Relative Sen's theory and locally analytic vectors

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Sen's theory

- $K/\mathbf{Q}_p$ finite extension, $C = \hat{\bar{K}}$, and $K_{\mathrm{cyc}} = K(\mu_{p^\infty})$.

Theorem (Sen, 1980)

For any f.d. semi-linear C-representation V of $Gal(\bar{K}/K)$, we have

1. $V = C \otimes_{K_{cyc}} D_{Sen}(V)$ *where*

$$D_{Sen}(V) = V^{Gal(\bar{K}/K_{cyc})-inv,\ K-fin}.$$

2. *Sen's operator Θ_V given by the infinitesimal action of $Gal(K_{cyc}/K)$ on $D_{Sen}(V)$. And $\Theta_V \in C \otimes_{\mathbf{Q}_p} Lie(G)$ where G is the image of $Gal(\bar{K}/K) \to GL(V)$.*

- May replace K_{cyc} by other infinitely ramified Galois extensions over K with Galois group a 1-dim p-adic Lie group.
- Counterexample (for decompletion) in higher dimensional case.

Sen's theory

Colmez (2001), Berger-Colmez (2008): Tate-Sen's formalism.

Theorem (Berger-Colmez, 2016)

Let K_∞/K be a Galois extension with $G = Gal(K_\infty/K)$ a p-adic Lie group, which is infinitely ramified.

1. *For any f.d. semi-linear $\hat{K}_\infty$-representation W of G, we have $W = \hat{K}_\infty \otimes_{\hat{K}_\infty^{G-la}} W^{G-la}$;*
2. *$\hat{K}_\infty^{G-la}$ is annihilated by Sen's operator $\Theta_V \in C \otimes_{\mathbf{Q}_p} Lie(G)$ (associated with $C \otimes_{\mathbf{Q}_p} V$, where V is a faithful $\mathbf{Q}_p$-representation of G).*

Locally analytic vectors

- $f : U \to \mathbf{Q}_p$ is *locally analytic* if for any $x_0 \in U$, one may expand
$$f(x) = \sum_{\alpha \in \mathbf{N}^d} b_\alpha (x - x_0)^\alpha$$
for all x in some small disk $B(x_0, \varepsilon) \subset U$.
- Define p-adic (locally analytic) manifolds (modelled on $\mathbf{Z}_p^d$) and locally analytic functions on it.
- Let G be a p-adic Lie group, V be a $\mathbf{Q}_p$-Banach representation of G. We call $v \in V$ locally analytic if the function $G \to V,\ g \mapsto g \cdot v$ is locally analytic.

Locally analytic vectors

Let B be a $\mathbf{Q}_p$-Banach representation of G.

- $\mathrm{Lie}(G)$ acts naturally on $B^{G-\mathrm{la}}$.
- Fact: if B is finite dimensional over $\mathbf{Q}_p$, then $B^{G-\mathrm{la}} = B$. So K-finite vectors are G-locally analytic.
- If $G \simeq \mathbf{Z}_p$ and there exists m such that $(\gamma - \mathrm{Id})^m B^\circ \subset p^2 B^\circ$ for all $\gamma \in G$, then B is locally analytic.

Main theorem

Theorem (Lue Pan, 2020)

Let $Spa(A, A^+)$ be a small 1-dim smooth affinoid adic space over C, and $Spa(B, B^+) \to Spa(A, A^+)$ an affinoid perfectoid pro-étale Galois covering with Galois group G a p-adic Lie group. Then there exists $\theta \in B \otimes_{\mathbf{Q}_p} Lie(G)$, unique up to $A^\times$), which annihilates B^{G-la}.

Applicable to affinoid perfectoid modular curve $\mathcal{X}_{K^p}$ over $\mathcal{X}_{K^pK_p}$ (need log case); helps study the locally analytic sections of $\pi_{HT*}\mathcal{O}_{\mathcal{X}_{K^p}}$ over $\mathscr{F}\ell$ where $\pi_{HT} : \mathcal{X}_{K^p} \to \mathscr{F}\ell$ is the Hodge-Tate period map (Scholze).

Tate-Sen's formalism

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Tate-Sen's conditions (Colmez):

1. $B^+_{G_0,\infty}$ is almost étale over $B^+_{G_1,\infty}$ for open subgroups $G_0 \subset G_1$ of G (almost purity).
2. There are "trace maps" $\overline{\mathrm{tr}}_{G_0,n} : B_{G_0,\infty} \to B_{G_0,n},\ \ n \in \mathbf{N}$ such that
   - $\overline{\mathrm{tr}}_{G_0,n}$ is $B_{G_0,n}$-linear and fix $B_{G_0,n}$;
   - $\overline{\mathrm{tr}}_{G_0,n}(B^+_{G_0,\infty}) \subset p^{-\varepsilon_n} B^+_{G_0,n}$ with $\lim_n \varepsilon_n = 0$;
   - $\lim_n \overline{\mathrm{tr}}_{G_0,n}(x) = x$ for $x \in B_{G_0,n}$;
   - compatible with taking subgroups, conjugation and the action of G.
3. For any $n \in \mathbf{N}$, there is a sequence $\varepsilon_m \to 0$ such that if γ_n is a generator of $p^n\Gamma$,
   - $\gamma_n - 1$ is invertible on $X_{G_0,m} = \ker \overline{\mathrm{tr}}_{G_0,m} \subset B_{G_0,\infty}$;
   - $(\gamma_n - 1)^{-1}(X_{G_0,m}) \subset p^{-\varepsilon_m} X_{G_0,m}$.

