Syntomic descent spectral sequence for p-adic rigid-analytic varieties

Yicheng Zhou*

9th April 2026

Abstract

For rigid-analytic varieties over a p-adic local field, we construct a natural syntomic descent spectral sequence which is compatible with the Hochschild-Serre spectral sequence. We deduce that the rigid-analytic étale regulator maps on the naive K-group of vector bundles factor through the geometric Selmer groups of Bloch-Kato when the rigid-analytic variety is proper.

Contents

*Email : yicheng.zhou@universite-paris-saclay.fr.

0 Introduction

We (re)examine the arithmetic syntomic cohomology for rigid-analytic varieties over p-adic local fields, define a rigid-analytic analogue of Soulé's étale regulators, and, in the proper case, study their images.

0.1 Main results

Let K be a p-adic local field of mixed characteristic $(0,p)$ with ring of integers $\mathcal{O}_K$, algebraic closure $\overline{K}$ and absolute Galois group $\mathscr{G}_K := \mathrm{Gal}(\overline{K}/K)$.

0.1.1. Motivation: algebraic setting. Recall that syntomic cohomology for proper and smooth schemes over $\mathcal{O}_K$ was introduced by Fontaine and Messing [34] in their proof of the crystalline-étale comparison theorem as a natural bridge between crystalline cohomology and étale cohomology. It was generalised later by Kato [46] to log-syntomic cohomology for semistable schemes over $\mathcal{O}_K$ (allowing horizontal divisors), and further extended by Nekovář and Nizioł [56] to a syntomic cohomology theory for arbitrary varieties over K or $\overline{K}$. Very roughly speaking, for $r \in \mathbf{N}$ and for schemes $\mathscr{X}$ semistable over $\mathcal{O}_K$, the r-th (log-)syntomic cohomology is, up to certain power of p, the filtered Frobenius eigenspace associated with the eigenvalue p^r in the absolute (log-)crystalline cohomology ; then it is rationalised and globalised to arbitrary varieties over K or $\overline{K}$ using alteration techniques. The syntomic cohomology could be thought of as a p-adic analogue of Deligne–Beilinson cohomology for complex manifolds X, which is defined as the cohomology of the Deligne complex

$$\mathbf{Z}(r)_{\mathscr{D}}\colon 0 \to \mathbf{Z}(r) \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{r-1}_X.$$

Indeed, since its introduction, log-syntomic cohomology has proved to be useful in the study of special values of p-adic L-functions and in formulating p-adic Beilinson conjectures.

Let us recall Nekovář and Nizioł's result [56, Theorem A].

Theorem (Nekovář–Nizioł). *For varieties X over K, there is a canonical graded commutative dg $\mathbf{Q}_p$-algebra $R\Gamma_{\mathrm{syn}}(X_h, *)$ satisfying the following properties:*

(i) *It is the unique extension of log-syntomic cohomology to varieties over K that satisfies h-descent.*

(ii) *It is a Bloch-Ogus cohomology theory*[1].

(iii) *For $X = \mathrm{Spec}\, K$, we have $H^*_{\mathrm{syn}}(X_h, r) \simeq H^*_g(\mathscr{G}_K, \mathbf{Q}_p(r))$, where $H^i_g(\mathscr{G}_K, -)$ denotes the* Ext*-group* $\mathrm{Ext}^i(\mathbf{Q}_p, -)$ *in the category of potentially semistable Galois $\mathbf{Q}_p$-representations of $\mathscr{G}_K$.*

(iv) *There are functorial syntomic-étale period maps*

$$\rho^{\mathrm{arith}}_{\mathrm{syn}}\colon R\Gamma_{\mathrm{syn}}(X_h, r) \to R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r)), \quad \rho^{\mathrm{geom}}_{\mathrm{syn}}\colon R\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r) \to R\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Q}_p(r))$$

compatible with product structures and inducing quasi-isomorphisms after taking the canonical truncation $\tau^{\leq r}$.

(v) *The Hochschild–Serre spectral sequence for étale cohomology*

$${}^{\mathrm{HS}}E_2^{i,j} = H^i_{\mathrm{cont}}(\mathscr{G}_K, H^j_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r))$$

has a syntomic analogue, the syntomic descent spectral sequence

$${}^{\mathrm{syn}}E_2^{i,j} = H^i_g(\mathscr{G}_K, H^j_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)).$$

(vi) *There is a canonical morphism of spectral sequences ${}^{\mathrm{syn}}E_t^{i,j} \Rightarrow {}^{\mathrm{\acute{e}t}}E_t^{i,j}$ compatible with the syntomic-étale period map.*

(vii) *There are syntomic Chern class maps*

$$c^{\mathrm{syn}}_{i,j}\colon K_j(X) \to H^{2i-j}_{\mathrm{syn}}(X_h, i)$$

[1] More precisely, there exists a canonical syntomic Borel-Moore homology $H^{\mathrm{syn}}_*(-,*)$ such that the pair of functors $(H^*_{\mathrm{syn}}(-,*), H^{\mathrm{syn}}_*(-,*))$ is a *twisted Poincaré duality theory with supports* in the sense of Bloch and Ogus [9, Definition (1.2), Definition (1.3)]; in particular, there are structures of cap product with supports (which is compatible with restriction map), projection formula, fundamental class and Poincaré duality.

compatible with Chern classes via the syntomic-étale period map.

For finite-dimensional $\mathbf{Q}_p$-representations V of $\mathscr{G}_K$, the extension groups $H^i_g(\mathscr{G}_K, V)$ that appear in (iii), so-called *geometric Selmer groups*, were introduced by Bloch and Kato [8] as part of the local tools for the Tamagawa Number Conjecture and the Tate–Shafarevich group of a motive. These extension groups play an important role in modern algebraic number theory, and one crucial fact is that there is a natural injection from $H^1_g(\mathscr{G}_K, V)$ into $H^1(\mathscr{G}_K, V)$, which is often strict. In light of (v) and (vi), syntomic cohomology groups can be viewed as a higher dimensional (or geometric) generalisation of these extension groups. They are used as an approximation of p-adic étale motivic cohomology (a refinement of p-adic étale cohomology capturing classes coming from geometry), and enter into the study of special values of p-adic L-functions, more precisely the p-adic regulators.

0.1.2. Rigid-analytic syntomic cohomology. For rigid-analytic varieties, which are deemed to be a suitable non-archimedean analogue of complex analytic spaces, the syntomic cohomology still serves as a useful tool for proving p-adic comparison theorems and even beyond them. We wish to extend the above theorem (0.1.1) to the rigid-analytic context.

Our principal object of interest has been well-defined: for smooth rigid-analytic varieties, the syntomic cohomology can be obtained by η-étale hyperdescent[2] from the (log-)syntomic cohomology of semistable models, as developped by Fontaine and Messing (and Kato) [21, 23]; the syntomic cohomology can be further defined for general (singular) rigid-analytic varieties by further éh-hyperdescent [11] thanks to the local smoothness of the éh-topology studied by Haoyang Guo [42]. The rigid-analytic syntomic cohomology was employed firstly by Colmez and Nizioł to prove a (potentially) semistable comparison theorem for smooth and proper semistable formal schemes over $\mathcal{O}_K$ (allowing horizontal divisors) [20], later generalised to the case of smooth proper rigid-analytic varieties [24], which has another proof by Bosco [11] using period sheaves and the Fargues–Fontaine curve. The syntomic method is important in the study of the Stein rigid-analytic varieties by Colmez, Dospinescu and Nizioł [17] in the semistable case, and continued by their other works [21, 23, 24] and by Bosco [10, 11].

