorème 3], there is an embedding $\mathrm{Lie}(G) \hookrightarrow \mathrm{End}_{\mathbf{Q}_p}(V)$ for some finite-dimensional $\mathbf{Q}_p$-vector space V. This integrates to a continuous morphism of groups $G_0 \to \mathrm{GL}(V)$ for some open subgroup $G_0 \subset G$ [Bou06b, Chapitre III, §7, n°1, Théorème 3 (i)], which becomes injective if we shrink G_0 further. □

4.1.10. Analytic functions on p-adic Lie groups. Let (G_0, ω) be as in (4.1.5) and let $G_n = G_0^{p^n}$. Fix an ordered basis $(g_1, \ldots, g_d)$ of (G_0, ω) and consider the map $c : \mathbf{Z}_p^d \to G_0$ defined by it (4.1.6.1). This is in fact a locally analytic chart for G_0 ([Sch11, Corollary 29.6, Theorem 29.8]), called *coordinate of the second kind.*

For a $\mathbf{Q}_p$-Banach space B, we define $(\mathscr{C}^{\mathrm{an}}(G_n, B), \|\cdot\|_{G_n})$ by pulling back $\left(\mathscr{C}^{\mathrm{an}}(p^n \mathbf{Z}_p, B), \|\cdot\|_{p^n \mathbf{Z}_p^d}\right)$ (4.1.3) via c. This is independent of the choice of the ordered basis. Indeed, consider the logarithm map

$$\log : G_0 \to \mathrm{Lie}(G).$$

Its image is a free sub-$\mathbf{Z}_p$-module with a basis $\log(g_1), \ldots, \log(g_d)$ [Sch11, Proposition 31.2]. Hence we obtain a homeomorphism

$$\tilde{c} : \mathbf{Z}_p^d \to G_0,\ (y_1, \ldots, y_d) \mapsto \exp\big(y_1 \log g_1 + \cdots + y_d \log g_d\big)$$

(called *coordinate of the first kind*). On the one hand, one can show that the transition map between the coordinates c and $\tilde{c}$ with respect to any fixed ordered basis is actually analytic (more than being only locally analytic) [Sch11, Proposition 34.1]. On the other hand, with respect to coordinates of the first kind, the change of ordered bases induces a $\mathbf{Z}_p$-linear (thus analytic) transition map. Therefore, by (4.1.3), the $\mathbf{Q}_p$-Banach space $\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ is independent of the choice of such ordered basis.

4.1.11. Notice that $\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$, $n \in \mathbf{N}$ are defined by some particular charts specified by (G_0, ω) rather than by all charts for G_n. Nevertheless, we could somehow get rid of such dependence. To be precise, let (G'_0, ω') be another pair verifying (4.1.5), then the restriction induces a well-defined map

$$\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p) \to \mathscr{C}^{\mathrm{an}}(G'_n, \mathbf{Q}_p)$$

for $n \gg 0$, where $G'_n = G_0'^{p^n}$. Moreover, it is continuous and injective.

Indeed, we may assume $G'_0 \subset G_0$. Let c (*resp.* c') be a chart for G_0 (*resp.* G'_0) as in (4.1.6.1). The transition map $c^{-1} \circ c' : \mathbf{Z}_p^d \to \mathbf{Z}_p^d$ is locally analytic, therefore analytic on $p^n \mathbf{Z}_p^d = c'^{-1}(G'_n)$ of 0 for some $n \in \mathbf{N}$. This implies that $\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p) \to \mathscr{C}^{\mathrm{an}}(G'_n, \mathbf{Q}_p)$ induced by restriction is a well-defined map and it is continuous. It is also injective by analyticity.

4.1.12. Lie algebras as right invariant derivations. The group structure of G_0 induces naturally the left and right translation actions of G_n on $\mathscr{C}^{\mathrm{an}}(G_n, B) = B \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$.

Reviewing (4.1.8) and the proof of [Sch11, Proposition 9.16], we see that the natural map $\mathrm{Lie}(G) \simeq \Gamma(G_n, TG_n)^{G_n} \to \mathrm{Der}_{\mathbf{Q}_p}(\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{G_n}$, $\xi \mapsto D_\xi$ is bijective; here G_n acts on $\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ by right translation. More generally, if B is a $\mathbf{Q}_p$-Banach algebra, then the natural map

$$B \hat{\otimes}_{\mathbf{Q}_p} \mathrm{Lie}(G) \to \mathrm{Der}_B(B \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{G_n}$$

is bijective, where G_n acts trivially on B. Indeed, by choosing an orthonormal basis $\{e_i\}_{i \in I}$ of B (see for example [Ber, Proposition 11.1]), any $\mathbf{Q}_p$-linear derivation $D : \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p) \to B \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ can be written uniquely as $\sum_i a_i e_i \otimes D_i$ where $a_i \in \mathbf{Q}_p$, $D_i \in \mathrm{Der}_{\mathbf{Q}_p}(\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))$. If D commutes with the right translation action of G_n, so do D_i, $i \in I$, which implies the claim.

4.1.13 - Lemma. *For $n \geq 0$, both the left and right translation actions of G_n on $\mathscr{C}^{\mathrm{an}}(G_n, B)$ preserve the norm $\|\cdot\|_{G_n}$.*

Proof. Let $g \in G_n$. Up to rescaling (G_0, ω) to $(G_n, \omega - n)$, we may assume $n = 0$; up to extracting p-power roots of g by saturation property, we may assume $g \in G_0 - G_1$. Then we may extend it to an ordered basis $(g_1 = g, g_2, \ldots, g_d)$ of G_0. Hence for $f \in \mathscr{C}^{\mathrm{an}}(G_n, B)$, writing $f \circ c = \sum_\alpha b_\alpha x^\alpha \in \mathscr{C}^{\mathrm{an}}(p^n \mathbf{Z}_p, B)$, we have

$$f(g^{-1}c(x)) = f(g^{-1+x_1} g_2^{x_2} \cdots g_d^{x_d}) = f \circ c(x_1 - 1, x_2, \ldots, x_d).$$

A calculation (or (4.1.3)) shows that the translation $(x_1, x_2 \ldots, x_d) \mapsto (x_1 + 1, x_2, \ldots, x_d)$ induces isometric automorphism of $\mathscr{C}^{\mathrm{an}}(\mathbf{Z}_p^d, B)$. This proves the lemma for the left translation action on $\mathscr{C}^{\mathrm{an}}(G_0, B)$. By using the ordered basis $(g_d, \ldots, g_2, g_1 = g)$, we get the result for the right translation action. □

4.1.14 - Lemma. *Let $n \geq 0$ be an integer and $\mathscr{C}^{\mathrm{an}}(G_n, B)^\circ \subset \mathscr{C}^{\mathrm{an}}(G_n, B)$ be the unit ball with respect to the norm $\|\cdot\|_{G_n}$. Then $\mathscr{C}^{\mathrm{an}}(G_n, B)^\circ / p$ is fixed by both left and right translation actions of G_{n+1}.*

Proof. Let $g \in G_{n+1}$. Similarly as above, we are reduced to the case $n = 0$ and $g = g_1^p$ with $g_1 \in G_0 - G_1$. We may extend it to an ordered basis $(g_1, \ldots, g_d)$ of G_0. Then we are reduced as above to verifying that the translation $(x_1, x_2, \ldots, x_d) \mapsto (x_1 - p, x_2, \ldots, x_d)$ induces identity on $\mathscr{C}^{\mathrm{an}}(\mathbf{Z}_p^d, B)/p$, which is obvious by calculation. □

4.1.15 - Lemma ([Pan20, Proposition 2.1.3]). *For $n \gg 0$, there exist finite-dimensional subspaces V_k, $k \in \mathbf{N}$ of $\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ stable under both left and right translation actions of G_n such that $V_k V_l \subset V_{k+l}$ and $\varinjlim_k V_k$ is dense in $\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$.*

Proof. Fix $V \simeq \mathbf{Q}_p^N$ as in (4.1.9). For $n \gg 0$, we have $G_n \hookrightarrow 1 + p^2 \mathrm{Mat}_N(\mathbf{Z}_p)$. Fix any such n. For $k \in \mathbf{N}$, let V_k be the space of functions on G_n that are restrictions of polynomials of degree $\leq k$ on $\mathrm{Mat}_N(\mathbf{Q}_p)$. We have a well-defined commutative diagram

$$\begin{array}{ccc}
\mathscr{C}^{\mathrm{an}}(1 + p^2 \mathrm{Mat}_N(\mathbf{Z}_p), \mathbf{Q}_p) & \xrightarrow{\mathrm{Res}} & \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p) \\
\uparrow \simeq & & \uparrow \simeq \\
\mathscr{C}^{\mathrm{an}}(p^2 \mathrm{Mat}_N(\mathbf{Z}_p), \mathbf{Q}_p) & \longrightarrow & \mathscr{C}^{\mathrm{an}}(\log(G_n), \mathbf{Q}_p)
\end{array}$$

Here, the vertical arrows are induced by charts log; the lower one is induced by the injective linear map of Lie algebras, hence has dense image. Since the set of polynomial functions on $p^2\mathrm{Mat}_N(\mathbf{Z}_p)$ is dense in $\mathscr{C}^{\mathrm{an}}(p^2\mathrm{Mat}_N(\mathbf{Z}_p),\mathbf{Q}_p)$, we deduce that $\varinjlim_k V_k$ is dense in $\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p)$. The stability properties of $\{V_k\}_k$ can be easily verified as they are for polynomials. □

4.2 Locally analytic vectors of $\mathbf{Q}_p$-Banach representations

Let G be a p-adic Lie group. We keep the notations (G_0,ω) and $G_n = G_0^{p^n}$ as in (4.1.10). Let $(B,\|\cdot\|_B)$ be a $\mathbf{Q}_p$*-Banach representation* of G_0, i.e. a $\mathbf{Q}_p$-Banach space equipped with a continuous representation of G_0.

4.2.1. Locally analytic vectors. We denote by $B^{G_n-\mathrm{an}} \subset B$ the subset of G_n*-analytic vectors*, which are those $b \in B$ such that the function $G_n \to B,\ g \mapsto g\cdot b$ is analytic, i.e. belongs to $\mathscr{C}^{\mathrm{an}}(G_n,B)$.

We can identify $B^{G_n-\mathrm{an}}$ with $\mathscr{C}^{\mathrm{an}}(G_n,B)^{G_n} \to B$ via the evaluation map at e_G, where the action of G_n on $\mathscr{C}^{\mathrm{an}}(G_n,B) \simeq B\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p)$ is induced by the natural action on B and the *left* translation action on $\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p)$. More precisely, $b \in B^{G_n-\mathrm{an}}$ is identified with $f_b : G_n \to B,\ g \mapsto g\cdot b$. The subset $B^{G_n-\mathrm{an}}$ is stable under the action of G_n, because the above evaluation map is G_n-equivariant with respect to the *right* translation action on $\mathscr{C}^{\mathrm{an}}(G_n,B)^{G_n}$. Via this identification, we endow $B^{G_n-\mathrm{an}}$ with the induced norm $\|\cdot\|_{G_n}$.

We denote by $B^{\mathrm{la}} = B^{G-\mathrm{la}} := \varinjlim_{n\in\mathbf{N}} B^{G_n-\mathrm{an}} \subset B$ the subset of G*-locally analytic vectors*, equipped with the direct limit topology. According to (4.1.11), this does not depend on the choice of (G_0,ω) in (4.1.5).

For any subspace $V \subset B$, possibly non-Banach, we denote $V^{\mathrm{la}} := V \cap B^{\mathrm{la}}$ as the subset of G-locally analytic vectors in V.

Taking locally analytic vectors is functorial: if $\varphi : B \to B'$ is a continuous morphism of $\mathbf{Q}_p$-Banach representations of G_0 and if $b \in B^{G_n-\mathrm{an}}$, then $\varphi(b) \in B'^{G_n-\mathrm{an}}$.

4.2.2. The $\mathbf{Q}_p$-Banach representation B of G is said to be G_n*-analytic* (*resp. locally analytic*) if $B^{G_n-\mathrm{an}} = B$ (*resp.* $B^{\mathrm{la}} = B$). For example, any finite-dimensional representation of G_0 is locally analytic, because this corresponds to a continuous morphism $G_0 \to \mathrm{GL}_d(\mathbf{Q}_p)$ and any continuous morphism of p-adic Lie groups is locally analytic [Bou06b, Chapitre III, §8, n°1, Théorème 1].

4.2.3 - Theorem (Amice). *Let B be a $\mathbf{Q}_p$-Banach representation of G_0. We have $B^{\mathrm{la}} = \varinjlim_{n\in\mathbf{N}} B^{(n)}$ with*

$$B^{(n)} := \{b \in B : \lim_{|\alpha|\to+\infty} p^{-\frac{|\alpha|}{p^{n-1}(p-1)}}(g_1-1)^{\alpha_1}\cdots(g_d-1)^{\alpha_d}b = 0\},$$

where $(g_1,\ldots,g_d)$ is a fixed ordered basis of (G_0,ω).

4.2.4 - Example. Suppose $G_0 = \mathbf{Z}_p$ and $\omega = v_p$, and that there exists an integer $m > 0$ such that $(\gamma-1)^m B^\circ \subset pB^\circ$ for any $\gamma \in \mathbf{Z}_p$. Then $B^{\mathrm{la}} = B^{(n)} = B$ for $n \gg 0$ by Amice's theorem.

In fact, one can check by hand that $B = B^{p^n\mathbf{Z}_p-\mathrm{an}}$ for any $n \in \mathbf{N}$ such that $\binom{p^n}{i}$ is divided by p^2 for $i = 1,\ldots,2m-1$. By writing

$$\gamma^{p^n} - 1 = \sum_{i=1}^{p^n}\binom{p^n}{i}(\gamma-1)^i,$$

we have $(\gamma^{p^n}-1)B^\circ \subset p^2B^\circ$ for any $\gamma \in \mathbf{Z}_p$. Now let γ_n be the $p^n \in \mathbf{Z}_p$, we have $(\gamma_n-1)B^\circ \subset p^2B^\circ$; for any $b \in B$, the series $f_b(x) := \sum_{i\geq 0}\binom{x/p^n}{i}(\gamma_n-1)^i b$ converges uniformly for $x \in p^n\mathbf{Z}_p$, defining an analytic function $p^n\mathbf{Z}_p \to B$ which verifies $f_b(x) = x\cdot b$. Hence $f_b \in \mathscr{C}^{\mathrm{an}}(p^n\mathbf{Z}_p,B)^{p^n\mathbf{Z}_p}$, so its image $b \in B^{p^n\mathbf{Z}_p-\mathrm{an}}$.

The results below will only be used in the subsection 5.7.

4.2.5 - Lemma. *If $b \in B^{G_n-\mathrm{an}}$, then*
(i) $b \in B^{G_{n+1}-\mathrm{an}}$,
(ii) $\|b\|_{G_{n+1}} \leq \|b\|_{G_n}$,
(iii) $\|b\|_{G_m} = \|b\|_B$ for $m \gg 0$.

Proof. This is immediate from (4.1.3) and the identification of b as $f_b \in \mathscr{C}^{\mathrm{an}}(G_n,B)$. □

4.2.6 - Lemma. *The Lie algebra* $\mathrm{Lie}(G)$ *acts on B^{la} by deriving the action of G_n on $B^{G_n-\mathrm{an}}$, $n \geq 0$. Moreover, for any $D \in \mathrm{Lie}(G)$ and $n \in \mathbf{N}$, there exists a constant $C_{D,n}$ such that $\|D(x)\|_{G_n} \leq C_{D,n}\|x\|_{G_n}$ for all $x \in B^{G_n-\mathrm{an}}$.*

Proof. The first sentence follows by the identification $b \leftrightarrow f_b$ and the construction of $\mathrm{Lie}(G)$ (4.1.8). The second part is a consequence of the Banach-Steinhaus theorem applied to $(B^{G_n-\mathrm{an}}, \|\cdot\|_{G_n})$, which is an analytic $\mathbf{Q}_p$-Banach representation of G_n, and to a family of bounded operators converging pointwisely to the derivation D. See also [ST02, Proposition 3.2]. □

4.2.7. For any $\mathbf{Q}_p$-Banach representation B of G, we put $\mathscr{LA}(B) := B^{\mathrm{la}}$ and define its "right derived functors" as

$$R^i\mathscr{LA}(B) = \varinjlim_{n\in\mathbf{N}} H^i_{\mathrm{cont}}(G_n, \mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p)\hat{\otimes}_{\mathbf{Q}_p} B).$$

We say that B is

- *$\mathscr{LA}$-acyclic* if $R^i\mathscr{LA}(B) = 0$ for all $i > 0$;
- *strongly $\mathscr{LA}$-acyclic* if for all $i > 0$, the direct system $\{H^i_{\mathrm{cont}}(G_n, B\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p))\}_{n\in\mathbf{N}}$ is essentially zero, that is for any $n \in \mathbf{N}$, the map $H^i_{\mathrm{cont}}(G_n, B\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p)) \to H^i_{\mathrm{cont}}(G_m, B\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_m,\mathbf{Q}_p))$ is zero for $m \gg 0$.