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Tate-Sen's formalism

Proposition (Colmez, 2001, specialised to our case)

Fix $c < \frac{1}{2}$ inside $|C^\times|$. Let T be a $\mathbf{Z}_p$-representation of G free of rank d, and $G_0 \subset G$ an open subgroup acting trivially $\bmod p$. Then for sufficiently large n, there exists a unique $B^+_{G_0,n}$-submodule $D^+_{G_0,n}(T) \subset B^\circ_\infty \otimes_{\mathbf{Z}_p} T$ free of rank d satisfying

- $D^+_{G_0,n}(T)$ *is fixed by G_0 and stable under $G \times \Gamma$;*
- $B^\circ_\infty \otimes_{B^+_{G_0,n}} D_{G_0,n}(T) \to B^\circ_\infty \otimes_{\mathbf{Z}_p} T$ *is an isomorphism;*
- $D^+_{G_0,n}(T)$ *has a $B^+_{G_0,n}$-basis such that the matrices of Γ is trivial $\bmod p^c$.*

In particular, there exists m independent of T such that for all $\gamma \in \Gamma$, $(\gamma - 1)^m D^+_{G_0,n}(T) \subset p^2 D^+_{G_0,n}(T)$, so that $D^+_{G_0,n}(T)$ is Γ-locally analytic.

Tate-Sen's formalism

Corollary

For any f.d. $\mathbf{Q}_p$-representation V of G, there exists a unique

$$\phi_V : Lie(\Gamma) \to End_{B_\infty}(B_\infty \otimes_{\mathbf{Q}_p} V)$$

extending the natural action of $Lie(\Gamma)$ on $(B_\infty \otimes_{\mathbf{Q}_p} V)^{G-inv,\ \Gamma-la}$. Moreover,

1. *ϕ_V commutes with $G \times \Gamma$;*
2. *ϕ_V is functorial in V;*
3. *$\phi_{V \otimes W} = \phi_V \otimes Id + Id \otimes \phi_W$.*

Proof of corollary

- Fix $c < \frac{1}{2}$ inside $|C^\times|$. Choose a G-stable lattice $T \subset V$ and $G_0 \subset G$ an open normal subgroup acting trivially mod p on T.
- By proposition,
  - $B_\infty^\circ \otimes_{\mathbf{Z}_p} T \simeq B_\infty^\circ \otimes_{B^+_{G_0,n}} D^+_{G_0,n}(T)$, compatible with G_0, n;
  - $D_{G_0,n}(T)$ is Γ-locally analytic.
- Galois descent: $D_{G_0,n}(T) \simeq B_{G_0,n} \otimes_A D_{G,n}(T)$.
- $\mathrm{Lie}(\Gamma)$ acts linearly on

$$(\cup_{n' \geq n} B_{G_0,n'}) \otimes_{B_{G_0,n}} D_{G_0,n}(T) \simeq (\cup_{n' \geq n} B_{G_0,n'}) \otimes_{A_n} D_{G,n}(T)$$

thus on

$$\begin{aligned}(\cup_{n' \geq n} A_{n'}) \otimes_{A_n} D_{G,n}(V) &= (A_\infty \otimes_{A_n} D_{G,n}(V))^{\Gamma-\mathrm{la}} \\ &= (B_\infty \otimes_{A_n} D_{G,n}(V))^{G-\mathrm{inv},\ \Gamma-\mathrm{la}} \\ &= (B_\infty \otimes_{\mathbf{Q}_p} V)^{G-\mathrm{inv},\ \Gamma-\mathrm{la}}.\end{aligned}$$

Calculate $A_\infty^{\Gamma-\mathrm{la}}$:

- $\overline{\mathrm{tr}}_n : A_\infty \to A_n$ is continuous, so maps $A_\infty^{\Gamma-\mathrm{an}}$ into $A_n^{\Gamma-\mathrm{an}}$.
- $A_n^{\Gamma-\mathrm{an}} = A_n^{\Gamma-\mathrm{inv}} = A$.
- Letting $n \to +\infty$, we get $A_\infty^{\Gamma-\mathrm{an}} = A$.
- In the same way, we get $A_\infty^{p^m\Gamma-\mathrm{an}} = A_m$.