0.1.3. Syntomic descent spectral sequence. What comes next in the rigid-analytic analogue of Theorem (0.1.1) are the syntomic-proétale period maps and a syntomic analogue of the Hochschild–Serre spectral sequence (see (3.1.3), (4.2.12), (3.1.8) *resp.* (3.2.22), (3.2.17), (3.3.7) for details).

Theorem (Syntomic descent spectral sequence). *Let X be a (quasi-separated, finite-dimensional and paracompact) proper or a smooth Stein rigid-analytic variety or a smooth dagger affinoid rigid-analytic variety over K. Let $r \in \mathbf{N}$.*

(i) *There are functorial syntomic-proétale period maps*

$$\rho^{\mathrm{arith}}_{\mathrm{syn}} \colon R\Gamma_{\mathrm{syn}}(X, r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$$

compatible with product structures and inducing quasi-isomorphisms after the canonical truncation $\tau^{\leq r}$.

(ii) *The Hochschild–Serre spectral sequence for proétale cohomology*

$${}^{\mathrm{HS}}E_2^{i,j} = H^i_{\mathrm{cont}}(\mathscr{G}_K, H^j_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$$

admits a natural map from the syntomic descent spectral sequence

$${}^{\mathrm{syn}}E_2^{i,j} \Rightarrow H^{i+j}_{\mathrm{syn}}(X, r)$$

compatible with the syntomic-proétale period maps.

(iii) *If X is proper over K, then the syntomic descent spectral sequence is identified with*

$${}^{\mathrm{syn}}E_2^{i,j} = H^i_g(\mathscr{G}_K, H^j_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{syn}}(X, r)$$

compatible with the natural maps $H^i_g(\mathscr{G}_K, -) \to H^i_{\mathrm{cont}}(\mathscr{G}_K, -)$.

Here, all the cohomology groups can be upgraded to condensed cohomology groups (see Section 1.3).

[2] This is the hyperdescent for the η-étale topology, whose covering families are those that become étale coverings on the generic fibres.

0.1.4 - Remark. We have restricted ourselves to the class of the rigid-analytic varieties that are proper or smooth Stein over K, as well as to smooth dagger affinoid varieties over K. This restriction relies essentially on our known instances of C_{st}-conjecture [24], which is to be applied to the base change from K to C of these analytic varieties (see Remarks (3.2.18) and (3.2.11)). If the C_{st}-conjecture were proved in a greater generality, for example, for ***small*** smooth dagger varieties over C, our theorem would likewise apply to the corresponding broader class of varieties, such as ***small*** smooth dagger varieties over K. Recall that a smooth dagger variety (over C, *resp.* over K) is said to be ***small*** [24, §1.2.3] if its de Rham cohomology groups are finite dimensional (over C, *resp.* over K). Small (smooth dagger) varieties include smooth proper varieties[3], quasi-compact smooth dagger varieties, analytifications of algebraic varieties (which are partially proper rigid-analytic varieties[3]), certain tubular neighbourhoods of subvarieties of proper varieties or complements of such tubular neighbourhoods. Currently, the only small varieties known to verify the C_{st}-conjecture are those with de Rham slopes ≥ 0 [24, Theorem 1.11].

0.1.5 - Remark. There is another collection of period maps, the *Fontaine–Messing period maps* α_r^{FM} [21], defined by globalising the Fontaine–Messing period maps on semistable models [34], which induce quasi-isomorphisms after the canonical truncation $\tau^{\leq r}$ [21, Corollary 7.3] as required by (i). However, ***a priori***, it is unclear whether the period maps α_r^{FM} fit for (ii) and eventually (iii). On the contrary, the arithmetic and geometric period maps ρ_{syn} that we will define in section 2.3 satisfy automatically the compatibility properties required by (ii) and (iii), although by construction, it is ***a priori*** unclear about their truncated quasi-isomorphisms. One of our key ingredients in combining the advantages of these two period maps is that there exist natural homotopies $\alpha_r^{\mathrm{FM}} \simeq \rho_{\mathrm{syn}}$ thanks to the uniqueness of geometric period morphisms [37].

0.1.6 - Remark. The reason for the properness condition in Theorem (0.1.3, iii) is twofold. Firstly, the groups $H^i_g(\mathscr{G}_K, -)$ were defined only for finite-dimensional $\mathbf{Q}_p$-representations of $\mathscr{G}_K$, and the proétale cohomology groups are only known to be finite-dimensional $\mathbf{Q}_p$-vector spaces for proper rigid-analytic varieties. It fails for Stein varieties in general, but it may not be the most important reason for which we require properness. More importantly, and likely related to the previous point, the non-proper case lacks the semistable comparison theorem. However, the proétale-to-de Rham comparison conjecture/theorem [24, §9] could help reformulate ${}^{\mathrm{syn}}E_2^{i,j}$ in terms of proétale cohomology for small varieties satisfying the C_{st}-conjecture (namely those ***with de Rham slopes*** ≥ 0 in the terminology of [24, §1.2.3]).

One possible logarithmic generalisation of this result could be allowing horizontal divisors, namely considering the Kummer proétale cohomology of proper log-smooth fs log-rigid spaces, whose foundation and finiteness were established by Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, Xinwen Zhu [29, Theorem 6.2.1]. An analogue [70] of Temkin's altered local uniformisation could be helpful. The author hopes to return in the future to these aspects beyond the proper case.

0.1.7. Bloch–Kato exponential. As a corollary of Theorem (0.1.3), we obtain a description of the Bloch–Kato exponential. Let X be as in Theorem (0.1.3). For this, recall that the arithmetic syntomic cohomology fits into a fibre sequence

$$R\Gamma_{\mathrm{syn}}(X,r) \to R\Gamma_{\mathrm{HK}}(X)^{\varphi=p^r, N=0} \xrightarrow{\iota_{\mathrm{HK}}^{\mathrm{arith}}} R\Gamma_{\mathrm{dR}}(X)/F^r,$$

where $\iota_{\mathrm{HK}}^{\mathrm{arith}}\colon R\Gamma_{\mathrm{HK}}(X) \to R\Gamma_{\mathrm{dR}}(X/K)$ denotes the rigid-analytic arithmetic Hyodo–Kato morphism. It yields boundary maps ∂ on cohomology groups.

Corollary. *Let X be as in Theorem* (0.1.3). *Let $i \in \mathbf{N}$. The composition*

$$H^i_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^{i+1}_{\mathrm{syn}}(X,r) \xrightarrow{\rho_{\mathrm{syn}}^{\mathrm{arith}}} H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to H^{i+1}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$$

is the zero map. The syntomic descent spectral sequence then induces a map

$$H^i_{\mathrm{dR}}(X)/F^r \to H^1_{\mathrm{cont}}(\mathscr{G}_K, H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))).$$

If X is proper over K, then this map is equal to the Bloch–Kato exponential map associated with the finite-dimensional [64] *$\mathbf{Q}_p$-representation $H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$ of $\mathscr{G}_K$.*

[3]When a dagger variety is partially proper, it can be equivalently viewed as a partially proper rigid-analytic variety: it admits a unique structure of rigid-analytic variety, which, conversely, determines its dagger structure.