4.2.8 - Lemma. *(i) Any short exact sequence $0 \to M^0 \to M^1 \to M^2 \to 0$ of $\mathbf{Q}_p$-Banach representations of G (with G-equivariant morphisms) induces long exact sequence*

$$0 \to (M^0)^{\mathrm{la}} \to (M^1)^{\mathrm{la}} \to (M^2)^{\mathrm{la}} \to R^1\mathscr{LA}(M^0) \to R^1\mathscr{LA}(M^1) \to R^1\mathscr{LA}(M^2) \to \cdots$$

(ii) Let $M^\bullet$ be a bounded chain complex of $\mathbf{Q}_p$-Banach representations of G with (G-equivariant) strict morphisms. If M^q and $H^q(M^\bullet)$ are $\mathscr{LA}$-acyclic for all q, then $H^q(M^\bullet)^{\mathrm{la}} = H^q((M^\bullet)^{\mathrm{la}})$.

(iii) Let $0 \to B \to M^\bullet$ be an exact chain complex of $\mathbf{Q}_p$-Banach representations of G (with G-equivariant morphisms). Assume M^q is $\mathscr{LA}$-acyclic for any q. Then $R^i\mathscr{LA}(B) = H^i((M^\bullet)^{\mathrm{la}})$. Moreover, if all M^q are strongly $\mathscr{LA}$-acyclic and the direct systems $\{H^i((M^\bullet)^{G_n-\mathrm{an}})\}_n$ are essentially zero for all $i > 0$, then B is strongly $\mathscr{LA}$-acyclic.

Proof. (i) For any $n \geq 0$, we have a short exact sequence (4.1.1)

$$0 \to M^0\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p) \to M^1\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p) \to M^2\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n,\mathbf{Q}_p) \to 0.$$

Taking continuous G_n-cohomology, we obtain a long exact sequence (1.1.4), which gives the desired long exact sequence when passing to the direct limit over $n \geq 0$.

The (ii) and (iii) follow from (i) by decomposing a complex into short exact sequences and by induction. □

5 Relative Sen's theory, after Lue Pan

5.1 Main theorem

5.1.1. Set-up. Let C be a complete algebraically closed non-archimedean field of characteristic zero and $\mathcal{O}_C$ its ring of integers. It is an extension of $\mathbf{Q}_p$ for some prime number p by Ostrowski's theorem. We may assume the norm on C restricts to the usual p-adic norm $|\cdot|_p = p^{-v_p(\cdot)}$ on $\mathbf{Q}_p$. We will start with the following data:

- $X = \mathrm{Spa}(A, A^+)$ be a one-dimensional smooth affinoid adic space over $\mathrm{Spa}(C, C^\circ)$ which admits a *toric chart*, i.e. an étale map $X \to \mathbf{T}^1$ that factors as a composition of rational embeddings and finite étale maps;
- G be a p-adic Lie group,
- $\widetilde{X} = \mathrm{Spa}(B, B^+)$be an affinoid perfectoid pro-étale G-Galois covering of X (2.4.7)[9].

5.1.2 - Theorem ([Pan20, 3.1.2]). *There is an element $\theta \in B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$, well-defined up to $A^\times$, satisfying the following properties:*

(i) $\theta(B^{\mathrm{la}}) = 0$*;*

(ii) θ is functorial in $(G, \widetilde{X})$: for H a closed normal subgroup of G such that $\widetilde{X}' = \mathrm{Spa}(B^H, B^{+H})$ is an affinoid perfectoid pro-étale G/H-Galois covering of X. Then we have $\theta_{\widetilde{X}'} = \theta_{\widetilde{X}} \bmod B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(H)$.

(iii) $\theta \neq 0$ if $\widetilde{X}$ is a locally analytic covering of X (see below for definition).

5.1.3 - Remark. Denote by Γ the Galois group of the pro-étale Galois covering $\mathbf{T}^1_\infty = \varprojlim_n \mathbf{T}^1_n \to \mathbf{T}^1$; it is canonically isomorphic to $\mathbf{Z}_p(1) := \varprojlim_n \mu_{p^n}(C)$ via $g \mapsto \{g(T^{1/p^n})/T^{1/p^n}\}_n$. We shall construct a linear map $\phi_{\widetilde{X}} : \mathrm{Lie}(\Gamma) \to B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$ from any toric chart $X \to \mathbf{T}^1$, and the element θ in the theorem will be the image of a generator $\mathrm{Lie}(\Gamma)$.

We shall see that different charts change $\phi_{\widetilde{X}}$ by $A^\times$. We can even use the module of differentials $\Omega^1_{A/C}$ to obtain a canonical element in

$$\mathrm{Lie}(\Gamma)^\vee \otimes_{\mathbf{Z}_p} (B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)) \otimes_A \Omega^1_{A/C} \simeq B \otimes_A \Omega^1_{A/C} \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)(-1).$$

5.1.4. Notation. According to (2.4.10), for any open subgroup G_i of G, $X_{G_i} := \mathrm{Spa}(B^{G_i}, (B^{G_i})^+)$ is finite étale over X, and is G/G_i-Galois if G_i is a normal subgroup, and $\widetilde{X}$ is the affinoid perfectoid space associated with $\varprojlim_i X_{G_i}$. We have a system of affinoids parametrised by open normal subgroups G_i of G and $n \in \mathbf{N} \cup \{\infty\}$, which fit into the following diagram of adic spaces

(5.1.4.1)
$$\begin{array}{ccccccc}
\widetilde{X}_\infty & \xrightarrow{G_i} & X_{G_i,\infty} & \longrightarrow & X_\infty & \longrightarrow & \mathbf{T}^1_\infty \\
{\scriptstyle p^n\Gamma}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{X}_n & \longrightarrow & X_{G_i,n} & \longrightarrow & X_n & \longrightarrow & \mathbf{T}^1_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{X} & \longrightarrow & X_{G_i} & \longrightarrow & X & \longrightarrow & \mathbf{T}^1
\end{array}$$

where each square is Cartesian[10]in the category of adic spaces by (2.4.6) and (2.4.11).

According to (2.4.11), $\widetilde{X}_\infty$ is associated with the system $\varprojlim_i X_{G_i,\infty}$ of the top row which is affinoid perfectoid pro-étale G-Galois over the affinoid perfectoid space $\mathbf{T}^1_\infty$, and $\widetilde{X}_\infty$ is also associated with the system $\varprojlim_n \widetilde{X}_n$ of the left column which is affinoid perfectoid pro-étale Γ-Galois over the affinoid perfectoid space $\widetilde{X}$. Hence the almost purity theorem allows us to apply (1.2.13, ii) to any p-adically complete B^+_∞-modules in two ways.

Also, all the other rows (*resp.* columns) gives an affinoid perfectoid G-Galois (*resp.* Γ-Galois) covering by (2.4.6).

[9]For the purpose of studying the cusps of modular curves, one also has to consider the logarithmic variant: X admits a log étale map to $\mathbf{B}^1$ that factors as a composite of rational embeddings and finite Kummer étale maps, whose ramification locus is a singleton S; here $\mathbf{B}^1 = \mathrm{Spa}(C\langle T\rangle, C^\circ\langle T\rangle)$ is given the canonical logarithmic structure associated with $\mathcal{M} = \mathbf{Z}[T]$. And $\widetilde{X} \sim \varprojlim_i X_i$ is an affinoid perfectoid G-Galois log pro-étale covering of X, such that the ramification index of each X_i over 0 is a p-power p^{n_i} and $\lim_i n_i = +\infty$.

[10]In the logarithmic case, not all squares are Cartesian, and we need to use normalisation process to construct the diagram (5.1.4.1), even at the finite level. Nevertheless, our assumption on the ramification degree of $\widetilde{X} \to X$ allows us to apply rigid Abhyankar's lemma [DLLZ19, Proposition 4.2.1] to kill ramification. As a result, the upper middle square in (5.1.4.1) would be Cartesian for $n \gg 0$, and the lower left square would be Cartesian for G_i sufficiently small.

Let's fix a topological generator γ of Γ for the discussion below.

5.1.5 - Lemma. *For any $b \in (B_\infty)^{G-\mathrm{la},\,\Gamma-\mathrm{la}}$, we have $\theta(b) = \lim_{m\to+\infty} \frac{\gamma^{p^m}(b)-b}{p^m}$ up to $A^\times$.*

5.1.6. Faltings extension. Let V be the following unipotent representation of Γ on $V = \mathbf{Q}_p^2$:

$$\Gamma \to \mathrm{GL}_2(\mathbf{Q}_p),\ \gamma \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5.1.6.1}$$

So V is an extension of the trivial representation $\mathbf{Q}_p$ by itself. By taking Γ-invariants, we get the following *ad hoc* Faltings extension[11]

$$0 \to B \to (B_\infty \otimes_{\mathbf{Q}_p} V)^\Gamma \to B \to 0, \tag{FE}$$

where the surjectivity comes from $H^1_{\mathrm{cont}}(\Gamma, B_\infty) = 0$ by almost étale descent.

5.1.7 - Proposition. *The following conditions are equivalent:*

(i) (FE) remains exact after taking G_0-analytic vectors for some open subgroup G_0 of G equipped with an integral valued, saturated p-valuation.

(i') (FE) remains exact after taking G-locally analytic vectors.

(ii) There exists a G-locally analytic vector $b \in B_\infty$ such that $\gamma(b) = b - 1$.

Proof. That (FE) remains exact after taking G_0-analytic (*resp.* G-locally analytic) vectors is equivalent to the surjectivity of $B^{G_0-\mathrm{an}}$-linear (*resp.* B^{la}-linear) map at $B^{G_0-\mathrm{an}}$ (*resp.* B^{la}) after taking G_0-analytic (*resp.* G-locally analytic) vectors; this is determined by whether $1 \in B$ lies in the image.

Write $B_\infty \otimes_{\mathbf{Q}_p} V = (B_\infty)^2$. Any element of $(B_\infty \otimes_{\mathbf{Q}_p} V)^\Gamma$ mapped to $1 \in B$ has the form $(b, 1)$, $b \in B_\infty$ such that $\gamma(b) = b - 1$. The analyticity of $(b, 1)$ is the same as that of b. Therefore (ii) is equivalent to (i) (*resp.* (i')), taking into account the definition of G-locally analytic vectors. □

5.1.8 - Definition. We say that $\widetilde{X}$ is a *locally analytic covering of X* if the equivalent conditions in (5.1.7) hold.

5.1.9 - Remark. The property of being a locally analytic covering is preserved when passed to any rational open subset $U \subset X$ and its preimage $\widetilde{U} \subset \widetilde{X}$. This is clear form the condition (iii).

Proof of (5.1.2, iii). By condition (5.1.7, ii), there exists $z \in (B_\infty)^{G-\mathrm{la}}$ such that $\gamma(b) = b - 1$. Then by induction and continuity, we have $\gamma^m(b) = b - m$ for all $m \in \mathbf{Z}_p$. Hence b is Γ-analytic, and $\theta(b) = 1$ up to $A^\times$ by (5.1.5). In particular, $\theta \neq 0$. □

5.2 Normalised traces

For notational purpose, we rewrite the diagram (5.1.4.1) in terms of completed Huber pairs

$$\begin{array}{ccccccc}
(B_\infty, B_\infty^+) & \xleftarrow{G_i} & (B_{G_i,\infty}, B_{G_i,\infty}^+) & \longleftarrow & (A_\infty, A_\infty^+) & \longleftarrow & (R_\infty, R_\infty^+) \\
{\scriptstyle p^n\Gamma}\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(B_n, B_n^+) & \longleftarrow & (B_{G_i,n}, B_{G_i,n}^+) & \longleftarrow & (A_n, A_n^+) & \longleftarrow & (R_n, R_n^+) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(B, B^+) & \longleftarrow & (B_{G_i}, B_{G_i}^+) & \longleftarrow & (A, A^+) & \longleftarrow & (R, R^+)
\end{array} \tag{5.2.0.1}$$

[11] Assume $X = (X_0)_C$ where X_0 is smooth over some discretely valued subfield $k \subset C$. In this case, Scholze constructed the Faltings extension [Sch12], whose base change to C is denoted as

$$0 \to \widehat{\mathscr{O}}_X(1) \to \mathscr{E} \to \widehat{\mathscr{O}}_X \otimes_{\mathscr{O}_X} \Omega^1_X \to 0. \tag{5.1.6.2}$$

With the notation of in the above diagram, we can recover the above *ad hoc* Faltings extension (FE) as the exact sequence (5.1.6.2) evaluated on $\widetilde{X}$.

Indeed, since $\widetilde{X}_\infty$ is affinoid perfectoid, $\widehat{\mathscr{O}}_X(X_\infty) = B_\infty$. Fix a (non-canonical) $G \times \Gamma$-equivariant isomorphism $B_\infty(1) \simeq B_\infty$ by choosing a generator of the Tate twist. Since $\widetilde{X}_\infty$ lives over X_∞, there is an element $v \in \mathscr{E}(\widetilde{X}_\infty)$ lifting a generator of $B_\infty \otimes_A \Omega^1_{A/C} \simeq B_\infty$ such that $\gamma(v) = v + t$ by [Sch13a, Section 6]. Therefore, evaluating (5.1.6.2) on $\widetilde{X}_\infty$, by vanishing of $H^i(\widetilde{X}_\infty, \widehat{\mathscr{O}}_X)$, $i > 0$, we get a $G \times \Gamma$-equivariant exact sequence

$$0 \to B_\infty \to B_\infty \otimes_{\mathbf{Q}_p} V \to B_\infty \to 0$$

where $V = \mathbf{Q}_p^2$ is the representation of Γ given by (5.1.6.1). Finally, we evaluate (5.1.6.2) over $\widetilde{X}$ using the covering $\widetilde{X}_\infty \to \widetilde{X}$, which amounts to taking Γ-invariants in the preceding sequence; so we recover (FE).

More generally, if X has a formally smooth model over $\mathscr{O}_C$ with a lifting over $A_{\mathrm{inf}}(\mathscr{O}_C)/(\ker\theta)^2$, we can also define a Faltings extension [Wan21].

5.2.1. Trace maps along the tower $\{\mathbf{T}^1_n\}_{n\in\mathbf{N}}$. Recall that for $n \in \mathbf{N} \cup \{\infty\}$, we have

$$(R_n, R_n^+) = (C\langle T^{\pm 1/p^n}\rangle, \mathcal{O}_C\langle T^{\pm 1/p^n}\rangle).$$

We equip R_∞ with the valuation associated with the Gauss norm

$$v_R(\sum_{\alpha\in\mathbf{Z}[\frac{1}{p}]} c_\alpha T^\alpha) := \inf_{\alpha\in\mathbf{Z}[\frac{1}{p}]} v_p(c_\alpha), \quad c_\alpha \in C.$$

Recall that the Galois group Γ of $\mathbf{T}^1_\infty$ over $\mathbf{T}^1$ acts isometrically on R_∞ by

$$\gamma(T^{l/p^k}) = \zeta_{p^k}^l T^{l/p^k}, \quad (l,p) = 1$$

where γ is a basis of $\Gamma \simeq \mathbf{Z}_p(1)$ corresponding to the sequence $\{\zeta_{p^n}\}_n$. For $n \geq m$, we have the usual Γ-equivariant normalised trace map $p^{-(n-m)}\operatorname{tr}_{n,m} : R_n^+ \to R_m^+$ sending T^{l/p^k}, $(l,p) = 1$ to T^{l/p^k} if $k \leq m$ and 0 otherwise. This is independent of n and extends by continuity to a R_m^+-linear map $\overline{\operatorname{tr}}_m^+ : R_\infty^+ \to R_m^+$ which is a retract of $R_m^+ \subset R_\infty^+$. After inverting p, we get a continuous Γ-equivariant R_m-linear map

$$\overline{\operatorname{tr}}_m : R_\infty \to R_m$$

which is a retract of $R_m \subset R_\infty$, so that $R_\infty = R_m \oplus \ker \overline{\operatorname{tr}}_m$. Moreover, we have $\overline{\operatorname{tr}}_m \circ \overline{\operatorname{tr}}_{m'} = \overline{\operatorname{tr}}_{\min\{m,m'\}}$. By explicit computation, we have for any $x \in R_\infty$

$$\lim_{n\to+\infty} \overline{\operatorname{tr}}_n(x) = x;$$

and for any topological generator γ of Γ, the operator $\gamma - 1$ is invertible on $\ker \overline{\operatorname{tr}}_m$ with

$$v_R((\gamma-1)^{-1}(x)) - v_R(x) \geq v_p(\zeta_{p^{m+1}} - 1) = -\frac{1}{p^m(p-1)}.$$

We fix from now on (unless specified otherwise) a toric chart $X \to \mathbf{T}^1$.