More on analytic functions

- There exists an open subgroup $G_0 < G$ and a continuous bijection

$$c : \mathbf{Z}_p^d \to G_0,\ (x_1, \ldots, x_d) \mapsto g_1^{x_1} \cdots g_d^{x_d}$$

such that $G_n := G_0^{p^n}$ is a subgroup and $c : p^n\mathbf{Z}_p^d \xrightarrow{\simeq} G_n$.
- For B a $\mathbf{Q}_p$-Banach representation of G, we may define $\mathcal{C}^{\mathrm{an}}(G_n, B)$ using the above homeomorphisms, samely $B^{G_n-\mathrm{an}}$.
- One has

$$\begin{aligned} B^{G_n-\mathrm{an}} &\simeq \mathcal{C}^{\mathrm{an}}(G_n, B)^{G_n},\ v \mapsto (f_v : g \mapsto g \cdot v). \\ &\simeq (B\hat{\otimes}_{\mathbf{Q}_p}\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p))^{G_n} \\ B^{G-\mathrm{la}} &= \cup_n B^{G_n-\mathrm{an}}. \end{aligned}$$

More on analytic functions

- Fact: the left and right translation actions of G_{n+1} on $\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)$ are trivial mod p.

- Fact: there exists finite dimensional $\mathbf{Q}_p$-subspaces $V_k,\ k \in \mathbf{N}$ of $\mathcal{C}^{\mathrm{an}}(G_n, \mathbf{Q}_p)^{\circ}$ stable under left and right translations such that $V_k V_l \subset V_{k+l}$ and $\cup_k V_k$ is a dense subspace.

Proposition

There exists a unique action

$$\phi_{G_0} : Lie(\Gamma) \to End_{B_\infty}(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{an}(G_0, \mathbf{Q}_p))$$

extending the natural one on $(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{an}(G_0, \mathbf{Q}_p))^{G_0-inv,\ \Gamma-la}$. *Moreover,*

1. ϕ_{G_0} *commutes with* Γ*;*
2. ϕ_{G_0} *commutes with the right translation action of* G_0 *on* $B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{an}(G_0, \mathbf{Q}_p)$*;*
3. ϕ_{G_0} *is a derivation:* $\forall \theta \in Im\ \phi_{G_0}$, $\theta(f_1 f_2) = \theta(f_1) f_2 + f_1 \theta(f_2)$.

Proof of proposition

- Fix $c < \frac{1}{2}$ inside $|C^\times|$. The G_0-stable lattices $V_k^\circ \subset V_k$ are such that and G_1 acts trivially mod p on V_k°, for all $k \in \mathbf{N}$.
- By Tate-Sen-Colmez,
  - $B_\infty^\circ \hat{\otimes}_{\mathbf{Z}_p} \mathcal{C}^{an}(G_0, \mathbf{Q}_p)^\circ \simeq B_\infty^\circ \hat{\otimes}_{B^+_{G_1,n}} D^+_{G_1,n}$, compatible with n;
  - $D_{G_1,n}$ is Γ-locally analytic.
- Galois descent: $D_{G_1,n} \simeq B_{G_1,n} \otimes_{B_{G_0,n}} D_{G_0,n}$.
- Lie(Γ) acts linearly on

$$(\cup_{n' \geq n} B_{G_0,n'}) \otimes_{B_{G_0,n}} D_{G_0,n} = (B_{G_0,\infty} \hat{\otimes}_{B_{G_0,n}} D_{G_0,n})^{\Gamma-\text{la}}$$
$$= (B_\infty \hat{\otimes}_{B_{G_0,n}} D_{G_0,n})^{G_0-\text{inv},\ \Gamma-\text{la}}$$
$$= (B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{an}(G_0, \mathbf{Q}_p))^{G_0-\text{inv},\ \Gamma-\text{la}}.$$

Existence

- ϕ_{G_0} extends uniquely the natural action of Lie(Γ) on

$$(B_\infty \hat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbf{Q}_p))^{G_0-\mathrm{inv},\ \Gamma-\mathrm{la}} = B_\infty^{G_0-\mathrm{an},\ \Gamma-\mathrm{la}}$$

  thus also the natural action on $B_\infty^{G-\mathrm{la},\ \Gamma-\mathrm{la}}$.
- Explicitly:

Uniqueness

- $f : \mathrm{Spa}(A, A^+) \to \mathbf{T}^1$ be a toric chart.
- $\Omega^1_{A/C} = A \cdot \mathrm{dlog}\, f^* T$ is free of rank 1.
- f' be another chart, then $\mathrm{dlog}\, f'^* T = a\, \mathrm{dlog}\, f^* T$.

Proposition

We have $\phi'_{G_0} = a^{-1}\phi_{G_0} : Lie(\Gamma) \mapsto B \otimes_{\mathbf{Q}_p} Lie(G)$. So the following map is independent of toric charts

$$\phi_{G_0} \otimes dlog\, f^* T : Lie(\Gamma) \mapsto \left(B \otimes_{Q_p} Lie(G)\right) \otimes_A \Omega^1_{A/C}.$$

Thank you!