To this end, for a general $X \in \mathrm{Rig}_K$, the boundary map ∂ that appears above can be considered as a higher dimensional analogue of Bloch–Kato exponential map.

0.1.8. Syntomic regulators. Dirichlet, followed by Dedekind, defined in the 19th century a regulator map (in fact, a logarithm map) from units in the ring integers $\mathcal{O}_E$ of an algebraic number field E to certain finite-dimensional real vector spaces; they showed that the image forms a lattice, whose covolume, also called a *regulator*, together with some other invariants of E are related to special values of ζ-functions (Dirichlet's class number formula), of which the regulator serves as the transcendental part.

The term *regulator* has since been used to denominate certain maps relating cycle class invariants, e.g., K-groups, Chow groups, to cohomological groups. One famous arhimedean example is the Beilinson regulator which takes values in Deligne–Beilinson cohomology groups. Intuitively speaking, the regulator plays the role of an abstract integration theory.

The syntomic cohomology has been considered as the p-adic analogue of Deligne–Beilinson cohomology (see for example [55]). Then, the syntomic regulator being regarded as an abstract p-adic integration, the Bloch–Kato exponential map above compares certain p-adic integrals to the values of the p-adic étale regulator.

The map of spectral sequences in Theorem (0.1.3, ii) allows us to study the image of p-adic (pro)étale regulator maps via syntomic regulators as follows.

0.1.9. Image of étale regulators: algebraic setting. Let us start with the algebraic setting. Let X be an algebraic variety over K. There are étale Chern class maps $c_i^{\text{ét}} \colon K_0(X) \to H^{2i}_{\text{ét}}(X, \mathbf{Q}_p(i))$ for $i \in \mathbf{N}$, where $K_0(X)$ denotes the Grothendieck group of the commutative monoid of all isomorphism classes of vector bundles on X with the monoid operation given by direct sum. It can be generalised to higher (connective) K-groups, so we have étale higher Chern class maps

$$c_{i,j}^{\text{ét}} \colon K_j(X) \to H^{2i-j}_{\text{ét}}(X, \mathbf{Q}_p(i))$$

for $i, j \in \mathbf{N}$. We may consider the subset of *homologically trivial elements of $K_j(X)$ along $c_{i,j}^{\text{ét}}$*, which is defined as

$$K_j(X)_0 := \ker(K_j(X) \xrightarrow{c_{i,j}^{\text{ét}}} H^{2i-j}_{\text{ét}}(X, \mathbf{Q}_p(i)) \to H^{2i-j}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))).$$

By the Hochschild–Serre spectral sequence, the map $c_{i,j}^{\text{ét}}$ induces Soulé's étale higher regulator map

$$r_{i,j}^{\text{ét}} \colon K_j(X)_0 \to H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))).$$

Nekovář and Nizioł proved the following result on the image of the regulators $r_{i,j}^{\text{ét}}$ [56, Theorem B], already known to Scholl, which generalised their own previous results with good or semi-stable reduction to arbitrary varieties over K.

Theorem (Scholl, Nekovář–Nizioł). *The regulator $r_{i,j}^{\text{ét}}$ factors through the subgroup*

$$H^1_g(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))) \subset H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))).$$

This follows directly from nice properties of syntomic cohomology (e.g., projective bundle formula and $\mathbf{A}^1$-homotopy invariance), their theorem on the syntomic descent spectral sequence (0.1.1) and compatibility of syntomic and étale Chern class maps with the syntomic-étale period maps.

0.1.10. Image of étale regulators: rigid-analytic setting. Similarly to Theorem (0.1.9), our Theorem on the syntomic descent sepctral sequence (0.1.3) should have a direct consequence concerning rigid-analytic étale regulators. Our goal is to establish a rigid analogue of Theorem (0.1.9). In the present article, we deduce only a basic case of it as follows (see (4.4.2) for details).

Theorem (Image of (pro)étale regulators). *Let X be as in Theorem* (0.1.3). *There are rigid-analytic* (*pro*)*étale regulators*

$$r_i^{\text{ét}} \colon K_0^{\text{naive}}(X)_0 \to H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-1}_{\text{proét}}(X_C, \mathbf{Q}_p(i))), \quad i \in \mathbf{N},$$

which factor through the map ${}^{\text{syn}}E_2^{1,2i-1} \to {}^{\text{HS}}E_2^{1,2i-1} = H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-1}_{\text{proét}}(X_C, \mathbf{Q}_p(i)))$. *In particular, if X is proper,* *then the regulators $r_i^{\text{ét}}$ factor through the geometric Selmer groups*

$$H^1_g(\mathscr{G}_K, H^{2i-1}_{\text{ét}}(X_C, \mathbf{Q}_p(i))) \subset H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-1}_{\text{ét}}(X_C, \mathbf{Q}_p(i))).$$

0.1.11 - Remark (Higher regulators). Let us make some comments on the higher syntomic regulators, the details of which is left to future work.

Presently there are two candidates for K-theory in the rigid-analytic setting: the (non-connective) analytic K-theory K^{an} à la Kerz–Saito–Tamme [47, 48] and the (non-connective) nuclear-continuous K-theory $\mathbf{K}^{\text{nuc}}_{\text{cont}}$ for analytic adic spaces à la Andreychev [2] (essentially equivalent to Morrow's continuous K-theory); the first is built from algebraic K-theory but "forcing" (pro-)$\mathbf{A}^{1,\text{an}}$-homotopy invariance, while the latter is built from the category of (derived) nuclear modules "just as" the algebraic K-theory is built from the category of perfect complexes. Both can be viewed as presheaves on Rig_K with values in $\text{Cond}^{\text{light}}(\text{Sp})$, the category of light condensed spectra, and there is a natural map of presheaves $\mathbf{K}^{\text{nuc}}_{\text{cont}} \to K^{\text{an}}$ of presheaves, inducing equivalence on qcqs varieties after $\mathbf{A}^1$-localising $\mathbf{K}^{\text{nuc}}_{\text{cont}}$.

In our future work, we are interested in generalising the above results to (higher Chern classes and) higher regulators. By the $\mathbf{A}^1$-homotopy invariance of étale cohomology, we may obtain rigid-analytic (pro)étale higher Chern class maps $c^{\text{ét}}_{i,j}$ (see [71, Appendix B] for its construction), whence étale higher regulators

$$r^{\text{ét}}_{i,j} \colon K^{\text{nuc}}_{\text{cont},j}(X)_0 \to H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{proét}}(X_C, \mathbf{Q}_p(i)))$$

for $i, j \in \mathbf{N}$. We hope that the latter factor through ${}^{\text{syn}}E_2^{1,2i-j-1} \to {}^{\text{HS}}E_2^{1,2i-j-1} = H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{proét}}(X_C, \mathbf{Q}_p(i)))$; in particular, if X is proper, then the regulators $r^{\text{ét}}_{i,j}$ would factor through the geometric Selmer groups

$$H^1_g(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_C, \mathbf{Q}_p(i))) \subset H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_C, \mathbf{Q}_p(i))).$$

One current obstacle is that, the preceding approach does not apply to the rigid-analytic syntomic cohomology, due to its failure of being $\mathbf{A}^1$-homotopy invariant. Nevertheless, it satisfies the projective bundle formula, and enjoys similar features with the motivic spectra of Annala–Iwasa [4]. It would be interesting to consider the framework of a rigid-analytic version of motivic spectra in order to construct reasonable syntomic higher Chern class maps.