5.2.2. Trace maps along the tower $\{X_n\}_{n\in\mathbf{N}}$. Recall that $X_n = \operatorname{Spa}(A_n, A_n^+)$ and that (A_n, A_n^+) is the pushout of $(R_n, R_n^+) \leftarrow (R, R^+) \to (A, A^+)$. By (2.4.6), we may extend A_m-linearly the normalised trace $\overline{\operatorname{tr}}_m$, first to $A_n^+ \otimes_{R_n^+} R_\infty^+$, $n \in \mathbf{N}$, and then get

$$\overline{\operatorname{tr}}_{X,m} : A_\infty \to A_m$$

by inverting p, which satisfies the following properties:

(a) $\overline{\operatorname{tr}}_{X,m}$ is a retract of the inclusion $A_m \subset A_\infty$;
(b) $\overline{\operatorname{tr}}_{X,m} \circ \overline{\operatorname{tr}}_{X,m'} = \overline{\operatorname{tr}}_{X,\min\{m,m'\}}$;
(c) the image $\overline{\operatorname{tr}}_{X,m}(A_\infty^+) \subset p^{-\epsilon_m} A_m^+$ for some constant $\epsilon_m \to 0$ as $m \to +\infty$, so that for any $x \in A_\infty$

$$v_A(\overline{\operatorname{tr}}_{X,m}(x)) \geq v_A(x) - \epsilon_m,$$

where v_A is the valuation on A_∞ defined by A_∞^+ as in (2.1.9).
(d) $\lim_{n\to+\infty} \overline{\operatorname{tr}}_{X,n}(x) = x$ for any $x \in A_\infty$;
(e) For any topological generator γ of Γ, the operator $\gamma - 1$ is invertible on $\ker \overline{\operatorname{tr}}_{X,m}$ with

$$v_A((\gamma-1)^{-1}(x)) - v_A(x) \geq -\frac{1}{p^m(p-1)} - 2\epsilon_m,$$

where $\lim_{m\to+\infty} \epsilon_m = 0$ Indeed, for any $x \in \ker \overline{\operatorname{tr}}_{X,m} \cap A_\infty^+$, we have $p^{\epsilon_m} x \in (A_m^+ \otimes_{R_m^+} \ker \overline{\operatorname{tr}}_m^+)^{\mathrm{tf},\wedge}$, so $(\gamma-1)^{-1}(p^{\epsilon_m} x)$ is well-defined and belongs to $p^{-1/p^m(p-1)}(A_m^+ \otimes_{R_m^+} \ker \overline{\operatorname{tr}}_m^+)^{\mathrm{tf},\wedge} \subset p^{-1/p^m(p-1)} A_\infty^+$, which implies the desired inequalities.

5.2.3. Trace maps along the tower $\{X_{G_i,n}\}_{n\in\mathbf{N}}$.

For any (compact) open subgroup G_i of G, $X_{G_i,n}$ is finite étale over X, hence small. Just as in the previous paragraph, we may extend $B_{G_i,n}$-linearly the normalised trace and get

$$\overline{\operatorname{tr}}_{X_{G_i},m} : B_{G_i,\infty} \to B_{G_i,m}$$

satisfying the properties (a)–(e) above, with B_{G_i} in place of A and $v_{B_{G_i}}$ in place of v_A.

5.2.4 - Proposition. *The set-up (3.1.1) and the Tate-Sen's conditions (3.1.3) are verified with*

- $G := G \times \Gamma$, χ *be the character induced by an embedding* $\Gamma \to \mathbf{Z}_p^\times$, *and* $H := \ker \chi = G \times \{1\}$;
- $\widetilde{\Lambda} = B_\infty$ *with the valuation canonically defined as in (2.1.9);*
- $\Lambda_{H_0,n} := B_{G_0,n}$ *and* $R_{H_0,n} := \overline{\mathrm{tr}}_{X_{G_0},n}$, *for any open subgroup of* $H_0 = G_0 \times \{1\}$ *of* $H = G \times \{1\} \simeq G$.
- $c_1, c_2, c_3 > 0$ *arbitrarily small.*

Proof. As for (TS1), we establish almost étaleness: for any open subgroups $H_1 \subset H_2$ of $G \times \{1\} \simeq G$, We know that $(B_{H_2}, B_{H_2}^+) \to (B_{H_1}, B_{H_1}^+)$ is finite étale by (2.4.10), so $(B_{H_2,\infty}, B_{H_2,\infty}^+) \to (B_{H_1,\infty}, B_{H_1,\infty}^+)$ is a finite étale map of perfectoid spaces by (2.4.6), hence $B_{H_2,\infty}^+ \to B_{H_1,\infty}^+$ is almost finite étale by almost purity (2.3.7, iii). The (TS2) and (TS3) follow directly from (5.2.2) and (5.2.3). □

5.2.5 - Remark. Recall that in the case $\widetilde{\Lambda} = \mathbf{C}_p$, $G = \mathscr{G}_{\mathbf{Q}_p}$ and $\chi = \chi_{\mathrm{cyc}}$, one can only allow $c_3 > 1$, which is different from our case.

5.2.6 - Lemma. *Let* $c < \frac{1}{2}$ *be a constant in* $|C^\times|$. *Let* T *be a* $\mathbf{Z}_p$*-representation of* G *free of rank* d, *and* $G_0 \subset G$ *be an open subgroup acting trivially on* T/pT. *Denote* $n(G_0) := n(G_0 \times \Gamma)$ *the constant given in (TS3), which depends only on* $c_3 := c$. *Then for any* $n \geq n(G_0)$, $B_\infty^\circ \otimes_{\mathbf{Z}_p} T$ *contains a unique sub-*$B_{G_0,n}^+$*-module* $D_{G_0,n}^+(T)$ *free of rank* d *satisfying:*

(i) $D_{G_0,n}^+(T)$ *is fixed by* G_0, *and stable under* $G \times \Gamma$;

(ii) The natural $G_0 \times \Gamma$*-equivariant map* $B_\infty^\circ \otimes_{B_{G_0,n}^+} D_{G_0,n}^+(T) \to B_\infty^\circ \otimes_{\mathbf{Z}_p} T$ *is an isomorphism;*

(iii) $D_{G_0,n}^+(T)$ *has a* $B_{G_0,n}^+$*-basis which is* c*-fixed, i.e. under which the matrices of the* Γ*-action is trivial modulo* p^c.

In particular, there is a constant $m(c, n)$ *(independent of* T*) such that* $(\gamma - 1)^m(D_{G_0,n}^+(T)) \subset pD_{G_0,n}^+(T)$ *for any* $\gamma \in \Gamma$ *and* $m \geq m(c, n)$.

Proof. Thanks to (5.2.4), this is essentially (3.1.6) with constants $c_1, c_2 > 0$ sufficiently small such that $c_1 + 2c_2 + 2c_3 < 1$. The last sentence follows from (iii). □

5.3 Sen's operator: construction, I

5.3.1 - Lemma. *Let* $G_0 \subset G$ *be an open subgroup. We have* $B_{G_0,\infty}^{p^m\Gamma-\mathrm{an}} = B_{G_0,m}$ *for any* $m \in \mathbf{N}$.

More generally, let D *be a* $\mathbf{Q}_p$*-Banach space with a* $B_{G_0,n}$*-module structure such that the natural map* $B_{G_0,\infty} \otimes_{B_{G_0,n}} D \to B_{G_0,\infty} \hat{\otimes}_{B_{G_0,n}} D$ *is injective (for example, a free* $B_{G_0,n}$*-module of finite rank). If* D *is equipped with an analytic* $p^{n'}\Gamma$*-action for some* $n' \geq n$, *then* $(B_{G_0,\infty} \hat{\otimes}_{B_{G_0,n}} D)^{p^m\Gamma-\mathrm{an}} = B_{G_0,m} \otimes_{B_{G_0,n}} D$ *for any* $m \geq n'$. *In particular, if* D *is equipped with a locally analytic* Γ*-action, then* $(B_{G_0,\infty} \hat{\otimes}_{B_{G_0,n}} D)^{\Gamma-\mathrm{la}} = (\varinjlim_m B_{G_0,m}) \otimes_{B_{G_0,n}} D$.

Proof. Without loss of generality, we may assume $G_0 = G$ so that $B_{G_0} = A$, since $X_{G_0} \to X$ is finite étale and the tower as well as the trace maps is constructed by base change.

Let's prove the general case. For $l \geq n$, the normalised trace $\overline{\mathrm{tr}}_{X,l} : A_\infty \to A_l$ induces a continuous Γ-equivariant morphism

$$T_l : A_\infty \hat{\otimes}_{A_n} D \to A_l \otimes_{A_n} D.$$

By (5.2.2, c), $\{T_l\}_{l \geq n}$ is an equicontinuous family of operators. Then we obtain maps

$$T_l : (A_\infty \hat{\otimes}_{A_n} D)^{p^m\Gamma} \to (A_l \otimes_{A_n} D)^{p^m\Gamma}.$$

The latter is equal to $A_l^{p^m\Gamma} \otimes_{A_n} D = A_m \otimes_{A_n} D$ because A_l is $p^m\Gamma$-smooth. We conclude by $\lim_{l \to +\infty} T_l(x) = x$. Indeed, this equality holds for x in the dense subset $A_\infty \otimes_{A_n} D \subset A_\infty \hat{\otimes}_{A_n} D$; we pass to general x by equicontinuity of $\{T_l\}_l$. □

5.3.2 - Proposition. *For each finite-dimensional continuous* $\mathbf{Q}_p$*-representation* V *of* G, *there exists a unique* B_∞*-linear action of* $\mathrm{Lie}(\Gamma)$ *on* $B_\infty \otimes_{\mathbf{Q}_p} V$

$$\phi_V : \mathrm{Lie}(\Gamma) \to \mathrm{End}_{B_\infty}(B_\infty \otimes_{\mathbf{Q}_p} V)$$

extending the natural action of $\mathrm{Lie}(\Gamma)$ *on* $(B_\infty \otimes_{\mathbf{Q}_p} V)^{G,\,\Gamma-\mathrm{la}}$. *Moreover, it satisfies the following properties:*

(i) ϕ_V *commutes with* $G \times \Gamma$;

(ii) ϕ_V *is functorial in* V;

(iii) $\phi_{V\otimes W} = \phi_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \phi_W$ *for* V, W *two finite-dimensional continuous* $\mathbf{Q}_p$*-representation of* G.

Proof. Since G is compact, V admits a G-stable lattice T. Choose $c < \frac{1}{2}$ a constant in $|C^\times|$ and an open subgroup $G_0 \subset G$ so small that G_0 acts trivially on T/pT, i.e. that the conditions (5.2.6) be verified. We are going to start by constructing an action $\phi_{V,G_0,n}$ for all sufficiently small open subgroup $G_0 \subset G$ and $n \geq n(G_0)$, and to prove it to be independent of G_0 and n. Once the existence of ϕ_V is proved, the implicit independence of $\phi_{V,G_0,n}$ on the choice of a lattice $T \subset V$ will be eliminated.

(1) Let $D^+_{G_0,n}(T)$ be the sub-$B^+_{G_0,n}$-module of $B^\circ_\infty \otimes_{\mathbf{Z}_p} T$ provided by (5.2.6). The last sentence of (5.2.6) and (4.2.4) imply that $D_{G_0,n}(V) := D^+_{G_0,n}(T)[\frac{1}{p}]$ is Γ-locally analytic for $n \geq n(G_0)$, inducing an action of $\mathrm{Lie}(\Gamma)$ on $D_{G_0,n}(V)$. This action is $B_{G_0,n}$-linear because the action of Γ is $B_{G_0,n}$-semi-linear and $B_{G_0,n}$ is $p^n\Gamma$-smooth; it is also $G\times\Gamma$-equivariant because the actions of $G \times \Gamma$ and Γ commute with each other.

We have $B_\infty \otimes_{B_{G_0,n}} D_{G_0,n}(V) = B_\infty \otimes_{\mathbf{Q}_p} V$ (5.2.6, ii), so we can extend B_∞-linearly the action of $\mathrm{Lie}(\Gamma)$ to one over $B_\infty \otimes_{B_{G_0,n}} D_{G_0,n}(V) \simeq B_\infty \otimes_{\mathbf{Q}_p} V$. We denote it by

$$\phi_{V,G_0,n} : \mathrm{Lie}(\Gamma) \to \mathrm{End}_{B_\infty}(B_\infty \otimes_{\mathbf{Q}_p} V).$$

(2) The action $\phi_{V,G_0,n}$ is independent of n. Indeed, for $n' \geq n$, the action $\phi_{V,G_0,n'}$ is uniquely determined by its restriction to

$$D_{G_0,n'}(V) = B_{G_0,n'} \otimes_{B_{G_0,n}} D_{G_0,n}(V), \tag{5.3.2.1}$$

where the equality follows from the uniqueness in (5.2.6). Deriving the locally analytic Γ-action and using the Γ-smoothness of $B_{G_0,n'}$, we obtain $\phi_{V,G_0,n'} = \phi_{V,G_0,n}$.

Now, let's show the independence of $\phi_{V,G_0} := \phi_{V,G_0,n}$ on G_0.

(3) First, we descend $D_{G_0,n}(V)$ to the G-invariant subspace as follows. Shrinking G_0, we may assume it to be normal in G. Then $A_n = B_{G,n} \to B_{G_0}$ is finite étale G/G_0-Galois, so that by Galois descent, $D_{G,n}(V) := D_{G_0,n}(V)^{G/G_0}$ is a free A_n-module with a locally analytic Γ-action and there is a Γ-equivariant isomorphism

$$D_{G_0,n}(V) \simeq B_{G_0,n} \otimes_{A_n} D_{G,n}(V).$$

The above Galois descent together with (5.3.2.1) also guarantees that

$$D_{G,n'}(V) = A_{n'} \otimes_{A_n} D_{G,n}(V). \tag{5.3.2.2}$$

(4) As a result of (3), ϕ_{V,G_0} is uniquely determined by its restriction to

$$(\varinjlim_m B_{G_0,m}) \otimes_{B_{G_0,n}} D_{G_0,n}(V) = (\varinjlim_m B_{G_0,m}) \otimes_{A_n} D_{G,n}(V) = (\varinjlim_m B_{G_0,m}) \otimes_{(\varinjlim_m A_m)} ((\varinjlim_m A_m) \otimes_{A_n} D_{G,n}(V)),$$

and $\phi_{V,G_0,n}$ restricted to $D_{G,n}(V)$ is the infinitesimal $\mathrm{Lie}(\Gamma)$-action. Using the Γ-smoothness of $A_{n'}$, we find that ϕ_{V,G_0} is even determined by its restriction to the following $(\varinjlim_m A_m)$-lattice of $B_\infty \otimes_{\mathbf{Q}_p} V$

$$(\varinjlim_m A_m) \otimes_{A_n} D_{G,n}(V) = (A_\infty \otimes_{A_n} D_{G,n}(V))^{\Gamma-\mathrm{la}} = (B_\infty \otimes_{A_n} D_{G,n}(V))^{G,\,\Gamma-\mathrm{la}} = (B_\infty \otimes_{\mathbf{Q}_p} V)^{G,\,\Gamma-\mathrm{la}},$$

where the first equality comes from (5.3.1). But this restriction coincides with the natural infinitesimal $\mathrm{Lie}(\Gamma)$-action, because we have $(\varinjlim_m A_m) \otimes_{A_n} D_{G,n}(V) = \varinjlim_m D_{G,m}$ by (5.3.2.2). This proves the independence of ϕ_{V,G_0} on G_0 as well as the uniqueness and the existence of ϕ_V.

(5) Finally, for properties (i)–(iii), it is enough to check them on $(B_\infty \otimes_{\mathbf{Q}_p} V)^{G,\,\Gamma-\mathrm{la}}$, which is immediate. □

5.3.3 - Remark. According to (3.1.4), We may also replace G by any open subgroup of it in the previous proposition.

5.3.4 - Corollary. *The action* ϕ_V *of* $\mathrm{Lie}(\Gamma)$ *extends the natural action of* $\mathrm{Lie}(\Gamma)$ *on* $(B_\infty \otimes_{\mathbf{Q}_p} V)^{G-\mathrm{sm},\,\Gamma-\mathrm{la}}$.

Proof. This follows from the previous remark and the uniqueness part of the previous proposition. □

5.4 Sen's operator: construction, II

5.4.1. Let (G_0, ω) be as in (4.1.5), i.e. $G_0 \subset G$ is a compact open subgroup of rank d equipped with an integral valued, saturated p-valuation ω defining the topology of G_0. In addition, by (4.1.15), up to replacing G_0 by some $G_n = G_0^{p^n}$, we may well assume that there exist finite-dimensional subspaces V_k, $k \geq 0$ of $\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$ stable under both the left and right translation actions of G_0 such that $V_k V_l \subset V_{k+l}$ and $\varinjlim_k V_k$ is dense in $\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$. We regard the V_k as representations of G_0 via its *left* translation action. It is also clear that such G_0 form a basis of neighbourhoods of $\mathbf{1}_G \in G$.

Recall that we equipped $\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$ with the norm $\|\cdot\|_{G_0}$. Let $\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^\circ$ be its unit ball and for each $k \geq 0$, $V_k^\circ := V_k \cap \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^\circ$. It is easy to see that V_k° is a G_0-stable lattice in V_k. Also, it follows from (4.1.14) that for all $k \geq 0$, V_k°/p is fixed by the open subgroup $G_1 = G_0^p$ of G_0.

5.4.2 - Proposition. *Let (G_0, ω) be as in (5.4.1). Then there exists a unique B_∞-linear action of* $\mathrm{Lie}(\Gamma)$ *on* $B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$

$$\phi_{G_0} : \mathrm{Lie}(\Gamma) \to \mathrm{End}_{B_\infty}(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))$$

extending the natural action of $\mathrm{Lie}(\Gamma)$ *on* $(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0,\, \Gamma-\mathrm{la}}$. *Moreover, it satisfies the following property:*

(i) ϕ_{G_0} commutes with $G_0 \times \Gamma$;

(ii) ϕ_{G_0} commutes with the right translation action of G_0 on $B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$;

(iii) ϕ_{G_0} is a derivation, i.e. for any $\theta \in \mathrm{Im}\, \phi_{G_0}$ and $f_1, f_2 \in \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$, one has $\theta(f_1 f_2) = \theta(f_1) f_2 + f_1 \theta(f_2)$.

In particular, ϕ_{G_0} factors as

$$\phi_{G_0} : \mathrm{Lie}(\Gamma) \to B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G_0) \hookrightarrow \mathrm{End}_{B_\infty}(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)),$$

where $\mathrm{Lie}(G_0)$ *acts B_∞-linearly and by deriving the* left *translation action on $\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$.*

Proof. In the same spirit as (5.3.2), we proceed as follows.

(1) Fix a constant $c < \frac{1}{2}$ in $|C^\times|$. Since $G_1 \subset G_0$ acts trivially on V_k°/p, $k \geq 0$, we may apply (5.2.6) to obtain, for any $n \geq n(G_1)$, $k \geq 0$, a G_1-invariant $G_0 \times \Gamma$-stable $B^+_{G_1,n}$-lattice $D^+_{G_1,n}(V_k^\circ) \subset (B_\infty)^\circ \otimes_{\mathbf{Q}_p} V_k^\circ$, i.e. such that

$$B_\infty^\circ \otimes_{B^+_{G_1,n}} D^+_{G_1,n}(V_k^\circ) = B_\infty^\circ \otimes_{\mathbf{Z}_p} V_k^\circ.$$

The uniqueness part of (5.2.6) makes $\{D^+_{G_1,n}(V_k^\circ)\}_{k \geq 0}$ a direct system, for any $n \geq n(G_1)$. So taking the directly limit of the above identity and then the p-adic completion, we obtain

$$B_\infty^\circ \hat{\otimes}_{B^+_{G_1,n}} D^+_{G_1,n} \simeq B_\infty^\circ \hat{\otimes}_{\mathbf{Z}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^\circ,$$

where $D^+_{G_1,n} \subset \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)^\circ$ is the p-adic completion of $\varinjlim_k D^+_{G_1,n}(V_k^\circ)$. After inverting p, this becomes

$$B_\infty \hat{\otimes}_{B_{G_1,n}} D_{G_1,n} \simeq B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$$

with $D_{G_1,n} := D^+_{G_1,n}[\frac{1}{p}]$. A side remark is that the natural map $B_\infty^\circ \otimes_{B^+_{G_1,n}} D^+_{G_1,n} \to B_\infty^\circ \hat{\otimes}_{B^+_{G_1,n}} D^+_{G_1,n}$ is injective.

According to (5.2.6), there is a constant $m(c,n)$ independent of k such that for $m \geq m(c,n)$, we have $(\gamma-1)^m D^+_{G_1,n}(V_k^\circ) \subset p D^+_{G_1,n}(V_k^\circ)$ for all $k \geq 0$, so that

$$(\gamma-1)^m D^+_{G_1,n} \subset p D^+_{G_1,n}.$$

Therefore $D_{G_1,n}$ is $p^{n'}\Gamma$-analytic for some $n' \in \mathbf{N}$ by (4.2.4). Its Lie algebra action extends B_∞-linearly to $B_\infty \hat{\otimes}_{B_{G_1,n}} D_{G_1,n} \simeq B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$, which we denote by

$$\phi_{G_0,n} : \mathrm{Lie}(\Gamma) \to \mathrm{End}_{B_\infty}(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)).$$

(2) We may argue as in the proof of (5.3.2), part (2) to show the independence of $\phi_{G_0,n}$ on n. Let $n' \geq n$. The uniqueness part of (5.2.6) implies

$$D^+_{G_1,n'}(V_k^\circ) = B^+_{G_1,n'} \otimes_{B^+_{G_1,n}} D^+_{G_1,n}(V_k^\circ),$$

so that after taking the direct limit over $k \geq 0$ and the p-adic completion, we obtain

$$D^+_{G_1,n'} = B^+_{G_1,n'} \hat{\otimes}_{B^+_{G_1,n}} D^+_{G_1,n}. \tag{5.4.2.1}$$

The independence of $\phi_{G_0,n}$ on n follows.

(3) Let's descend $D_{G_1,n}$ to its G_0-invariant subspace $D_{G_0,n}$. As $D_{G_1,n}(V_k) = B_{G_1,n} \otimes_{B_{G_0,n}} D_{G_0,n}(V_k)$, after taking the direct limit over $k \geq 0$ and the p-adic completion, we obtain

$$D_{G_1,n} = B_{G_1,n} \otimes_{B_{G_0,n}} D_{G_0,n}.$$

with $D_{G_0,n}$ the p-adic completion of $\varinjlim_k D_{G_0,n}(V_k)$, on which G_0 acts trivially. By Galois (faithfully flat) descent, we have $D_{G_0,n} = (D_{G_1,n})^{G_0/G_1}$, and thanks to (5.4.2.1), we have for $n' \geq n$

$$D_{G_0,n'} = B_{G_0,n'} \otimes_{B_{G_0,n}} D_{G_0,n}. \tag{5.4.2.2}$$

(4) From the natural isomorphism $B_\infty \hat{\otimes}_{B_{G_0,n}} D_{G_0,n} \simeq B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)$, we obtain an identification $B_{G_0,\infty} \hat{\otimes}_{B_{G_0,n}} D_{G_0,n} = (B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0}$ (2.4.9), so that by (5.3.1) and (5.4.2.2),

$$(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0,\, p^m\Gamma-\mathrm{an}} = D_{G_0,m},$$

$$(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0,\, \Gamma-\mathrm{la}} = (\varinjlim_m B_{G_0,m}) \otimes_{B_{G_0,n}} D_{G_0,n} = \varinjlim_m D_{G_0,m}.$$

But we know from (1) that the action ϕ_{G_0} of $\mathrm{Lie}(\Gamma)$ when restricted to $\varinjlim_n D_{G_0,m}$ is the natural action by derivation. This proves the existence and the uniqueness of ϕ_{G_0}.

(5) By construction, we see that ϕ_{G_0} restricted to $B_\infty \otimes_{\mathbf{Q}_p} V_k$ coincides with ϕ_{V_k}. The (i)–(iii) then follow from those of (5.3.2). The last sentence is a direct consequence of (i)–(iii), because we have $B_\infty \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G_0) \simeq \mathrm{Der}_{B_\infty}(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0}$ (4.1.12) and $(B_\infty \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G_0))^\Gamma = B_\infty^\Gamma \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G_0) = B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G_0)$ (2.4.9). □

5.4.3. We identify $\mathrm{Lie}(G_0) = \mathrm{Lie}(G)$ (4.1.8) and denote the map $\mathrm{Lie}(\Gamma) \to B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$ by ϕ_{G_0}. Then ϕ_{G_0} does not depend on the choice of (G_0, ω) verifying the conditions in (5.4.1). We denote this map henceforth by

$$\phi_{\widetilde{X}} : \mathrm{Lie}(\Gamma) \to B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G).$$

Indeed, let (G_0', ω') be another pair as in (5.4.1). If G_0' is contained in G_0 and if this inclusion induces a well-defined (necessarily $G \times \Gamma$-equivariant) continuous injective map $j : B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p) \to B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0', \mathbf{Q}_p)$, then $\phi_{G_0} = \phi_{G_0'}$ will follow from the uniqueness part of the previous proposition. In general, according to (4.1.11), there exists $n \in \mathbf{N}$ such that both inclusions $G_n' = G_0'^{p^n} \subset G_0'$ and especially $G_n' \subset G_0$ induce a well-defined j as above; therefore, $\phi_{G_0'} = \phi_{G_n'} = \phi_{G_0}$, showing the desired independence.

5.4.4. Sen's operator. Picking a topological generator γ of Γ, we define the associated Sen's operator

$$\theta_{\widetilde{X}} := \phi_{\widetilde{X}}\left(\frac{\partial}{\partial \gamma}\right) \in B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$$

(see (4.1.8) for definition of $\frac{\partial}{\partial \gamma}$). Different choices of γ modify θ by $\mathbf{Z}_p^\times$.

5.5 Sen's operator: properties

5.5.1. Let (G_0, ω) be a pair as in (5.4.1). By the construction of ϕ_{G_0} (5.4.2), $\theta_{\widetilde{X}}$ acts naturally on[12]

$$(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0,\, \Gamma-\mathrm{la}} \simeq B_\infty^{G_0-\mathrm{an},\, \Gamma-\mathrm{la}}.$$

The identification here is Γ-equivariant (4.2.1), so $\theta_{\widetilde{X}}$ acts on the right hand side by the natural infinitesimal operator $\frac{\partial}{\partial \gamma}$; one has more explicitely

$$\theta_{\widetilde{X}}(b) = \lim_{n \to +\infty} \frac{\gamma^{p^n}(b) - b}{p^n}, \quad b \in B_\infty^{G_0-\mathrm{an},\, \Gamma-\mathrm{la}}.$$

So, varying G_0, we see that $\theta_{\widetilde{X}}$ also acts on $B_\infty^{G-\mathrm{la},\, \Gamma-\mathrm{la}}$ by the infinitesimal action operator $\frac{\partial}{\partial \gamma}$. In particular, $\theta_{\widetilde{X}}$ annihilates $B_\infty^{G-\mathrm{la},\, \Gamma} = B^{G-\mathrm{la}} (= B^{\mathrm{la}})$, the equality by (2.4.9).

[12] On the left hand side, the action is induced by the natural infinitesimal *left* translation action of G: writing $\theta = \sum_i b_i \otimes X_i \in B_\infty \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)$ and $b \in B_\infty^{G_0-\mathrm{an}}$, we have $(\theta(f_b))(g) = -\sum_i b_i X_i(f_b(g))$. But $\theta(f_b)$ is *not necessarily* of the form $f_{b'}$ for some $b' \in B_\infty^{G_0-\mathrm{an}}$. Nevertheless, if furthermore $b \in B_\infty^{G_0-\mathrm{an},\, \Gamma-\mathrm{la}}$ and $\theta \in \mathrm{Im}\, \phi_{\widetilde{X}}$, then $\theta(f_b)$ is of the form $f_{b'}$, because θ can now be interpreted as an infinitesimal action of Γ by (5.4.2). Evaluating $\theta(f_b)$ at e_G, we get $b' = -\sum_i b_i X_i(b)$. In conclusion, θ really acts on $B_\infty^{G_0-\mathrm{an},\, \Gamma-\mathrm{la}}$ by

$$\theta(b) = -\sum_i b_i X_i(b), \quad b \in B_\infty^{G_0-\mathrm{an},\, \Gamma-\mathrm{la}}.$$

Proof of (5.1.5) and (5.1.2, i). By the previous paragraphs. □

5.5.2 - Proposition. *For any finite-dimensional continuous $\mathbf{Q}_p$-representation (V,ρ) of G, we have a commutative diagram*

$$\begin{array}{ccc} & & B\otimes_{\mathbf{Q}_p}\mathrm{Lie}(G) \\ & \nearrow^{\phi_{\widetilde{X}}} & \downarrow d\rho \\ \mathrm{Lie}(\Gamma) & \xrightarrow{\phi_V} & \mathrm{End}_{B_\infty}(B_\infty\otimes_{\mathbf{Q}_p} V) \end{array}$$

where $d\rho$ is induced by the natural infinitesimal action $\mathrm{Lie}(G)\to\mathrm{End}_{\mathbf{Q}_p}(V)$.

Therefore, to calculate $\phi_{\widetilde{X}}$, we may take a faithful finite-dimensional $\mathbf{Q}_p$-representation V of some open subgroup G_0 (4.1.9) and calculate ϕ_V.

Proof. Let $t\in\mathrm{Lie}(\Gamma)$, it is enough to compare $\phi_{\widetilde{X}}(t)$ and $\phi_V(t)$. For simplicity, we will omit t below.

Let $G_0\subset G$ be an open subgroup acting trivially mod p on some lattice of V.

Any finite-dimensional representation is locally analytic, so shrinking if need, there exists a pair (G_0,ω) as in (5.4.1) such that the matrix coefficients of ρ are G_0-analytic. This induces a map $m_V: V\otimes_{\mathbf{Q}_p}V^*\to\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p)$, $m_V(v\otimes\lambda)(g)=\lambda(g^{-1}v)$, which is easily seen to be G_0-equivariant with respect to the action $\rho\otimes\mathrm{id}_{V^*}$ on $V\otimes V^*$ and the left translation action on $\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p)$. This induces a diagram

$$\begin{array}{ccc} B_\infty\otimes_{\mathbf{Q}_p}V\otimes_{\mathbf{Q}_p}V^* & \xrightarrow{\mathrm{id}_{B_\infty}\otimes m_V} & B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p) \\ \downarrow \phi_V\otimes\mathrm{id}_{V^*} & & \downarrow \phi_{\widetilde{X}} \\ B_\infty\otimes_{\mathbf{Q}_p}V\otimes_{\mathbf{Q}_p}V^* & \xrightarrow{\mathrm{id}_{B_\infty}\otimes m_V} & B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p) \end{array}$$

Every map here is B_∞-linear and is determined by its restriction to G_0-fixed Γ-locally analytic vectors, and one checks that the diagram commutes on such vectors (see (5.3.2) and (5.4.2)), so the diagram commutes in general.

On the other hand, one has a diagram

$$\begin{array}{ccc} B_\infty\otimes_{\mathbf{Q}_p}V\otimes_{\mathbf{Q}_p}V^* & \xrightarrow{\mathrm{id}_{B_\infty}\otimes m_V} & B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p) \\ \downarrow d\rho\circ\phi_{\widetilde{X}}\otimes\mathrm{id}_{V^*} & & \downarrow \phi_{\widetilde{X}} \\ B_\infty\otimes_{\mathbf{Q}_p}V\otimes_{\mathbf{Q}_p}V^* & \xrightarrow{\mathrm{id}_{B_\infty}\otimes m_V} & B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p) \end{array}$$

whose commutativity can be checked again on G_0-fixed Γ-locally analytic vectors.