0.2 Outline of proofs

0.2.1. Usage of condensed mathematics. To have a better control on p-adic cohomology theories, which have huge cohomology groups and are thus difficult to handle as plain groups, one needs to take their natural, if not canonical, topological structures into account. Our point of view will be upgrading objects and statements to the condensed world without much loss of information. The advantage of condensed mathematics that we make use of in this work lies essentially in the good homological algebraic properties of solid p-adic functional analysis, especially those of nuclear modules over $\mathbf{Q}_p$, such as behaviours of interactions between countable limits and solid tensor products.

We believe that many results could have been done in the more classical language of locally convex topological $\mathbf{Q}_p$-vector spaces or in other potentially adequate models of topological structures on algebras, but we are not going to pursue this point of view.

0.2.2. Consider first Theorem (0.1.3). As observed above in (0.1.5), it is easy to find a candidate, namely the Fontaine–Messing period maps α_r^{FM} satisfy the assertion (i), however it is not clear whether they fulfill (ii) and (iii); on the other hand, using (condensed) proétale period sheaves as in [11], one may construct Galois equivariant geometric period maps $\rho^{\text{geom}}_{\text{syn}}$, with the help of which the statements (ii) and (iii) become more accessible; the uniqueness proved by Sally Gilles [37] reunites these two sets of period maps.

Now let us focus on (ii) and (iii). Following the path of proof of Nekovář and Nizioł, we find that the key is the compatibility between arithmetic and geometric Hyodo–Kato morphisms, not only after taking derived fixed points and monodromy-trivial elements but also before it. In other words, we should have the following

Galois equivariant commutative diagram

(0.2.2.1)
$$\begin{tikzcd} R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{F} B^+_{\mathrm{st}} \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}} \otimes \iota_p"] \arrow[d] & R\Gamma_{\mathrm{dR}}(X/K) \otimes^{\blacksquare}_{K} B^+_{\mathrm{dR}} \arrow[d, "\simeq"] \\ R\Gamma_{\mathrm{HK}}(X_C) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(X_C/B^+_{\mathrm{dR}}) \end{tikzcd}$$

for $X \in \mathrm{Rig}_K$, where $R\Gamma_{\mathrm{inf}}(X_C/B^+_{\mathrm{dR}})$ denotes the condensed enrichment (1.3.8) of the infinitesimal cohomology of X_C over B^+_{dR} [41, Theorem 1.2.7, Definition 7.2.1], and where the vertical maps are the obvious ones, while $\iota_p \colon B^+_{\mathrm{st}} \to B^+_{\mathrm{dR}}$ is the canonical embedding given by $\log[p^\flat] \mapsto \log\left(\frac{[p^\flat]}{p}\right)$. Let us discuss the Hyodo–Kato morphisms appearing in the horizontal maps. Recall that Colmez and Nizioł defined globally for rigid-analytic varieties (as well as for dagger varieties) an arithmetic Hyodo–Kato morphism in [21] using a zig-zag construction, which we denote for the moment by $\iota^{\mathrm{arith,CN}}_{\mathrm{HK}}$, and defined a geometric Hyodo–Kato morphism in [23] using Beilinson's method, which we denote by $\iota^{\mathrm{geom,CN}}_{\mathrm{HK}}$.

It is unclear on the first sight whether these two Hyodo–Kato morphisms are compatible via the obvious maps; if it is the case, then our work should have been greatly shortened on this issue[4]; otherwise, we are at least able to construct by Galois descent a seemingly new arithmetic Hyodo–Kato morphisms $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ making the diagram (0.2.2.1) commutative and Galois equivariant. Indeed, this follows essentially from our construction.

It is then natural to ask whether there is a homotopy $\iota^{\mathrm{arith}}_{\mathrm{HK}} \simeq \iota^{\mathrm{arith,CN}}_{\mathrm{HK}}$. We prove this in the semistable reduction case, however, the homotopy that we construct does not seem to be natural (due to potential higher associativity issues), since it depends on the choice of uniformiser of a finite extension L of K. Despite this shortcoming, the syntomic cohomology defined by this new arithmetic Hyodo–Kato morphisms is naturally isomorphic to the usual one defined by Colmez and Nizioł in [21].

Once the Galois equvariant compatibility (0.2.2.1) is established, the (ii) and (iii), namely, the existence of the syntomic descent spectral sequence mapping naturally to the Hochschil-Serre spectral sequence and the identification of terms of its E_2-page in the proper case with Selmer groups, follow as in [56].

0.2.3. Now we turn to the second Theorem (0.1.10). We have proétale first Chern class maps for p-adic proétale cohomology, which induces, by the projective bundle formula, Chern classes for vector bundles. Similarly, we obtain a theory of Chern classes taking values in syntomic cohomology groups. These two different Chern classes are compatible via the period maps ρ_{syn}, and even better, compatible with the map from the syntomic descent spectral sequence to the Hochschild–Serre spectral sequence, from which Theorem (0.1.10) then follows immediately.

0.3 Organisation of the article

In Section 1, we are going to collect some preliminary results from higher topos theory and condensed mathematics, recall condensed group cohomology, which generalises Tate's continuous group cohomology, and finally discuss what condensed structures we will consider on typical p-adic cohomology theories. Readers familiar with condensed mathematics may skip this on their first reading.

In Section 2, we "redefine" the arithmetic Hyodo–Kato morphism for rigid-analytic varieties to be compatible with the geometric Hyodo–Kato morphisms defined in [23], which induces the same arithmetic syntomic cohomology as defined in [21]. We will also extend the construction to overconvergent situation by standard procedures.

In Section 3, we will define the arithmetic syntomic-proétale period map, compare it with the Fontaine–Messing period maps, and construct morphisms of spectral sequences under specific conditions, namely the C_{st}-conjecture.

In the last section (Section 4), we define and study the étale Chern class maps for rigid-analytic varieties. Along the way, we provide details of the projective bundle formula for syntomic and integral p-adic (pro)étale

[4]The compatibility holds for the overconvergent Hyodo–Kato morphisms, thanks to a motivic construction of the p-adic monodromy operator, more precisely based on [7, 12]. The proof idea has been communicated to the author by Alberto Vezzani after an earlier version of the present article has been written.

cohomology, and produce compatible higher Chern class maps by standard methods.

0.4 Acknowledgement

This work has been done during the author's PhD studies. He would like to express sincere gratitudes towards his advisor Wiesława Nizioł for her suggestion on this subject, helpful conversations and comments on earlier versions of this article. Ideas of this manuscript are greatly indebted to her joint work with Jan Nekovář, as well as her joint work with Pierre Colmez.

The author would like to thank Kęstutis Česnavičius et Alberto Vezzani for their precious remarks and insights on this work. Discussions with Christian Dahlhausen, Can Yaylali, Grigory Andreychev and Dustin Clausen around the analytic K-theory have also greatly benefited the author, and have especially enriched the appendix of an earlier version of this article.