Now for any $v\in V$, there exists $\lambda\in V^*$ such that $m_V(v\otimes\lambda)\neq 0$. So the induced map

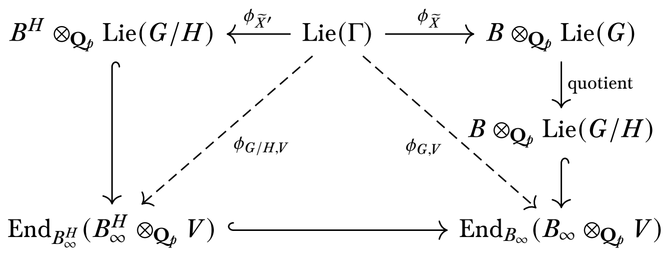
$$\mathrm{id}_{B_\infty}\otimes m_V(-\otimes\lambda): B_\infty\otimes_{\mathbf{Q}_p}\mathbf{Q}_p v\to B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0,\mathbf{Q}_p)$$

is injective. Then we deduce from the commutative diagrams that ϕ_V and $d\rho\circ\phi_{\widetilde{X}}$ agree on $B_\infty\otimes_{\mathbf{Q}_p}\mathbf{Q}_p v$. Hence they are equal, since $v\in V$ is arbitrary. □

5.5.3 - Corollary (*id est (5.1.2, ii)*). *Let H be a closed normal subgroup of G such that $\widetilde{X}'=\mathrm{Spa}(B^H,B^{+H})$ is an affinoid perfectoid G/H-Galois pro-étale covering of X. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Lie}(\Gamma) & \xrightarrow{\phi_{\widetilde{X}}} & B\otimes_{\mathbf{Q}_p}\mathrm{Lie}(G) \\ \downarrow \phi_{\widetilde{X}'} & & \downarrow \textit{quotient} \\ B^H\otimes_{\mathbf{Q}_p}\mathrm{Lie}(G/H) & \xrightarrow{\subset} & B\otimes_{\mathbf{Q}_p}\mathrm{Lie}(G/H) \end{array}$$

Proof. By the construction of $\phi_{\widetilde{X}}$ and $\phi_{\widetilde{X}'}$, we may shrink G as much as possible. So we may assume that G/H admits a faithful finite-dimensional $\mathbf{Q}_p$-representation (V,ρ) (4.1.9). It is then enough to check the following solid commutative diagram



The upper left and upper right triangles are commutative by the previous proposition; and according to (5.3.2), the lower triangle commutes by simple check about the actions of $\mathrm{Lie}(\Gamma)$ on $(B^H_\infty \otimes_{\mathbf{Q}_p} V)^{G/H-\mathrm{sm},\,\Gamma-\mathrm{la}}$ and $(B_\infty \otimes_{\mathbf{Q}_p} V)^{G-\mathrm{sm},\,\Gamma-\mathrm{la}}$ respectively. □

5.6 (In)dependence on charts

Conserving our notations in (5.1.1), we want to examine the dependence of $\phi_{\widetilde{X}}$ on the choice of a toric chart $X \to \mathbf{T}^1$ (*resp.* $\mathbf{B}^1$) in the smallness assumption. For a second toric chart $X \to \mathbf{T}^1$, let's denote by X_∞ the fibre product $X \times_{\mathbf{T}^1} \mathbf{T}^1_\infty$ using this chart. Similarly for anything associated with this tower we emphasise it by the orange colour, e.g. $\Gamma = \mathrm{Gal}(X_\infty/X)$ and $\phi_{\widetilde{X}}$. There is a canonical isomorphism $\Gamma \simeq \Gamma$. Let's fix a topological generator $\gamma \in \Gamma$ and the corresponding element $\gamma \in \Gamma$.

5.6.1. There exists a unique $a \in A^\times$ such that $a \mathrm{dlog} T = \mathrm{dlog} T$, for $\Omega^1_{R/C}$ is a free A-module of rank 1 with basis $\mathrm{dlog} T$ or $\mathrm{dlog} T$.

Notice that in the geometric case, we have $H^1_{\mathrm{cont}}(\Gamma, A_\infty(1)) \simeq H^1_{\mathrm{cont}}(\Gamma, A_\infty)(1)$ and likewise for other ones with A_∞. So according to (2.4.12), we have natural isomorphisms of A-modules

$$\Omega^1_{A/C} \overset{\delta}{\simeq} H^1(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X)(1) \simeq H^1_{\mathrm{cont}}(\Gamma \times \Gamma, A_{\infty,\infty})(1) \simeq \begin{cases} H^1_{\mathrm{cont}}(\Gamma, A_\infty)(1) \simeq H^1_{\mathrm{cont}}(\Gamma, A)(1), \\ H^1_{\mathrm{cont}}(\Gamma, A_\infty)(1) \simeq H^1_{\mathrm{cont}}(\Gamma, A)(1), \end{cases} \tag{5.6.1.1}$$

where the isomorphisms on the right result from properties of the normalised traces (5.2.2). By the explicit discussion of (2.4.12), we have more precisely

$$\Omega^1_{A/C} \simeq H^1_{\mathrm{cont}}(\Gamma, A(1)), \quad \mathrm{dlog} T \mapsto (g \mapsto \{g(T^{1/p^n}))/T^{1/p^n}\}_n),$$
$$\Omega^1_{A/C} \simeq H^1_{\mathrm{cont}}(\Gamma, A(1)), \quad \mathrm{dlog} T \mapsto (g \mapsto \{g(T^{1/p^n}))/T^{1/p^n}\}_n).$$

Hence the composed isomorphism

$$\Omega^1_{A/C} \simeq H^1_{\mathrm{cont}}(\Gamma, A) \simeq H^1_{\mathrm{cont}}(\Gamma \times \Gamma, A_{\infty,\infty}) \simeq H^1_{\mathrm{cont}}(\Gamma, A) \simeq \Omega^1_{A/C}$$

must be given by the multiplication by $a \in A^\times$.

Consequently, there exists $b \in A_{\infty,\infty}$ providing a coboundary connecting the cocycles $(\gamma, -) \mapsto 1$ and $(-, \gamma) \mapsto a$, i.e. such that

$$1 + (\gamma, e)(b) - b = 0 \tag{5.6.1.2}$$
$$0 + (e, \gamma)(b) - b = a. \tag{5.6.1.3}$$

In particular, $b \in A^{\Gamma-\mathrm{la},\,\Gamma-\mathrm{la}}_{\infty,\infty}$ and

$$\frac{\partial}{\partial \gamma} \cdot b = -1, \quad \frac{\partial}{\partial \gamma} \cdot b = a.$$

We point out that one may always perturb b by some $b_0 \in \bigcup_n A_{n,n} = (A_{\infty,\infty})^{\Gamma-\mathrm{sm},\,\Gamma-\mathrm{sm}}$ to make $\|b\|$ sufficiently small without changing the above formula.

5.6.2 - Remark. If G_0 is an open subgroup of G and we consider B_{G_0} in place of A, then the change of charts induced from those of A will produce the same element $a \in B^\times_{G_0}$.

5.6.3 - Proposition (Uniqueness part of (5.1.2)). *We have $\phi_{\widetilde{X}} = a\phi_{\widetilde{X}}$.*

Proof. We may shrink G such that G admits a faithful finite-dimensional $\mathbf{Q}_p$-representation V (4.1.9). Moreover, we may assume that there is a pair (G_0, ω) as in (4.1.5) with $G_0 = G$, so that every notation about B_{G_0} can be simplified as A; this does not change the element $a \in A^\times$ by the previous remark.

It suffices to check that $\phi_V = a\phi_V$. By functoriality result, cf. the proof of (5.5.3), we may use the affinoid perfectoid pro-étale $G \times \Gamma$-Galois (*resp.* $G \times \Gamma$-Galois) covering $\widetilde{X}_\infty$ (*resp.* $\widetilde{X}_\infty$) and the chart $X \longrightarrow \mathbf{T}^1$ (*resp.* $X \longrightarrow \mathbf{T}^1$). Then ϕ_V and ϕ_V are calculated respectively by the following data

$$\begin{array}{ccccc} (B_\infty)_\infty & \longleftarrow & A_\infty & \longleftarrow & R_\infty \\ \uparrow & & \uparrow & & \uparrow \\ B_\infty & \longleftarrow & A & \longleftarrow & R \end{array} \qquad \begin{array}{ccccc} (B_\infty)_\infty & \longleftarrow & A_\infty & \longleftarrow & R_\infty \\ \uparrow & & \uparrow & & \uparrow \\ B_\infty & \longleftarrow & A & \longleftarrow & R \end{array}$$

Notice that canonically $(B_\infty)_\infty \simeq (B_\infty)_\infty$ by symmetry; we denote it by $B_{\infty,\infty}$. Denote also $\theta = \phi_V(\frac{\partial}{\partial\gamma})$ and $\theta = \phi_V(\frac{\partial}{\partial\gamma})$. It suffices to show $\theta = a\theta$.

Now consider the operators

$$\Phi = \sum_{n\geq 0} \frac{(-a^{-1}b)^n}{n!}\theta^n, \quad \Psi = \sum_{n\geq 0} \frac{b^n}{n!}\theta^n \in \mathrm{End}_{B_{\infty,\infty}}(B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V),$$

where we assume $\|b\|$ to be sufficiently small (5.6.1) to guarantee the convergence. Using (5.6.1.2) and (5.6.1.3), we check that $\theta, \theta, \Phi, \Psi$ all commute, and we have a commutative diagram

$$\begin{array}{ccc} (B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V)^{G\times\Gamma,\, \Gamma-\mathrm{la}} & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & (B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V)^{G\times\Gamma,\, \Gamma-\mathrm{la}} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle a\theta} \\ (B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V)^{G\times\Gamma,\, \Gamma-\mathrm{la}} & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & (B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V)^{G\times\Gamma,\, \Gamma-\mathrm{la}} \end{array}$$

where Φ and Ψ are mutually inverse. As both sides generate $B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V$ as a $B_{\infty,\infty}$-module, we see immediately that $\theta = a\theta$ as follows: for any $z' \in (B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V)^{G\times\Gamma,\, \Gamma-\mathrm{la}}$, write $z' = \Phi(z)$ for $z \in (B_{\infty,\infty} \otimes_{\mathbf{Q}_p} V)^{G\times\Gamma,\, \Gamma-\mathrm{la}}$; then we have

$$\theta(z') = \theta(\Phi(z)) = \Phi(\theta(z)) = a\theta(\Phi(z)) = a\theta(z'),$$

where the third equality comes from the above commutative diagram. □

5.6.4 - Remark. By the proposition and the definition of a, the map

$$\phi_{\widetilde{X}} \otimes \mathrm{dlog}T : \mathrm{Lie}(\Gamma) \to (B \otimes_{\mathbf{Q}_p} \mathrm{Lie}(G)) \otimes_A \Omega^1_{A/C}$$

is independent of the toric chart $X \to \mathbf{T}^1$ chosen, hence it is a canonical map.

5.7 $\mathscr{L}\mathscr{A}$-acyclicity

Let G be a p-adic Lie group. Let $X = \mathrm{Spa}(A, A^+)$ be a smooth affinoid adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$, and To remove?
$\pi : \widetilde{X} = \mathrm{Spa}(B, B^+) \to X$ be an affinoid perfectoid pro-étale G-Galois covering.

5.7.1 - Theorem. *Suppose that X is small. Then $R^i\mathscr{L}\mathscr{A}(B) = 0$, $i > 0$ if and only if $\widetilde{X}$ is a locally analytic covering of X; in this case, B is strongly $\mathscr{L}\mathscr{A}$-acyclic.*

Fix (G_0, ω) as in (4.1.5) and denote $G_n = G_0^{p^n}$ as usual. Denote $\widetilde{\mathcal{O}} = \pi_*\mathcal{O}_{\widetilde{X}}$. Consider the subsheaves $\widetilde{\mathcal{O}}^n \subset \widetilde{\mathcal{O}}$ of G_n-analytic sections, $n \in \mathbf{N}$, and the subsheaf $\widetilde{\mathcal{O}}^{\mathrm{la}} \subset \widetilde{\mathcal{O}}$ of G-locally analytic sections. By left exactness of $\mathscr{L}\mathscr{A}$, we have $\widetilde{\mathcal{O}}^n(U) = \widetilde{\mathcal{O}}(U)^{G_n-\mathrm{an}}$ and $\widetilde{\mathcal{O}}^{\mathrm{la}}(U) = \widetilde{\mathcal{O}}(U)^{\mathrm{la}}$ for U quasi-compact open. Clearly we have $\lim_n \widetilde{\mathcal{O}}^n = \widetilde{\mathcal{O}}^{\mathrm{la}}$.

5.7.2 - Proposition. *Suppose that X admits a covering $\mathcal{U}_0 = \{X_1, \ldots, X_k\}$ by small rational open subsets X_i, $i = 1, \ldots, k$ whose preimage $\pi^{-1}(X_i) \subset \widetilde{X}$ is a locally analytic covering of X_i.*

(i) We have $R^\bullet\mathscr{L}\mathscr{A}(B) \simeq H^\bullet(X, \widetilde{\mathcal{O}}^{\mathrm{la}}) \simeq \check{H}^\bullet(X, \widetilde{\mathcal{O}}^{\mathrm{la}})$. In particular, B is $\mathscr{L}\mathscr{A}$-acyclic if and only if $\check{H}^\bullet(X, \widetilde{\mathcal{O}}^{\mathrm{la}}) = 0$, $i > 0$.

(ii) If the direct system $\{\check{H}^i(\mathcal{U}_0, \widetilde{\mathcal{O}}^n)\}_{n\in\mathbf{N}}$ is essentially zero for all $i > 0$, then B is strongly $\mathscr{L}\mathscr{A}$-acyclic.

Proof. Let $\mathscr{B}$ be the basis of open subsets of X consisting of small rational open subsets of X that are contained in some X_i. By assumption on X_i, for any $U \in \mathscr{B}$, the preimage $\pi^{-1}(U)$ is also a locally analytic covering of U. We claim that for any $U \in \mathscr{B}$,

$$\check{H}^i(U, \widetilde{\mathscr{O}}^{\mathrm{la}}) = 0, \quad i > 0.$$

Granting this, we finish the proof formally: by classical results [Gro57, p.175–176], there is a natural isomorphism $\check{H}^\bullet(U, \widetilde{\mathscr{O}}^{\mathrm{la}}) \simeq H^\bullet(U, \widetilde{\mathscr{O}}^{\mathrm{la}})$ for any open subset $U \subset X$, and there is a natural isomorphism

$$\check{H}^\bullet(\mathscr{U}_0, \widetilde{\mathscr{O}}^{\mathrm{la}}) \simeq H^\bullet(X, \widetilde{\mathscr{O}}^{\mathrm{la}}).$$

We have a G-equivariant resolution $0 \to \widetilde{\mathscr{O}}(X) \to \mathscr{C}^\bullet(\mathscr{U}_0, \widetilde{\mathscr{O}})$ by (2.3.10); each term is a $\mathbf{Q}_p$-Banach space by finiteness of $\mathscr{U}_0$. The remaining statements then follow from (4.2.8, iii) and the strong $\mathscr{LA}$-acyclicity in the theorem.

To prove the claim, since U is quasi-compact, it is enough to show that for any *finite* cover $\mathscr{U}$ of U by open subsets in $\mathscr{B}$, we have

$$\check{H}^i(\mathscr{U}, \widetilde{\mathscr{O}}^{\mathrm{la}}) = 0, \quad i > 0.$$

We obtain an exact sequence $0 \to \widetilde{\mathscr{O}}(U) \to \mathscr{C}^\bullet(\mathscr{U}, \widetilde{\mathscr{O}})$ of $\mathbf{Q}_p$-Banach representations of G by applying (2.3.10) to the rational open cover $\pi^{-1}(\mathscr{U})$ of $\pi^{-1}(U)$ (and by finiteness of $\mathscr{U}$). The claim follows by applying (4.2.8, ii) and the first part of the above theorem. □

Proof of (5.7.1). The "only if" part is clear by applying $-\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ to (FE), taking $H^\bullet_{\mathrm{cont}}(G_n, -)$ and then passing to the direct limit over $n \in \mathbf{N}$.

For the converse, we only sketch the proof. We are to prove the strong $\mathscr{LA}$-acyclicity of B. By almost étale descent and the degenerate Lyndon-Hochschild-Serre spectral sequence (1.1.3) (twice), we obtain a natural isomorphism functorial in G_0

$$H^i_{\mathrm{cont}}(G_0, B\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)) \simeq H^i_{\mathrm{cont}}(G_0 \times \Gamma, B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p)) \simeq H^i_{\mathrm{cont}}(\Gamma, B_\infty^{G_0-\mathrm{an}}), \quad i > 0.$$

Fix a topological generator γ of Γ. Then the last cohomology group is calculated by the complex

$$B_\infty^{G_0-\mathrm{an}} \xrightarrow{\gamma-1} B_\infty^{G_0-\mathrm{an}},$$

so it vanishes for $i \geq 2$.

We need a decompletion lemma.

5.7.2.1 - Lemma. *For sufficiently large k, there is a natural isomorphism functorial in G_0*

$$H^1_{\mathrm{cont}}(\Gamma, B_\infty^{G_0-\mathrm{an}}) \simeq H^1_{\mathrm{cont}}(\Gamma, B_\infty^{G_0\times p^k\Gamma-\mathrm{an}}).$$

Proof. Let's go back to the proof of (5.4.2) and use the notation there: $c < \frac{1}{2}$, and $D^+_{G_1,n}$, $D_{G_1,n}$, $D_{G_0,n} = (D_{G_1,n})^{G_0}$ for n large enough, etc. Denote $D^+_{G_0,n} = (D^+_{G_1,n})^{G_0}$. In part (4), we obtained for some $n \geq 0$

$$B_\infty^{G_0-\mathrm{an}} \simeq (B_\infty\hat{\otimes}_{\mathbf{Q}_p}\mathscr{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0} \simeq B_{G_0,\infty}\hat{\otimes}_{B_{G_0,n}} D_{G_0,n}.$$

Since normalised trace maps are retracts, we obtain

$$H^1_{\mathrm{cont}}(\Gamma, B_\infty^{G_0-\mathrm{an}}) \simeq H^1_{\mathrm{cont}}(\Gamma, B_{G_0,k} \otimes_{B_{G_0,n}} D_{G_0,n}) \oplus H^1_{\mathrm{cont}}(\Gamma, \ker \overline{\mathrm{tr}}_{X_{G_0,k}} \hat{\otimes}_{B_{G_0,n}} D_{G_0,n}).$$

The first direct factor is the same as $H^1_{\mathrm{cont}}(\Gamma, D_{G_0,k}) = H^1_{\mathrm{cont}}(\Gamma, B_\infty^{G_0-\mathrm{an},\ p^k\Gamma-\mathrm{an}})$. It remains to show that According to part (1), for $m \geq m(c,n)$, we have

$$(\gamma - 1)^m D^+_{G_0,n} \subset p D^+_{G_0,n}.$$

Recall (5.2.3) that for k sufficiently large (depending on m), $\gamma - 1$ is invertible on $\ker \overline{\mathrm{tr}}_{X_{G_0,k}}$ with

$$(\gamma - 1)^{-2m}(\ker \overline{\mathrm{tr}}_{X_{G_0,k}})^+ \subset (\ker \overline{\mathrm{tr}}_{X_{G_0,k}})^+,$$

where $(\ker \overline{\mathrm{tr}}_{X_{G_0,k}})^+ := \ker \overline{\mathrm{tr}}_{X_{G_0,k}} \cap B^+_\infty$. It is then not hard to show $H^1_{\mathrm{cont}}(\Gamma, \ker \overline{\mathrm{tr}}_{X_{G_0,k}} \hat{\otimes}_{B_{G_0,n}} D_{G_0,n}) = 0$. □

To show the $\mathcal{LA}$-acyclicity of B, it suffices to fix $k \in \mathbf{N}$ and show that for any element $x \in B_\infty^{G_0 \times p^k \Gamma-\mathrm{an}}$, we can find $n \geq 0$ and $l \geq k$ such that

$$x \in (\gamma - 1) B_\infty^{G_n \times p^l \Gamma-\mathrm{an}}.$$

For this, denote $\delta = \frac{\partial}{\partial \gamma} \in \mathrm{Lie}(\Gamma)$.

5.7.2.2 - Lemma. *Suppose $\widetilde{X}$ is a locally analytic covering of X. Then, for sufficiently large n and l depending on k but not on x, there exists $y \in B_\infty^{G_n \times p^l \Gamma-\mathrm{an}}$ such that $\delta(y) = x$.*

Proof. By (4.2.5) and (4.2.6), there exists a constant $C = C(k)$ such that for all $n \geq 0$ and $l \geq k$,

$$\|\delta(x)\|_{G_n \times p^l \Gamma} \leq C \|x\|_{G_0 \times p^k \Gamma}.$$

uniformly for $x \in B_\infty^{G_0 \times p^k \Gamma-\mathrm{an}}$.

By definition (5.1.7, iii), there is $b \in B_\infty^{G-\mathrm{la}}$ such that $\gamma(b) = b - 1$, hence

$$\delta(b) = -1.$$

Furthermore, similarly as in the proof of (5.6.3), by perturbing b by an element of $B_\infty^{G \times \Gamma-\mathrm{sm}}$, we can make $\|b\|_{B_\infty} = \|b\|_{G_n \times p^l \Gamma}$ so small that we obtain a well-defined element

$$y = -\sum_{m \geq 0} \frac{\delta^m(x)}{(m+1)!} b^{m+1}$$

which converges in $(B_\infty^{G_n \times p^l \Gamma-\mathrm{an}}, \|\cdot\|_{G_n \times p^l \Gamma})$ for $n, l \gg 0$. One checks easily that

$$\delta(y) = -\sum_{m \geq 0} \frac{\delta^{m+1}(x)}{(m+1)!} b^{m+1} - \sum_{m \geq 0} \frac{\delta^m(x)}{(m+1)!} (m+1) \delta(b) b^m = x. \qquad \square$$

Fix such n, l as in the lemma. By the proof of (5.4.2), part (1), after enlarging l if needed, one has

$$(\gamma - 1)^m D_{G_n,l}^+ \subset p D_{G_n,l}^+ \tag{5.7.2.3}$$

for some integer $m > 0$. Also, $D_{G_n,l}^+$ is an open bounded and Γ-stable sub-$\mathbf{Z}_p$-module of the $\mathbf{Q}_p$-Banach space $D_{G_n,l} = B_\infty^{G_n \times p^n \Gamma-\mathrm{an}}$.

5.7.2.4 - Lemma. *We have $\delta(D_{G_n,l}) \subset (\gamma - 1) D_{G_n,l}$.*

Proof. Let $z \in D_{G_n,l}$. By definition,

$$\delta(x) = \lim_{n \to +\infty} \frac{(\gamma^{p^n} - 1)(x)}{p^n} = (\gamma - 1) \lim_{n \to +\infty} \prod_{k=0}^{n-1} \frac{\sum_{i=0}^{p-1} \gamma^{i p^k}}{p} (x)$$

once we show that this last limit exists. For this, one deduce from (5.1.7) that there exists some $l > 0$ such that $(\gamma^{p^k} - 1) D_{G_n,l}^+ \subset p^{k-l+1} D_{G_n,l}^+$ for any $k \geq l$, thus $(\frac{1}{p} \sum_{i=0}^{p-1} \gamma^{i p^k} - 1) D_{G_n,l}^+ \subset p^{k-l} D_{G_n,l}^+$. Hence the limit exists. $\square$

This proves the $\mathcal{LA}$-acyclicity of B. Notice that the n, l chosen in (5.7.2.2) depends only on k but not on $x \in B_\infty^{G_0 \times p^k \Gamma-\mathrm{an}}$, so B is even strongly $\mathcal{LA}$-acyclic. $\square$

6 Towards a p-adic Simpson correspondence

Let k be a complete discretely valued field over $\mathbf{Q}_p$ with perfect residue field κ. let $\mathcal{O}_k$ be its ring of integers. Let C be the completed algebraic closure of k with ring of integers $\mathcal{O}_C$.

6.1 Faltings's approach

6.1.1. Notation for local study. Let $\mathrm{Spec}(R)$ be a smooth affine scheme over $\mathcal{O}_k$ which is *small*, that is R is étale over $\mathbf{G}^d_{m,k} = \mathrm{Spec}(\mathcal{O}_k[T_1^{\pm 1}, \ldots, T_d^{\pm 1}])$[13]. Consider the following tower

$$\overline{R} \longleftarrow R_\infty \overset{\Delta_\infty}{\longleftarrow} R_1 \longleftarrow R,$$

where $R_1 = R \otimes_{\mathcal{O}_k} \mathcal{O}_{\bar{k}}$, $\overline{R}$ is the integral closure of R in the maximal étale cover of $R \otimes_{\mathcal{O}_k} \bar{k}$ which is "Galois" over R_1 with group Δ, and R_∞ is the base change R by $\mathcal{O}_{\bar{k}}[T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}]$, which is "Galois" over R_1 with group $\Delta_\infty \simeq \mathbf{Z}_p(1)^d$. Fix a $\mathbf{Z}_p$-basis $\zeta = (\zeta_{p^n})_n$ of $\mathbf{Z}_p(1)$. We associate with it the $\mathbf{Z}_p$-basis $\{\gamma_1, \ldots, \gamma_d\}$ of Δ_∞ such that $\gamma_i(T_j^{1/p^n}) = \zeta_{p^n}^{\delta_{ij}} T_j^{1/p^n}$.

6.1.2. A *generalised representation* of Δ is a finite projective $\widehat{\overline{R}}$-module with the canonical topology (2.3.2) equipped with a continuous semi-linear action of Δ. Such a representation M is called α*-small* for some rational number $\alpha > 0$ if M is a finite free $\widehat{\overline{R}}$-module with a basis on which the action is trivial modulo p^α, and is called *small* if M is α-small for some $\alpha > \frac{2}{p-1}$. Similarly, we define the notion of small representation of Δ_∞ by considering $\widehat{R}_1$-modules and also the p-torsion versions of generalised representations.

We remark that the $\frac{2}{p-1}$ in the definition of smallness is on the one hand related to the element $\zeta_p - 1 \in \bar{k}$, which annihilates $H^1(\Delta_\infty, M \otimes R_\infty)$ (by decomposing it into a direct sum corresponding to the eigenspace decomposition of R_∞), and on the other hand related to the radius of convergence of the exponential function.

A *Higgs module* over $\widehat{R}_1$ is a pair (M, θ) consisting of a finite free $\widehat{R}_1$-module M and a Higgs field $\theta : M \to M \otimes_R \Omega^1_{R/\mathcal{O}_k}(-1)$ such that $\theta \wedge \theta = 0$. We say that (M, θ) is α*-small* if θ is divisible by p^α, and *small* if it is α-small for some $\alpha > \frac{1}{p-1}$.

6.1.3. Descent and decompletion. There is a natural functor

(6.1.3.1)
$$\begin{array}{rcl} \{\text{small } \widehat{R}_1\text{-representations of } \Delta_\infty\} & \to & \{\text{small } \widehat{\overline{R}}\text{-representations of } \Delta\} \\ M' & \mapsto & M' \otimes_{\widehat{R}_1} \widehat{\overline{R}}. \end{array}$$

By Faltings' almost purity theorem [Fal02, Section 2, Theorem 4], each $\widehat{\overline{R}}$-module with a continuous semi-linear action of $\Delta = \mathrm{Gal}(\overline{R}/R)$ is almost induced from an $\widehat{R_\infty}$-module with an action of Δ_∞. Using this and the remark above on eigenspace decomposition of R_∞, it is not hard to prove the following descent and decompletion lemma:

6.1.3.2 - Lemma ([Fal05, Lemma 1]). *Suppose $\alpha > \frac{1}{p-1}$ is a rational number. Let $\overline{M} \simeq \overline{R}^r/p^s$ be a generalised representation.*

(i) If $\overline{M}$ is $p^{2\alpha}$-small, then its reduction modulo $p^{s-\alpha}$ is induced from an $R_1/p^{s-\alpha}$-representation of Δ_∞.

(ii) Let M, N be two semi-linear R_1/p^s-representations of Δ_∞. If $f : M \otimes \overline{R} \to N \otimes \overline{R}$ is a Δ-equivariant $\overline{R}$-linear map, then its reduction modulo $p^{s-\alpha}$ is given by a Δ_∞-equivariant R_1-linear map.

By gluing mod p^s representations for various $s > \frac{1}{p-1}$, we see that the functor (6.1.3.1) is an equivalence of categories.

6.1.4. Let $\alpha > \frac{1}{p-1}$. Let M be an α-small $\widehat{R}_1$-representation of Δ_∞. This induces the infinitesimal action $\theta : M \to M \otimes_R \Omega^1_{R/\mathcal{O}_k}(-1)$ by taking logarithm. More precisely, we have

$$\theta = \sum_i \log(\gamma_i) \otimes \mathrm{dlog} T_i \cdot \zeta^{-1},$$

where the convergence of logarithms is guaranteed by the smallness of M. Since γ_i commute with each other, so do $\log(\gamma_i)$, hence θ is a Higgs field on M.

[13] In [Fal05], Faltings allowed R to have toroidal singularities, and similarly for the global case.

Conversely, let (M, θ) be an α-small Higgs $\widehat{R}_1$-module. Then writing $\theta = \sum_i \theta_i \otimes \mathrm{dlog} T_i \cdot \zeta^{-1}$ or equivalently defining $\theta_i := \theta(\frac{\partial}{\partial \log T_i} \cdot \zeta)$, we obtain operators on M

$$\gamma_i(x) = \exp(\theta_i)(x) := \sum_n \frac{1}{n!} \theta_i^n(x), \quad x \in M \tag{6.1.4.1}$$

whose convergence is assured by the smallness, and we extend them continuously to an $\widehat{R}_1$-linear action of Δ_∞ on M by integrability of θ.

6.1.5 - Lemma. *For any toric chart of* $\mathrm{Spec}(R)$*, the formulae (6.1.4.1) and (6.1.3.1) induce a bijection between small Higgs modules over* $\widehat{R}_1$ *and small generalised representations of* Δ.

6.1.6. Globalisation. Now we turn to the global case. Let X be a smooth and proper scheme over $\mathcal{O}_k$ and $\mathfrak{X}$ be the associated formal scheme. A *small generalised representation* of the geometric étale fundamental group $\pi_1(\mathfrak{X}_C)$ is defined as a compatible system of small generalised representations on a covering of $\mathfrak{X}$ by $\{\mathfrak{U}_i\}_i$ with $U_i = \mathrm{Spec}(R_i)$ small affines of X. A *small Higgs bundle* on $\mathfrak{X}_{\mathcal{O}_C}$ is a vector bundle $\mathfrak{E}$ together with a morphism $\theta : \mathfrak{E} \to \mathfrak{E} \otimes \Omega^1_{\mathfrak{X}_{\mathcal{O}_C}/\mathcal{O}_C}(-1)$ which is divisible by p^α for some rational number $\alpha > \frac{2}{p-1}$, and such that $\theta \wedge \theta = 0$.

First, let's fix a toric chart for each U_i. To any small Higgs bundle $(\mathfrak{E}, \theta)$, the local correspondence associates a small representation ρ_i on each $\mathfrak{U}_i$. In order to glue these local Higgs field to a global one, we need to study auto-equivalences of the category of Higgs fields (especially those induced by change of charts). This will be done by choosing a lifting of $\mathfrak{X}_{\mathcal{O}_C}$ over A_2.

6.1.7. Fontaine's rings. Recall that Fontaine's A_{inf} construction gives a ring $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$ by Witt vector construction, which admits a surjection onto $\mathcal{O}_C$ whose kernel is a principal ideal generated by some ξ. We define $A_2 := A_{\mathrm{inf}}/\xi^2$, which sits in the extension

$$0 \to \mathcal{O}_C \xi \to A_2 \to \mathcal{O}_C \to 0.$$

We have canonically $\mathcal{O}_C(1) \simeq p^{1/(p-1)} \mathcal{O}_C \xi$ and Galois equivariantly. Similarly, we define $A_2(R) = W(\overline{R}^\flat)/\xi^2$, which is a lifting of $\widehat{\overline{R}}$ over A_2 and sits in the extension

$$0 \to \widehat{\overline{R}}\xi \to A_2(R) \to \widehat{\overline{R}} \to 0.$$

Fix a system $\{\zeta_{p^n}\}_n$ of p-power roots of unity and let $\epsilon = (\zeta_{p^n})_n \in C^\flat$. Then the image $t \in A_2(R)$ of $\log([\epsilon]) \in A_{\mathrm{inf}}(R)$ is an generator of $p^{1/(p-1)} \widehat{\overline{R}} \xi = \widehat{\overline{R}}(1)$.

Suppose we have a diagram

$$\begin{array}{ccccc} A_2 & \longrightarrow & \widetilde{R} & \dashrightarrow & A_2(R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_C & \longrightarrow & \widehat{R}_1 & \longrightarrow & \widehat{\overline{R}} \end{array}$$

where the first row is a lifting of the second row over A_2. By classical deformation theory, the set $\mathcal{L}$ of all possible dashed arrows is a torsor under $\mathrm{Hom}_{\widehat{R}_1}(\widehat{\Omega}^1_{\widehat{R}_1/\mathcal{O}_C}, \widehat{\overline{R}}\xi)$.

6.1.8 - Lemma. *We keep the notation of (6.1.6) Let* $\widetilde{\mathfrak{X}}$ *be a lifting* $\mathfrak{X}_{\mathcal{O}_C}$ *over* A_2*. To each* $U = \mathrm{Spec}(R)$ *small affine of* X*, we associate* $\mathcal{T}(\mathfrak{U}) = \mathrm{Hom}_{\widehat{R}_1}(\widehat{\Omega}^1_{\widehat{R}_1/\mathcal{O}_C}, \widehat{\overline{R}}\xi)$.

(i) There exists a natural 1-Čech cocycle $\phi_{ij} \in \mathcal{T}(\mathfrak{U}_i \cap \mathfrak{U}_j)$ *such that* $\exp\big(\theta(\phi_{ij})\big)$ *interwines* ρ_i *and* ρ_j*, which necessarily glue together to a small generalised representation* ρ.

(ii) For another choice of toric charts for U_i*, denote the corresponding cocycles by* ϕ'_{ij}*. Then there exists a natural cochain* ψ_i *such that* $\phi' - \phi = d\psi$*, so that necessarily* $\exp(\theta(\psi_i))$ *interwines* ρ *and* ρ'.

Furthermore, these constructions are going to be functorial. So we obtain

6.1.9 - Theorem (Faltings). *Let* X *be a smooth and proper scheme over* $\mathcal{O}_k$ *and* $\mathfrak{X}$ *be the associated formal scheme. Suppose* $\mathfrak{X}_{\mathcal{O}_C}$ *is liftable to* A_2*, Then there exists an equivalence between the category of small Higgs bundles over* $\mathfrak{X}_{\mathcal{O}_C}$ *and that of small generalised representations of* $\pi_1(\mathfrak{X}_C)$.