During the preparation of this work, the author was supported by Fondation de l'École Polytechnique and Simons Collaboration on Perfection in Algebra, Geometry, and Topology, and has been supported by Fondation Mathématique Jacques Hadamard during the final stages of his writing.

1 Preliminaries

1.0.1. Notation and conventions. Fix a prime number p. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$, with the ring of integers $\mathcal{O}_K$ and perfect residue field k. Let $\overline{K}$ be an algebraic closure of K and $C := \widehat{\overline{K}}$ be its p-adic completion, which is still algebraically closed[5]. Let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K, which is a profinite group. Let $\mathcal{O}_F := W(k)$ be the ring of Witt vectors over k, and $F := \mathcal{O}_F[\frac{1}{p}]$, which is the maximal unramified subfield of K. Let F^{nr} be the maximal unramified extension of F inside $\overline{K}$ and $\breve{F}$ be its p-adic completion (so that $\breve{F} \subset C$).

Let L be any other complete nonarchimedean field of mixed characteristic $(0,p)$. We denote by k_L its residue field, by F_L its maximal unramified subfield with ring of integers $\mathcal{O}_{F_L}$. We have three log-structures: the trivial log-structure $\mathcal{O}_L^{\mathrm{triv}} = (\mathcal{O}_L, \mathcal{O}_L^\times)$, the canonical log-structure $\mathcal{O}_L^\times = (\mathcal{O}_L, \mathcal{O}_L\backslash\{0\})$ defined by the closed point of $\mathrm{Spec}(\mathcal{O}_L)$ (and given by the inclusion $\mathcal{O}_L\backslash\{0\} \subset \mathcal{O}_L$), and the fat (or hollow) log-structure $\mathcal{O}_{F_L}^0$ associated with the pre-log-structure $(\mathcal{O}_{F_L}, \mathcal{O}_L\backslash\{0\})$ sending $0 \neq a \in \mathcal{O}_L$ to $[\bar{a}] \in W(k_L)$. We have reductions $\mathcal{O}_{L,n} := \mathcal{O}_L/p^n$ for $n \geq 1$ and $\mathcal{O}_{L,0} := \mathcal{O}_L/\mathfrak{m}_L \simeq \mathcal{O}_{F_L,1}$, which could be decorated to designate the corresponding induced log-structures.

On the level of derived categories, we will principally work with ∞-categories in the sense of Lurie [51, 52, 53], which naturally capture higher homotopical structures and coherences. However, many computations can still be performed using ordinary categories, where classical techniques and tools apply.

1.0.2. Rigid spaces. Let L be a complete nonarchimdean field of mixed characteristic $(0,p)$. All the rigid-analytic varieties over L that we will consider are supposed to be quasi-separated, of finite dimension, and *paracompact*, i.e., admitting an admissible locally finite affinoid covering; in particular, they are admissible disjoint unions of *connected* paracompact rigid-analytic varieties *of countable type* (i.e., having a countable admissible affinoid covering) [27, Lemma 2.5.7]. When we regard them as adic spaces, we may use the term *rigid spaces*.

Let Rig_L (*resp.* RigSm_L) denote the category of rigid spaces (*resp.* smooth rigid spaces) over L.

1.1 Condensed mathematics

Topological structures on $\mathbf{Q}_p$-vector spaces will be used in many of our arguments, particularly in dealing with continuous group cohomology on the derived level. As is well-known, many categories of very interesting topological $\mathbf{Q}_p$-vector spaces lack certain nice properties of abelian categories, making it tricky to define derived functors on them. Nevertheless, Colmez–Dospinescu–Nizioł managed to employ certain *quasi-abelian*

[5] This follows as an application of [57, Krasner's Lemma (8.1.6)]; see [57, Proposition (10.3.2)] for a detailed argument.

category of topological spaces (see (1.1.7) to study p-adic cohomology of Stein spaces. On the other hand, Bosco was able to overcome this difficulty in his PhD thesis thanks to the convenient framework of condensed mathematics, especially its solid aspects[6], developed by Clausen and Scholze in [15, 14, 16]. Moreover, Clausen and Scholze have proposed a more recent ***light*** condensed theory, though not well documented, during their lectures on Analytic Stacks jointly held in IHES and Bonn in winter 2023/2024.

We are now going to recall some basics of the theory of condensed mathematics (both its original and light versions) and some solid p-adic functional analysis.

1.1.1. Condensed mathematics. Let us recall some definitions in condensed mathematics:

(i) Let $\mathcal{C}$ be an ∞-category that admits small limits. For any uncountable strong limit cardinal κ, one defines the *∞-category of κ-condensed objects* of $\mathcal{C}$ as the ∞-category of hypersheaves on the site of κ-condensed sets valued in $\mathcal{C}$, i.e.

$$\mathrm{Cond}_\kappa(\mathcal{C}) := \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{ProFin}_{<\kappa}, \mathcal{C}).$$

It is the same as the ∞-category of contravariant functors from κ-small extremally disconnected profinite sets to $\mathcal{C}$ that take finite coproducts to products, denoted by

$$\mathrm{Cond}_\kappa(\mathcal{C}) \simeq \mathrm{Fun}^\times(\mathrm{EDS}^{\mathrm{op}}_{<\kappa}, \mathcal{C}).$$

(ii) Light condensed theory, introduced by Dustin Clausen and Peter Scholze in their lectures on Analytic Stacks jointly held in IHES and Bonn in winter 2023/2024, are of interest to us, since most objects that concern us live in the light setting. A profinite set is called *light* if it can be written as a *countable* inverse limit of finite sets. One defines the *∞-category of light condensed objects* of $\mathcal{C}$ as

$$\mathrm{Cond}^{\mathrm{light}}(\mathcal{C}) := \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{ProFin}^{\mathrm{light}}, \mathcal{C}).$$

(iii) Both $\mathrm{Cond}_\kappa(\mathcal{C})$ and $\mathrm{Cond}^{\mathrm{light}}(\mathcal{C})$ are stable (*resp.* presentable) ∞-categories if $\mathcal{C}$ is stable (*resp.* presentable) [26, Lemma A.4 (*resp.* Lemma A.3)].

(iv) The κ-condensed and light condensed theories can be related via the adjunctions $L_\kappa \dashv \mathrm{Res}^{\mathrm{light}} \dashv R_\kappa$, where $L_\kappa\colon \mathrm{Fun}(\mathrm{ProFin}^{\mathrm{light}}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{ProFin}_{<\kappa}, \mathcal{C})$ is the left Kan extension $L_\kappa X(S) := \varinjlim_{S\to T\in \mathrm{ProFin}^{\mathrm{light}}} X(T)$, and $\mathrm{Res}^{\mathrm{light}}\colon \mathrm{Fun}(\mathrm{ProFin}_{<\kappa}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{ProFin}^{\mathrm{light}}, \mathcal{C})$ is the restriction of the functor to the subcategory $\mathrm{ProFin}^{\mathrm{light}} \subset \mathrm{ProFin}_{<\kappa}$, and $R_\kappa\colon \mathrm{Fun}(\mathrm{ProFin}^{\mathrm{light}}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{ProFin}_{<\kappa}, \mathcal{C})$ is the right Kan extension (or "sheafification") $R_\kappa X(S) := \lim_{\mathrm{ProFin}^{\mathrm{light}} \ni T \to S} X(T)$. The formula shows that $\mathrm{Res}^{\mathrm{light}} \circ L_\kappa \simeq \mathrm{id}$, so L_κ is fully faithful; and $\mathrm{Res}^{\mathrm{light}} \circ R_\kappa \simeq \mathrm{id}$, so R_κ is also fully faithful; Moreoever, both $\mathrm{Res}^{\mathrm{light}}$ and R_κ preserve sheaves, so they restrict to an adjunction $\mathrm{Res}^{\mathrm{light}} \dashv R_\kappa$ between $\mathrm{Cond}_\kappa(\mathcal{C})$ and $\mathrm{Cond}^{\mathrm{light}}(\mathcal{C})$.