6.1.10 - Remark. (i) The set of all liftings $\widetilde{\mathfrak{X}}$ is a torsor under $H^1(\mathfrak{X}_{\mathcal{O}_C}, \mathcal{H}om_{\mathfrak{X}_{\mathcal{O}_C}}(\widehat{\Omega}^1_{\mathfrak{X}_{\mathcal{O}_C}/\mathcal{O}_C}, \mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C}}\xi))$. This group acts on the set of equivalences provided by the theorem, in a compatible way with its action on the set of liftings.

(ii) So far, our construction is integral. This extends also to the rational level by inverting p (by well defining the categories). Besides, one can compare cohomologies through this correspondence.

(iii) The smallness of generalised representations and Higgs bundles can be dropped for curves by some descent argument.

We sketch a proof of the last lemma.

6.1.11. Now let (M,θ) be a small Higgs module over $\widehat{R}_1$ and let's define an action of Δ on M. On the one hand, by integrability and smallness of θ, the formula $\exp(\theta(u))$ is well-defined and induces an $\widehat{\overline{R}}$-linear action of $u \in \mathrm{Hom}_{\widehat{R_1}}(\widehat{\Omega}^1_{\widehat{R_1}/\mathcal{O}_C}, \widehat{\overline{R}}\xi)$ on $M \otimes \widehat{\overline{R}}$. On the other hand, any $\phi \in \mathscr{L}$ induces a group homomorphism

$$\Delta \to \mathrm{Hom}_{\widehat{R_1}}(\widehat{\Omega}^1_{\widehat{R_1}/\mathcal{O}_C}, \widehat{\overline{R}}\xi), \quad g \mapsto g(\phi) - \phi,$$

hence an $\widehat{\overline{R}}$-semi-linear action of Δ on $M \otimes \widehat{\overline{R}}$, that is

$$\rho_\phi(\theta)(g) := \exp\left(\theta(g(\phi) - \phi)\right) \circ g, \quad g \in \Delta.$$

Any two $\phi_1, \phi_2 \in \mathscr{L}$ differ by some $\phi_{12} \in \mathrm{Hom}_{\widehat{R_1}}(\widehat{\Omega}^1_{\widehat{R_1}/\mathcal{O}_C}, \widehat{\overline{R}}\xi)$ and the corresponding representations $\rho_{\phi_1}(\theta)$ and $\rho_{\phi_2}(\theta)$ are interwined by the automorphism $\exp(\theta(\phi_{12}))$ of $M \otimes \widehat{\overline{R}}$.

Let's specialise to toric charts. For any toric chart and elements $\widetilde{T_1}, \ldots, \widetilde{T_d} \in \widetilde{R}$ lifting $T_1, \ldots, T_d$, there is a distinguished element $\phi_{T,\widetilde{T}} \in \mathscr{L}$ defined by $\widetilde{R} \to A_2(R)$, $\widetilde{T_i} \mapsto [T_i^\flat]$, where $T_i^\flat = (T_i^{1/p^n}) \in \overline{R}^\flat$. The corresponding action of Δ factors through Δ_∞ and is described exactly by the same formula as (6.1.4.1) with $\theta_i = \theta(\frac{\partial}{\partial \log T_i} \cdot t)$.

Now the lemma follows immediately.

6.2 Liu-Zhu's approach

Let X be a d-dimensional ***smooth*** rigid analytic variety over $\mathrm{Spa}(k, \mathcal{O}_k)$, i.e. locally admitting toric charts $U \to \mathbf{T}^d = \mathrm{Spa}(k\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle, \mathcal{O}_k\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle)$ which can be written as a composition of rational subsets and finite étale maps. In [LZ17], Liu and Zhu defined a functor from the category of $\mathbf{Q}_p$-local systems on $X_{\text{ét}}$ to the category of nilpotent Higgs bundles on X_K (local systems play the role of representations of the fundamental group). Here, the condition of being smooth over a ***discretely valued*** field is needed to make sense of the period sheaves especially $\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ [Sch13a, Definition 6.8] and the Faltings extension; this is also ***a posteriori*** related to the nilpotency of target Higgs fields.

6.2.1. Let $\nu : X_{\text{proét}} \to X_{\text{ét}}$ and $\nu' : (X_K)_{\text{proét}} \to (X_K)_{\text{ét}}$ denote the canonical projections. A $\mathbf{Z}_p$*-local system* $\mathbf{L}$ on $X_{\text{ét}}$ is an inverse system of sheaves of $\mathbf{Z}/p^n$-modules $\mathbf{L}_n$ on $X_{\text{ét}}$ such that each $\mathbf{L}_n$ is étale locally a constant sheaf associated to a finitely generated $\mathbf{Z}/p^n$-module and that this inverse system is isomorphic in the pro-category to an inverse system for which $\mathbf{L}_{n+1}/p^n \simeq \mathbf{L}_n$. We associate with $\mathbf{L}$ the sheaf $\hat{\mathbf{L}} = \varprojlim_n \nu^* \mathbf{L}_n$ on $X_{\text{proét}}$. For example, $\mathbf{Z}_p = \{\mathbf{Z}/p^n\}_n$ is a $\mathbf{Z}_p$-local system on $X_{\text{ét}}$ and we obtain the sheaf $\hat{\mathbf{Z}}_p$ on $X_{\text{proét}}$; inverting p, we obtain the sheaf $\hat{\mathbf{Q}}_p$.

A $\mathbf{Q}_p$*-local system* on $X_{\text{ét}}$ is a descent datum for the étale topology in the isogeny category of $\mathbf{Z}_p$-local systems. We can associate to a $\mathbf{Q}_p$-local system $\mathbf{L}$ a sheaf $\hat{\mathbf{L}}$ on $X_{\text{proét}}$, which is a locally free $\hat{\mathbf{Q}}_p$-module.

6.2.2. Let X be a smooth scheme or rigid analytic variety over k. A *Higgs bundle* is a pair $(\mathcal{H}, \theta)$ on $X_{\text{ét}}$ consisting of a vector bundle $\mathcal{H}$ and an $\mathcal{O}_X$-linear map $\mathcal{H} \to \mathcal{H} \otimes \Omega^1_{X/k}(-1)$ verifying the integrability condition $\theta \wedge \theta = 0$, where everything is on the étale site[14]. The θ is called a *Higgs field* or *flat connection* on $\mathcal{H}$. The *Higgs cohomology* of $(\mathcal{H}, \theta)$ is the cohomology of the complex

$$\mathcal{H} \xrightarrow{\theta} \mathcal{H} \otimes \Omega^1_{X/k}(-1) \xrightarrow{\theta} \mathcal{H} \otimes \Omega^2_{X/k}(-2) \to \cdots.$$

[14]By [Sch13a, Lemma 7.3], these data descend to the analytic topology over X.

Locally we can write $\theta = (\theta_1, \ldots, \theta_d)$ with $\theta_i \in \mathrm{End}(\mathscr{H})$, corresponding to a set of generators $t_1, \ldots, t_d$ of $\Omega^1_{X/k}(-1)$. The integrability condition $\theta \wedge \theta = 0$ is equivalent to the commutativity condition $[\theta_i, \theta_j] = 0$.

We say that θ is ***nilpotent*** if each θ_i is nilpotent; this definition is independent of the generators chosen. Similarly, we can define a Higgs module (M, θ) over a ring R smooth over a ring containing k_∞ or over its ring of integers.

The category of Higgs bundles admits a tensor product given by $(\mathscr{H}_1, \theta_1) \otimes (\mathscr{H}_2, \theta_2) := (\mathscr{H}_1 \otimes \mathscr{H}_2, \theta_1 \otimes 1 + 1 \otimes \theta_2)$, and a natural dual. For a morphism $f : Y \to X$ between smooth rigid analytic varieties over k, the pullback induces a functor from Higgs bundles on $X_{\text{ét}}$ to Higgs bundles on $Y_{\text{ét}}$. One can ask further for such as nilpotency or, for Higgs bundles on $(X_K)_{\text{ét}}$, a semi-linear $\mathrm{Gal}(K/k)$-action, these are preserved under these operations.

6.2.3. A period sheaf. In [Sch13a, Section 6] and [Sch16], Scholze constructed the de Rham period sheaf $\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ on $X_{\text{proét}}$, which is a sheaf of $\mathcal{O}_X$-algebras with a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{O}\mathbf{B}_{\mathrm{dR}}$ and an integrable connection $\nabla_{\mathcal{O}\mathbf{B}_{\mathrm{dR}}} : \mathcal{O}\mathbf{B}_{\mathrm{dR}} \to \mathcal{O}\mathbf{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$ satisfying Griffiths transversality. We will only use its zeroth graded piece

$$\nabla = \mathrm{gr}^0 \nabla_{\mathcal{O}\mathbf{B}_{\mathrm{dR}}} : \mathcal{O}\mathbf{C} \to \mathcal{O}\mathbf{C} \otimes_{\mathcal{O}_X} \Omega^1_{X/k}(-1),$$

where $\mathcal{O}\mathbf{C} = \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}$. Here the Tate twist (-1) comes from $\mathrm{gr}^{-1} \mathcal{O}\mathbf{B}_{\mathrm{dR}} \simeq \mathcal{O}\mathbf{C}(-1)$. Fix a system $\{\zeta_{p^n}\}_n$ of p-power roots of unity and let $\epsilon = (\zeta_{p^n})_n \in K^\flat$, $t = \log([\epsilon]) \in \mathbf{B}_{\mathrm{dR}}$. Then t^{-1} is a generator of the Tate twist.

We have a friendly local description of $\mathcal{O}\mathbf{C}$ on $X_{\text{proét}}$ ([Sch13a, Section 6]). Let Y be étale over X such that there is an étale map $Y \to \mathbf{T}^d$ and denote $Y_{K,\infty} = Y_K \times_{\mathbf{T}^d_K} \mathbf{T}^d_{K,\infty} \in (X_K)_{\text{proét}}$. Choose a basis $\{\gamma_1, \ldots, \gamma_d\}$ of the Galois group of $\mathbf{T}^d_{K,\infty}$ over $\mathbf{T}^d_K$, which is canonically isomorphic to $\mathbf{Z}_p(1)^d$, such that $\gamma_i(T_j^{1/p^n}) = \zeta_{p^n}^{\delta_{ij}} T_j^{1/p^n}$. Then we have over $X_{\text{proét}}/Y_{K,\infty}$ that

$$\mathcal{O}\mathbf{C} = \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}} \simeq \widehat{\mathcal{O}}_{Y_K}[V_1, \ldots, V_d], \tag{6.2.3.1}$$

where $V_i = t^{-1} \log\Big([T_i^\flat]/T_i\Big)$ and $\gamma_i(V_j) = V_j + \delta_{ij}$; also, over $X_{\text{proét}}/Y_{K,\infty}$, the connection ∇ is simply the derivation with respect to all T_i, so that

$$\nabla = -\sum_i \frac{\partial}{\partial V_i} \otimes \mathrm{dlog} T_i \cdot t^{-1}. \tag{6.2.3.2}$$

Alternatively, one can also express $(\mathcal{O}\mathbf{C}, \nabla)$ using the Faltings extension

$$0 \to \widehat{\mathcal{O}}_X \to \mathscr{E} \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \Omega^1_{X/k}(-1) \to 0,$$

which is the first graded piece of the integral Poincaré lemma [Sch13a, Corollary 6.13]. Locally over $X_{\text{proét}}/Y_{K,\infty}$, $\mathscr{E}$ admits an $\widehat{\mathcal{O}}_X$-basis $\{1, y_1, \ldots, y_d\}$ such that $\gamma_i(1) = 1$, $\gamma_i(y_j) = y_j + \delta_{ij}$ (cf. footnote 11). Then

$$\mathcal{O}\mathbf{C} = \varinjlim_n \mathrm{Sym}^n_{\widehat{\mathcal{O}}_X} \mathscr{E}$$

with ∇ induced by the surjection in the Faltings extension (up to a sign -1, just for later convenience), which matches the previous description via $y_j \mapsto V_j$.

6.2.4 - Theorem ([LZ17, Theorem 2.1]). *Let X be a d-dimensional smooth rigid analytic variety over k.*

(i) Let $\mathbf{L}$ be a $\mathbf{Q}_p$-local system on $X_{\text{ét}}$ of rank r. Then $R^i \nu'_*(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}) = 0$ for $i > 0$, and

$$\mathscr{H}(\mathbf{L}) := \nu'_*(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})$$

is a vector bundle of rank r on X_K together with a nilpotent Higgs field

$$\theta_{\mathbf{L}} := \nu'_*(\nabla) : \mathscr{H}(\mathbf{L}) \to \mathscr{H}(\mathbf{L}) \otimes \Omega^1_{X_K/K}(-1)$$

and a natural semi-linear $\mathrm{Gal}(K/k)$*-action.*

(ii) There is a canonical isomorphism

$$\nu'^* \mathscr{H}_L \otimes_{\mathcal{O}_{X_K}} \mathcal{O}\mathbf{C}\big|_{(X_K)_{\acute{e}t}} \simeq (\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})\big|_{(X_K)_{\acute{e}t}},$$

which is compatible with the Higgs fields on both sides.

(iii) The functor $\mathscr{H} : \mathbf{L} \mapsto (\mathscr{H}(\mathbf{L}), \theta)$ from the category of $\mathbf{Q}_p$-local systems on $X_{ét}$ to the category of nilpotent Higgs bundles on $(X_K)_{ét}$ respects the tensor product and the dual, and is compatible with the pullback by a morphism $Y \to X$ of smooth rigid analytic varieties over k.

(iv) Assume X is smooth proper over k. For any $\mathbf{Z}_p$-local system $\mathbf{L}$ on $X_{ét}$, there is a canonical isomorphism

$$H^i_{ét}(X_C, \mathbf{L}) \otimes C \simeq H^i_{ét}(X_C, (\mathscr{H}(\mathbf{L}) \otimes \Omega^\bullet_{X_C/C}, \theta_{\mathbf{L}})),$$

where the last complex is $\mathscr{H}(\mathbf{L}) \xrightarrow{\theta_{\mathbf{L}}} \mathscr{H}(\mathbf{L}) \otimes \Omega^1_{X_C/C}(-1) \xrightarrow{\theta_{\mathbf{L}}} \mathscr{H}(\mathbf{L}) \otimes \Omega^2_{X_C/C}(-2) \to \cdots$.

6.2.5 - Remark. (i) In the situation of (iv), if $\mathbf{L} = \mathbf{Z}_p$, then $(\mathscr{H}(\mathbf{L}), \theta_{\mathbf{L}}) = (\mathscr{O}_{X_C}, 0)$, and we recover the classical Hodge-Tate decomposition (although in fact, the tools to be seen in the proof can imply more directly this decomposition).

(ii) If $X = \{x\} = \mathrm{Spa}(k, \mathscr{O}_k)$ is a point, then for any $\mathbf{Q}_p$-local system $\mathbf{L}$ on $X_{\text{ét}}$, the geometric fibre $\mathbf{L}_{\bar{x}}$ is a continuous $\mathbf{Q}_p$-representation of $\mathrm{Gal}(K/k)$ and $\mathscr{H}(\mathbf{L}) = \mathbf{L}_{\bar{x}} \otimes K$; the Higgs field is zero since we are in dimension zero. We know that the scalar extension loses information, so that in this case the functor $\mathscr{H}$ is not fully faithful.

(iii) The statement (iv) can be more generally stated in the relative case: let $f : X \to Y$ be smooth proper morphism of rigid analytic varieties over k and $\mathbf{L}$ be a $\mathbf{Z}_p$-local system on $X_{\text{ét}}$. If $R^i f_* \mathbf{L}$ is a $\mathbf{Z}_p$-local system on $Y_{\text{ét}}$, then there is a canonical isomorphism $(\mathscr{H}(R^i f_* \mathbf{L}), \theta_{R^i f_* \mathbf{L}}) \simeq R^i f_{\mathrm{Higgs},*}(\mathscr{H}(\mathbf{L}) \otimes \Omega^\bullet_{X/Y}, \theta_{\mathbf{L}})$.

6.2.6 - Remark. Working with a formally smooth formal scheme $\mathfrak{X}$ over $\mathscr{O}_C$ (that is, the geometric case) which admits a lifting $\widetilde{\mathfrak{X}}$ to $A_{2,k}$[15], Wang constructed a pair $(\mathscr{O}\widehat{\mathbf{C}}^\dagger, \nabla)$ consisting of an overconvergent period sheaf and a connection on it, defined the functor $\mathbf{L} \mapsto (\mathscr{H}^\dagger(\mathbf{L}), \theta_{\mathbf{L}}) := \nu_*(\mathbf{L} \otimes \mathscr{O}\widehat{\mathbf{C}}^\dagger, 1 \otimes \nabla)$ from locally finite free $\widehat{\mathscr{O}}_X$-modules on $X_{\text{proét}}$ to $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules on $\mathfrak{X}_{\text{ét}}$, where ν denotes the natural projection $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$, and proved it to be an equivalence between small generalised representations and small Higgs bundles [Wan21]. This functor is compatible with that of Liu-Zhu in the case where $\mathfrak{X}$ is actually defined over $\mathscr{O}_k$. This construction has the advantage of being global just as that of Liu-Zhu; similarly, a more detailed study of this functor is done by passing to toric charts.

The period sheaf $\mathscr{O}\widehat{\mathbf{C}}^\dagger$ is constructed, in a similar way as in the last paragraph of (6.2.3), from an integral faltings extension $0 \to \rho_k^{-1}\widehat{\mathscr{O}}^+_X \to \mathscr{E}^+ \to \widehat{\mathscr{O}}^+_X \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathscr{O}_C}(-1) \to 0$ on the site $X_{\text{proét}}$, for certain $\rho_k \in \mathscr{O}_C$. From this, we get via pullback an inverse system of smaller extensions

$$0 \to \widehat{\mathscr{O}}^+_X \to \mathscr{E}^+_\rho \to \rho\widehat{\mathscr{O}}^+_X \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathscr{O}_K}(-1) \to 0$$

parametrised by $\rho \in \rho_k \mathscr{O}_K$. We define successively

$$\mathscr{O}\mathbf{C}^+_\rho = \varinjlim_n \mathrm{Sym}^n \mathscr{E}^+_\rho, \quad \mathscr{O}\widehat{\mathbf{C}}^+_\rho = p\text{-adic completion of } \mathscr{O}\mathbf{C}^+_\rho, \quad \mathscr{O}\widehat{\mathbf{C}}^{\dagger,+} = \varinjlim_\rho \mathscr{O}\widehat{\mathbf{C}}^+_\rho,$$

and a natural connection ∇ on all of them. Inverting p, we get the period sheaves $\mathscr{O}\mathbf{C}_\rho = \mathscr{O}\mathbf{C}$, $\mathscr{O}\widehat{\mathbf{C}}_\rho$ and $\mathscr{O}\widehat{\mathbf{C}}^\dagger$. These are not surprising objects: indeed, we have a similar local description as in (6.2.3). Choose a toric chart $U = \mathrm{Spa}(R, R^+) \to \mathbf{T}^d$ and let $U_\infty = \mathrm{Spa}(\hat{R}_\infty, \hat{R}^+_\infty) = U \times_{\mathbf{T}^d} \mathbf{T}^d_\infty$. There are sections $1, y_1, \ldots, y_d \in \mathscr{E}^+(U_\infty)$ such that $\gamma_i(1) = 1$, $\gamma_i(y_j) = y_j + \delta_{ij}$ and $y_1, \ldots, y_d$ map to a basis of $\widehat{\Omega}^1_{\mathfrak{X}/\mathscr{O}_K}$. Then, denoting V_i the corresponding images of y_i, we have

$$\mathscr{O}\mathbf{C}^+_\rho(U_\infty) \simeq \hat{R}^+_\infty[\rho V_1, \ldots, \rho V_d], \; \mathscr{O}\widehat{\mathbf{C}}^+_\rho(U_\infty) \simeq \hat{R}^+_\infty\langle \rho V_1, \ldots, \rho V_d\rangle, \; \mathscr{O}\widehat{\mathbf{C}}^{\dagger,+}(U_\infty) \simeq \varinjlim_\rho \hat{R}^+_\infty\langle \rho V_1, \ldots, \rho V_d\rangle,$$

and ∇ is the same as (6.2.3.2).

The rest of this subsection is dedicated for a sketch of the proof of the theorem.

First, we introduce a nice basis $\mathscr{B}$ of $(X_K)_{\text{ét}}$, whose objects consist of those étale maps to X_K that are the base changes of *standard étale morphisms* $Y \to X_{k'}$ (i.e. a finite composition of finite étale maps and rational localisations) over some finite extension $k \subset k' \subset K$ where Y is small after some finite extension of k', and whose morphisms are the base changes of étale morphisms over some finite extension of k in K.

We start from the statements concerning $R\nu'_*(\hat{\mathbf{L}} \otimes \mathscr{O}\mathbf{C})$: it is enough to prove that

(i) if $X = \mathrm{Spa}(A, A^+)$ admits a toric chart to $\mathbf{T}^d$, then $H^0(X_{\text{proét}}/X_K, \hat{\mathbf{L}} \otimes \mathbf{C})$ is a finite projective A_K-module of rank r;

[15] Here, the ring $A_{2,k}$ is a $\mathscr{O}_k$-typical variant of A_2 (6.1.7): let $A_{\mathrm{inf},k} := W(\mathscr{O}^\flat_C) \otimes_{W(\kappa)} \mathscr{O}_k$; it surjects naturally onto $\mathscr{O}_C$ with kernel a principal ideal generated by ξ_k; then we define $A_{2,k} := A_{\mathrm{inf},k}/\xi^2$.

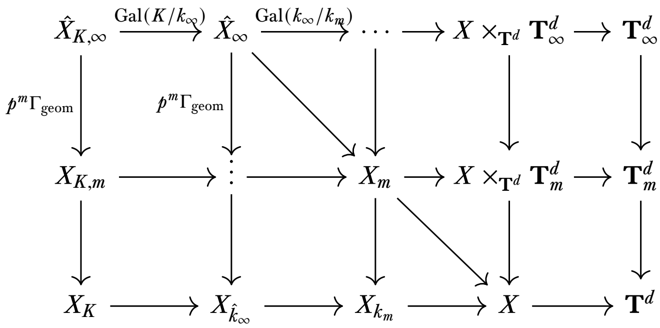
(ii) if in addition $Y = \mathrm{Spa}(B,B^+) \to X$ is a standard étale morphism, then

$$H^i(X_{\text{proét}}/Y_K, \hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}) = 0, \quad i > 0$$

$$H^0(X_{\text{proét}}/Y_K, \hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}) = B_K \otimes_{A_K} H^0(X_{\text{proét}}/Y_K, \hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}).$$

Thanks to their local nature, we may well assume that $\mathbf{L}$ is a $\mathbf{Z}_p$-local system.

6.2.7. Notation for local computation. Let $k_\infty = \bigcup_m k_m$ be the cyclotomic extension of k in $\bar{k}$. . We construct the following diagram



where $m \geq 0$, each square is Cartesian and the labelled arrows are affinoid perfectoid pro-étale Galois with corresponding Galois groups; also, $\hat{X}_{K,\infty}$ is affinoid perfectoid pro-étale $\mathrm{Gal}(K/k) \ltimes \Gamma_{\text{geom}}$ over X; here $\Gamma_{\text{geom}} \simeq \mathbf{Z}_p(1)^d$ is essentially the "Galois group" of $\mathbf{T}^d_\infty$ over $\mathbf{T}^d$. We denote

$$X_\infty = \varprojlim_m X_m = \varprojlim_m \mathrm{Spa}(A_m, A_m^+) \in X_{\text{proét}}, \quad \hat{X}_\infty = \mathrm{Spa}(\hat{A}_\infty, \hat{A}_\infty^+)$$
$$X_{K,\infty} = \varprojlim_m X_{K,m} = \varprojlim_m \mathrm{Spa}(\hat{A}_{K,m}, \hat{A}^+_{K,m}) \in (X_K)_{\text{proét}}, \quad \hat{X}_{K,\infty} = \mathrm{Spa}(A_{K,\infty}, A^+_{K,\infty}).$$

Then $X_{K,\infty}$ is affinoid perfectoid pro-étale Γ_{geom}-Galois over X_K and X_∞ is affinoid perfectoid pro-étale $\mathrm{Gal}(k_\infty/k) \ltimes \Gamma_{\text{geom}}$-Galois over X.

Similarly, if $Y \to X$ is a standard étale morphism, then we may define $Y_m = \mathrm{Spa}(B_m, B_m^+)$, $Y_{K,m} = \mathrm{Spa}(B_{K,m}, B^+_{K,m})$, Y_∞, $\hat{Y}_\infty = \mathrm{Spa}(\hat{B}_\infty, \hat{B}^+_\infty)$, $Y_{K,\infty}$ and $\hat{Y}_{K,\infty} = \mathrm{Spa}(\hat{B}_{K,\infty}, \hat{B}^+_{K,\infty})$ by pullback.

6.2.8. The vanishing of $H^i(X_{\text{proét}}/Y_{K,\infty}, \hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})$, $i > 0$ allows an almost étale descent:

6.2.8.1 - Lemma. *For $i \geq 0$, the natural map*

$$H^i(\Gamma_{\text{geom}}, (\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})(Y_{K,\infty})) \simeq H^i(X_{\textit{proét}}/Y_K, \hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})$$

is an isomorphism.

We are reduced to examine the left hand side. The local description of $\mathcal{O}\mathbf{C}$ over $Y_{K,\infty}$ allows to write

$$(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})(Y_{K,\infty}) \simeq \mathcal{M}(Y_{K,\infty})[V_1, \ldots, V_d],$$

where we denote $\mathcal{M} := \hat{\mathbf{L}} \otimes \widehat{\mathcal{O}}_X$. The only mysterious part is $\mathcal{M}(Y_{K,\infty})$, which is a finite projective $\hat{B}_{K,\infty}$-module of rank r. By decompletion results for toric towers in [KL19, 7], we obtain:

6.2.8.2 - Lemma. *There exists a unique finite projective B_K-module $M_K(Y)$ (necessarily of rank r) of $\mathcal{M}(Y_{K,\infty})$, which is stable under Γ, such that*

(a) $M_K(Y) \otimes_{B_K} \hat{B}_{K,\infty} = \mathcal{M}(Y_{K,\infty})$;

(b) the B_K-linear representation of Γ_{geom} on $M_K(Y)$ is unipotent.

Furthermore, $M_K(Y)$ has the following properties:

(P1) We can descend $M_K(Y)$ further to a finite projective B_{k_m}-module $M_{k_m}(Y)$ such that $M_{k_m}(Y) \otimes_{B_{k_m}} B_K = M_K(Y)$. Moreover, the construction of $M_{K_m}(Y)$ is compatible with base change along standard étale morphisms.

(P2) The natural maps $H^i(\Gamma_{\text{geom}}, M_K(Y)) \to H^i(\Gamma_{\text{geom}}, \mathcal{M}(Y_{K,\infty}))$, $i > 0$ are isomorphisms.

According to (P1) and by Tate's acyclicity theorem and Kiehl's glueing theorem for coherent sheaves, the rule $(Y = \mathrm{Spa}(B,B^+) \to X_{k'}) \in \mathcal{B} \mapsto M_K(Y)$ extends to a vector bundle $\mathcal{H}$ on $(X_K)_{\text{ét}}$.

The (a) means that globally,

$$\nu'^*\mathcal{H} \otimes_{\mathcal{O}_{X_K}} \mathcal{O}\mathbf{C}\big|_{(X_K)_{\text{ét}}} \simeq (\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})\big|_{(X_K)_{\text{ét}}}, \tag{6.2.8.3}$$

which is $\mathrm{Gal}(K/k) \ltimes \Gamma_{\mathrm{geom}}$-equivariant.

By (b), one can define operators $\log(\gamma_i)$ on $M_K(Y)$, which gives a Higgs field

$$\theta : M_K(Y) \to M_K(Y) \otimes_B \Omega^1_{B/k}(-1),\ x \mapsto \sum_i \log(\theta_i)(x) \otimes \mathrm{dlog} T_i \cdot t^{-1}.$$

Thus, we obtain a Higgs bundle $(\mathcal{H}, \theta)$ on $(X_K)_{\text{ét}}$.

Now, (P2) amounts to saying that $H^\bullet(X_{\text{proét}}/Y_K, \hat{\mathbf{L}} \otimes \widehat{\mathcal{O}}_X)$ can be calculated by the Higgs cohomology $(M_K(Y), \theta)$. This could be useful for some computation.

Everything becomes quite explicit now. After a little computation, we obtain:

6.2.8.4 - Lemma. *For $i \geq 0$, there is a canonical isomorphism*

$$H^i(\Gamma_{\mathrm{geom}}, (\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})(Y_{K,\infty})) \simeq \begin{cases} M_K(Y) & i = 0 \\ 0 & i > 0, \end{cases}$$

which is $\mathrm{Gal}(K/k)$*-equivariant.*

Therefore, one can identify $\mathcal{H}(\mathbf{L})(Y_K)$ with $M_K(Y) = \mathcal{H}(Y_K)$, hence $\mathcal{H}(\mathbf{L}) \simeq \mathcal{H}$.

Now we verify that the Higgs field θ constructed above is identified with $\nu'_*(\nabla) = \theta_{\mathbf{L}}$. Indeed, deriving the action of Γ_{geom} on $(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})(Y_{K,\infty})$, which is also unipotent, we find that $H^\bullet(X_{\text{proét}}/Y_K, \hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})$ is calculated by the Higgs cohomology of $((\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C})(Y_{K,\infty}), \Theta)$ with

$$\Theta = \theta \otimes 1 + 1 \otimes \sum_i \frac{\partial}{\partial V_i} \otimes \mathrm{dlog} T_i \cdot t^{-1} = \theta \otimes 1 - 1 \otimes \nabla.$$

Therefore, $\theta = \nabla$ on $\ker \Theta = M_K(Y)$.

Combining with (6.2.8.3), we have shown (i) and (ii) of the theorem.

6.2.9. The proof of (iii) is rather formal. For example, let $f : Z \to X$ be a morphism of smooth rigid analytic varieties over k and let $f_{\text{proét}}$ be the induced map of pro-étale sites; then we have a natural map

$$f^*\mathcal{H}(\mathbf{L}) = f^*\nu'_{X,*}(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}_X) \to \nu'_{Z,*} f^*_{\text{proét}}(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}_X) \simeq \nu'_{Z,*}(\widehat{f^*\mathbf{L}} \otimes \mathcal{O}\mathbf{C}_Z) = \mathcal{H}(f^*\mathbf{L}),$$

where $f^*_{\text{proét}}\mathcal{M} = f^{-1}_{\text{proét}}\mathcal{M} \otimes_{f^{-1}_{\text{proét}}\widehat{\mathcal{O}}_X} \widehat{\mathcal{O}}_Z$ for any $\widehat{\mathcal{O}}_X$-module $\mathcal{M}$ on $X_{\text{proét}}$, left adjoint to $f_{\text{proét},*}$. That this map is an isomorphism is verified locally and would follow from the two lemmas above.

6.2.10. Let's sketch a proof of (iv). Let $g : X \to \mathrm{Spa}(k, \mathcal{O}_k)$ be the structural smooth proper morphism and $\mathbf{L}$ be a $\mathbf{Z}_p$-local system on $X_{\text{ét}}$. Consider the following diagram

$$\begin{array}{ccc} (X_C)_{\text{proét}} & \xrightarrow{g_{\text{proét}}} & \mathrm{Spa}(C, \mathcal{O}_C)_{\text{proét}} \\ \downarrow{\scriptstyle \nu'_X} & & \downarrow{\scriptstyle \nu'_C} \\ (X_C)_{\text{ét}} & \xrightarrow{g_{\text{ét}}} & \mathrm{Spa}(C, \mathcal{O}_C)_{\text{ét}} \end{array}$$

First, we have a quasi-isomorphism by (i)

$$Rg_{\text{ét},*}(\mathcal{H}(\mathbf{L}) \otimes \Omega^\bullet_{X_C/C}, \theta_{\mathbf{L}}) \simeq Rg_{\text{ét},*}R\nu'_{X,*}(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}, \nabla) \simeq R\nu'_{C,*}Rg_{\text{proét},*}(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}, \nabla).$$

Using the explicit local description of $\mathcal{O}\mathbf{C}$ and ∇ in terms of polynomial algebras (6.2.3), we find that $(\hat{\mathbf{L}} \otimes \mathcal{O}\mathbf{C}, \nabla)$ is a resolution of $\hat{\mathbf{L}} \otimes \widehat{\mathcal{O}}_X$. So

$$H^\bullet_{\text{ét}}(\mathcal{H}(\mathbf{L}) \otimes \Omega^\bullet_{X_C/C}, \theta_{\mathbf{L}}) \simeq H^\bullet_{\text{proét}}(X_C, \hat{\mathbf{L}} \otimes \widehat{\mathcal{O}}_X).$$

By properness, one can apply the primitive comparison theorem to $\hat{\mathbf{L}}$ over X_C [Sch13a, Theorem 5.1], and a limit argument shows $H^i_{\text{ét}}(X_C, \mathbf{L}) \otimes C \simeq H^i_{\text{proét}}(X_C, \hat{\mathbf{L}} \otimes \widehat{\mathcal{O}}_X)$. So we conclude.

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