As opposed to the κ-condensed case, colimits of light condensed objects may not be computed pointwise. Nevertheless, most statements about $\mathrm{Cond}_\kappa(\mathcal{C})$ can be transferred to the $\mathrm{Cond}^{\mathrm{light}}(\mathcal{C})$, sometimes requiring extra countability control on the index sets.

(v) There is a functor $\mathrm{const}\colon \mathcal{C} \to \mathrm{Cond}_\kappa(\mathcal{C})$, $X \mapsto \underline{X}$, which is defined as the composition of the constant functor $\mathcal{C} \to \mathrm{Fun}(\mathrm{ProFin}_{<\kappa}, \mathcal{C})$ and the hypersheafification. There is an adjunction $\mathrm{const} \dashv \mathrm{ev}_*$ where $\mathrm{ev}_*\colon \mathrm{Cond}_\kappa(\mathcal{C}) \to \mathcal{C}$, $X \mapsto X(*)$ is the functor taking underlying objects. Similarly in the light setting.

(vi) There is a comparison functor $\gamma_\kappa\colon \mathrm{pro}(\mathcal{C}) \to \mathrm{Cond}_\kappa(\mathcal{C})$, "$\lim$"$_{i\in I} X_i \mapsto \lim_{i\in I} \underline{X_i}$, which preserves small limits, similarly we have γ^{light} in the light setting; they are related by $\gamma^{\mathrm{light}} = \mathrm{Res}^{\mathrm{light}} \circ \gamma_\kappa$[7]. This could be thought of as "putting the discrete topology on objects of $\mathcal{C}$ and then the profinite topology on pro-objects". For our purposes, it will be indifferent to choose between the κ- and the light condensed settings. However, since the light condensed setting tends to have its own interest, we intend to keep these two frameworks in parallel throughout the remainder of this work. All statements will be true both in the κ- and the light condensed settings, unless otherwise mentioned.

[6]We should point out that it might not be substantial to use the condensed mathematics in our work. For example, we might consider the quasi-abelian category used in the works of Colmez–Dospinescu–Nizioł, and might consider only pro-spectra instead of condensed spectra in the K-theoretic part, but condensed mathematics will simplify greatly our treatment of these theories.

[7]However, it is not true in general that we have $L \circ \gamma^{\mathrm{light}} = \gamma_\kappa$, even when restricted to the subcategory of light profinite sets and with $\mathcal{C} = \mathrm{Ab}$.

(vii) The restriction functor to light pro-spectra $\gamma^{\text{light}}\colon \operatorname{pro}^{\text{light}}(\mathrm{Sp}^+) \to \mathrm{Cond}_\kappa(\mathrm{Sp})$ is conservative by Clausen-Scholze [26, Theorem A.12][8]. Here Sp^+ denotes the category of bounded above spectra.

1.1.2 - Remark. The statement of [26, Theorem A.12] (Clausen-Scholze) might not be fully correct; indeed, their proof actually proves that γ_κ and γ^{light} are conservative, but one could not deduce directly from that the conservativity of their functor $\gamma_\kappa^{\text{DY}} := R_\kappa \circ \gamma^{\text{light}}$, since it is not equal to the functor $\gamma_\kappa\colon \operatorname{pro}^{\text{light}}(\mathrm{Sp}^+) \to \mathrm{Cond}_\kappa(\mathrm{Sp})$, which defined as $``\lim_i" X_i \mapsto \lim_i \underline{X_i}$.

Moreover, we record a contribution of the author to the proof of [26, Theorem A.12]. The original version of the proof (cf. version 1 of *op. cit.*) established only the conservativity of γ_κ. Nonetheless, their argument applies *verbatim* in the light condensed setting, thanks to the following lemma, which we provide as an analogue of [26, Lemma A.15] adapted to the light condensed context:

1.1.2.1 - Lemma. *For a tower $(M_n)_{n\in\mathbf{N}}$ of abelian groups, the following are equivalent:*

(i) *The tower is Mittag–Leffler and* $\lim_n M_n = 0$.

(ii) *The tower is pro-zero, i.e.* $``\lim"_n M_n = 0$ *in* $\operatorname{pro}(\mathrm{Ab})$.

(iii) *We have* $\lim_n \underline{M_n} = R^1\lim_n \underline{M_n} = 0$ *in* $\mathrm{Cond}^{\text{light}}(\mathrm{Ab})$.

Proof. The proof of (i) ⇒ (ii) is the same as that of [26, Lemma A.15].

The proof of (iii) ⇒ (i) is also the same as in [26, Lemma A.15], except that we have to establish the isomorphisms of abelian groups

$$(\clubsuit)\qquad \lim_n \Gamma(S,\underline{M_n}) \simeq \Gamma(S,\lim_n \underline{M_n}),\qquad R^1\lim_n \Gamma(S,\underline{M_n}) \simeq \Gamma(S,R^1\lim_n \underline{M_n})$$

for *the* light profinite set $S := \mathbf{N}\cup\{\infty\}$. For this, recalling that for this special profinite set S, the object $\mathbf{Z}[S] \in \mathrm{Cond}^{\text{light}}(\mathrm{Ab})$ is an (internally) projective object, so that

$$R\Gamma(S,\underline{M_n}) \simeq \Gamma(S,\underline{M_n})[0].$$

Using the isomorphisms $R\lim_n R\Gamma(-,\underline{M_n}) \simeq R\Gamma(-,R\lim_n \underline{M_n})$ on $\mathrm{ProFin}^{\text{light}}$ for general countable projective systems $\{M_n\}_{n\in\mathbf{N}}$ in $\mathrm{Cond}^{\text{light}}(\mathrm{Ab})$, we obtain

$$R\lim_n(\Gamma(S,\underline{M_n})[0]) \simeq R\lim_n R\Gamma(S,\underline{M_n}) \simeq R\Gamma(S,R\lim_n \underline{M_n}) \simeq \Gamma(S,R\lim_n \underline{M_n}).$$

Taking cohomology groups, using again the projectivity of $\mathbf{Z}[S]$, one finally obtains (♣).

As for the proof of (ii) ⇒ (iii), we apply [26, Lemma A.15] to obtain that $R\lim_n \underline{M_n} = 0$ in $\mathrm{Cond}_\kappa(\mathscr{D}(\mathrm{Ab}))$; but the functor $\mathrm{Res}^{\text{light}}\colon \mathrm{Cond}_\kappa(\mathscr{D}(\mathrm{Ab})) \to \mathrm{Cond}^{\text{light}}(\mathscr{D}(\mathrm{Ab}))$ preserves limits (cf. [26, Remark A.7]) and commutes with the functor $M \mapsto \underline{M}$; thus, $R\lim_n \underline{M_n} = 0$ remains true in $\mathrm{Cond}^{\text{light}}(\mathscr{D}(\mathrm{Ab}))$. □

Let Spc be the ∞-category of spaces and Sp be the ∞-category of spectra.

1.1.3 - Lemma. *Let $R \in \mathrm{Cond}_\kappa(\mathrm{CAlg})_{\geq 0}$*[9] *be a connective condensed ring spectrum, and let* $\mathrm{Mod}_R^{\mathrm{cond},\kappa} := \mathrm{Mod}_R(\mathrm{Cond}_\kappa(\mathrm{Sp}))$ *denote the symmetric monoidal ∞-category of R-module objects of* $\mathrm{Cond}_\kappa(\mathrm{Sp})$.

(i) *The forgetful functor* $\mathrm{Mod}_R^{\mathrm{cond},\kappa} \to \mathrm{Cond}_\kappa(\mathrm{Sp})$ *is conservative and preserves (small) limits and colimits.*

(ii) *The t-structure on* $\mathrm{Cond}_\kappa(\mathrm{Sp})$ *induces a t-structure* $(\mathrm{Mod}_{R,\geq 0}^{\mathrm{cond},\kappa}, \mathrm{Mod}_{R,\leq 0}^{\mathrm{cond},\kappa})$ *on* $\mathrm{Mod}_R^{\mathrm{cond},\kappa}$, *where* $\mathrm{Mod}_{R,\geq 0}^{\mathrm{cond},\kappa}$ (resp. $\mathrm{Mod}_{R,\leq 0}^{\mathrm{cond},\kappa}$) *is the inverse image of* $\mathrm{Shv}(\mathcal{C},\mathrm{Sp})_{\geq 0}$ (resp. *of* $\mathrm{Shv}(\mathcal{C},\mathrm{Sp})_{\leq 0}$) *under the forgetful functor* $\mathrm{Mod}_R^{\mathrm{cond},\kappa} \to \mathrm{Cond}_\kappa(\mathrm{Sp})$. *This forgetful functor is t-exact. The subcategories* $\mathrm{Mod}_{R,\geq 0}^{\mathrm{cond},\kappa}$ *and* $\mathrm{Mod}_{R,\leq 0}^{\mathrm{cond},\kappa}$ *of* $\mathrm{Mod}_R^{\mathrm{cond},\kappa}$ *are stable under (small) filtered colimits.*

(iii) *If R is discrete, then the functor π_0 induces an equivalence*

$$\mathrm{Mod}_R^{\mathrm{cond},\kappa,\heartsuit} \xrightarrow{\simeq} \mathrm{Mod}_R(\mathrm{Cond}_\kappa(\mathrm{Set})),$$

where the latter is (the nerve of) the ordinary category of (discrete) condensed R-modules.

[8] The proof *loc. cit.* is not correct. Please refer to (1.1.2) for one possible correct proof.

[9] Cohomological indices are denoted by superscripts, while homological indices are denoted by subscripts. They could be quite confusing when we mix homotopy theories with cohomological notation of derived categories.

(iv) *If the condensed ring spectrum R is discrete, then there is a canonical colimit-preserving and t-exact equivalences of ∞-categories*

$$\iota\colon \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond},\kappa,\heartsuit}) \xrightarrow{\simeq} \mathrm{Mod}_R^{\mathrm{cond},\kappa}.$$

The same holds for the light condensed setting.

Proof. This follows from applying [53, Proposition 2.1.0.3 (iii), Proposition 2.1.1.1, Remark 2.1.2.1, Corollary 2.1.2.3] (see [71, Lemma B.1.7]) to the ∞-topos $\mathcal{X} = \mathrm{Cond}_\kappa(\mathrm{Spc})$, which is hypercomplete by definition. The same works in the light condensed setting. □

1.1.4. Solid mathematics. It will be particularly useful to employ "complete" objects, under the name of *solid* objects [15, Lectures V–VIII].

(i) There is a full subcategory Solid $\subset$ CondAb consisting of *solid abelian groups*, which is a localisation with left adjoint $(-)^{\blacksquare}$ the solidification functor. It is stable under all limits and colimits, equipped with a tensor product $-\otimes^{\blacksquare}- := (-\otimes-)^{\blacksquare}$, compactly generated under colimits by $\mathbf{Z}[S]^{\blacksquare}$ for κ-small extremally disconnected sets S, hence compactly generated under colimits by their retracts $\prod_I \mathbf{Z}$ (in particular, $\mathbf{Z}[S]^{\blacksquare}$, isomorphic to a product of $\mathbf{Z}$, is a compact projective for any profinite set S). For any $M \in$ CondAb and $N \in$ Solid, we have $\mathrm{Ext}^i_{\mathrm{CondAb}}(M,N) \in$ Solid for all $i \in \mathbf{Z}$.

(ii) There is also a derived version of solid theory: there is a notion of *solid objects* in $\mathscr{D}(\mathrm{CondAb})$, which form a full subcategory stable under all limits and colimits. The natural functor $\mathscr{D}(\mathrm{Solid}) \to \mathscr{D}(\mathrm{CondAb})$ is fully faithful, whose essential image consists precisely of the solid objects of $\mathscr{D}(\mathrm{CondAb})$; this functor has a left adjoint $(-)^{L\blacksquare}$, called the derived solidification functor. There is a unique way to endow $\mathscr{D}(\mathrm{Solid})$ with a symmetric monoidal tensor product $-\otimes^{L\blacksquare}-$ such that the derived solidification functor is symmetric monoidal. The derived solidification functor and the derived solid tensor product coincide with the derived functors of the corresponding functors in (i). Moreover, we have $\mathbf{Z}[S]^{L\blacksquare} \simeq \mathbf{Z}[S]^{\blacksquare}[0]$ by [15, Proposition 5.6].

(iii) For any (noncommutative) condensed ring R (e.g., a commutative ring, a group ring), we denote by $\mathrm{Mod}_R^{\mathrm{cond}} \subset$ CondAb the full subcategory of R-modules, and $\mathrm{Mod}_R^{\blacksquare} := \mathrm{Mod}_R^{\mathrm{cond}} \cap$ Solid the category of R-modules that are solid abelian groups, which are the same as solid $R^{\blacksquare}$-modules. Similarly as Solid, the category $\mathrm{Mod}_R^{\blacksquare}$ carries a tensor product $-\otimes_R^{\blacksquare}- := (-\otimes_R-)^{\blacksquare}$, and is compactly generated under colimits by $(\prod_I \mathbf{Z})\otimes_{\mathbf{Z}}^{\blacksquare} R$.

(iv) For example, there is a canonical isomorphism $(\prod_I \mathbf{Z})\otimes_{\mathbf{Z}}^{L\blacksquare} \underline{M} \xrightarrow{\simeq} \prod_I \underline{M}$ for any profinite abelian group M [10, Lemma A.19]. In particular, when the residue field of K is finite[10], we obtain $(\prod_I \mathbf{Z})\otimes_{\mathbf{Z}}^{L\blacksquare} \underline{\mathcal{O}_K} \xrightarrow{\simeq} \prod_I \underline{\mathcal{O}_K}$ and then $(\prod_I \mathbf{Z})\otimes_{\mathbf{Z}}^{L\blacksquare} \underline{K} \xrightarrow{\simeq} (\prod_I \underline{\mathcal{O}_K})[\frac{1}{p}]$.

(v) When the situation is clear, for example for usual topological abelian groups M such as $\mathbf{Z}$, $\mathcal{O}_K$, K, C, etc., we will not distinguish the notations $\underline{M}$ and M, unless we want to stress some results in the classical topological setting.

1.1.5. Let $\mathcal{C}$ be a site. We record two consequences of [53, Corollary 2.1.2.3]:

(i) For any ordinary ring R, there is an equivalence of ∞-categories

$$\mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}}) \xrightarrow{\simeq} \mathrm{Shv}^{\mathrm{hyp}}(*_{\mathrm{pro\acute{e}t}}, \mathscr{D}(\mathrm{Mod}_R))$$

sending M to $S \mapsto R\Gamma(S,M)$ on profinite sets S, which simplifies to $M(S)$ on extremally disconnected sets S.

(ii) For any condensed ring R, we have equivalences of ∞-categories

$$\mathscr{D}(\mathrm{Shv}(\mathcal{C},\mathrm{Mod}_R^{\mathrm{cond}})) \xrightarrow{\simeq} \mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C},\mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})), \quad \mathscr{D}(\mathrm{Shv}(\mathcal{C},\mathrm{Mod}_R^{\blacksquare})) \xrightarrow{\simeq} \mathrm{Shv}^{\mathrm{hyp}}(\mathcal{C},\mathscr{D}(\mathrm{Mod}_R^{\blacksquare}))$$

induced by sending a sheaf $F \in \mathrm{Shv}(\mathcal{C},\mathrm{Mod}_R^{\mathrm{cond}})$ to its global section functor $R\Gamma(-,F)$.

1.1.6. Convention. For simplicity, we will denote by $\mathrm{Hom}(-,-)$ the internal Hom (bi)functor $\mathrm{Hom}_{\mathrm{Cond(Set)}}(-,-)$ or $\mathrm{Hom}_{\mathrm{CondAb}}(-,-)$ depending on the context, and also denote $\mathrm{Hom}_R(-,-) :=$

[10] This is the situation in [10, Appendix A], see *loc. cit.*, Lemma A.17. Special thanks to Alberto Vezzani for pointing this out.

$\mathrm{Hom}_{\mathrm{Mod}_R^{\mathrm{cond}}}(-,-)$ for any condensed ring R; similarly for the derived internal Hom. However, the internal Hom in categories of solid modules will keep the full notation $\mathrm{Hom}_{\mathrm{Mod}_R^{\blacksquare}}(-,-)$.

1.1.7. Topological structures on p-adic cohomology theories: a review. We may understand the p-adic proétale cohomology of rigid-analytic varieties over K or C by comparisons with other (integral or rational) p-adic cohomology theories such as Hyodo–Kato cohomology and de Rham cohomology. However, in order to control maps between their cohomology groups, which are typically large (with no hope of finite-dimensionality, except in the dagger qcqs case), it would be useful to endow these groups with a suitable topological structure.

Firstly, Colmez, Nizioł and Dospinescu [17] have considered the category C_K of locally convex topological K-vector spaces, which is a quasi-abelian category[11] by [62, Proposition 2.1.11]. Its left bounded derived ∞-category $\mathscr{D}(C_K)$ admits a t-structure $(\mathscr{D}^{\le 0}, \mathscr{D}^{\ge 0})$, called the *left t-structure*, where $\mathscr{D}^{\le 0}$ (*resp.* $\mathscr{D}^{\ge 0}$) is the full subcategory of $\mathscr{D}(C_K)$ formed by the complexes being strictly exact in each positive (*resp.* negative) degree. The corresponding heart $LH(C_K)$ (the so-called *left heart*) consists of complexes represented (up to equivalence) by a monomorphism $f : E \to F$, where F sits in degree 0. The cohomology groups of an object $X \in \mathscr{D}(C_K)$ are given by

$$\widetilde{H}^n(X) := \tau^{\le n}\tau^{\ge n}(X) = (\mathrm{coim}\, d^{n-1} \to \ker d^n) \in LH(C_K),$$

while there are also naive cohomology groups

$$H^n(X) := (\ker d^n/\mathrm{coim}\, d^{n-1}) \in C_K$$

endowed with the quotient topology. An object $(E \to F) \in LH(C_K)$ is called *classical* if the natural morphism $(E \to F) \to F/E = H^0(E \to F)$ is an equivalence.

More recently, in his PhD thesis [10, 11], Guido Bosco has drawn on condensed mathematics to investigate the rational p-adic Hodge theory, which is related to the previous perspective by the *condensification* functor [19, Section 4]

$$C_K^{\mathrm{Hcg}} \to \mathrm{Mod}_K^{\mathrm{cond}}, \quad V \mapsto \underline{V} \tag{1.1.7.1}$$

as well as its natural extension to derived categories, where $C_K^{\mathrm{Hcg}} \subset C_K$ denotes the full subcategory of spaces that are Hausdorff and compactly generated, and $\mathrm{Mod}_K^{\mathrm{cond}}$ the category of (underived) condensed $\underline{K}$-modules.

The condensification functor sends strict exact sequences of K-Fréchet spaces (*resp.* of spaces of compact type) to exact sequences in $\mathrm{Mod}_K^{\blacksquare}$ [19, Lemma 2.18]. For example, this applies to rational p-adic cohomology theories such as (pro)étale cohomology, de Rham cohomology, Hyodo–Kato cohomology; indeed, $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(i))$ is represented by a complex of $\mathbf{Q}_p$-Banach spaces if $X \in \mathrm{Rig}_K^{\mathrm{qcqs}}$ [17, §3.3.2], and is represented by a complex of $\mathbf{Q}_p$-vector spaces of compact type if $X \in \mathrm{Rig}_K^{\dagger,\mathrm{qcqs}}$ [19, Proposition 4.23]; similarly for varieties over C instead of over K, likewise for de Rham cohomology and Hyodo–Kato cohomology.

Moreover, for K-Fréchet spaces, the condensification functor transforms projective tensor products in C_K to solid tensor products [10, Proposition A.68].

1.1.8. Solid p-adic functional analysis. It seems that the condensed, especially solid, mathematics has better homological behaviour than the classical p-adic functional analysis. For this reason, we will always stick to the condensed point of view unless arguments demand intervention of the classical one.

We gather the following nice features of solid p-adic functional analysis.

(i) There is a particular class of *nuclear K-vector spaces* inside $\mathrm{Mod}_K^{\mathrm{cond}}$ [10, §A.6] (see [16, Lecture VIII] for a more abstract framework). To avoid conceptual confusion with the classical nuclear K-vector spaces à la Grothendieck [39], we shall call the former *solid-nuclear K-vector spaces.* The classcial nuclear K-vector spaces are somewhat orthogonal to the concept of K-Banach spaces, as their common objects are finite-dimensional. On the contrary, any K-Banach space is solid-nuclear.

(ii) Let $\mathrm{Mod}_K^{\mathrm{nuc}} \subset \mathrm{Mod}_K^{\blacksquare}$ denote the full subcategory of solid-nuclear K-vector spaces. It is stable under finite limits, countable products (hence countable limits), all colimits, and the solid tensor product [10,

[11] We refer to [62, §1] for a review of quasi-abelian homological algebra. Let $\mathcal{A}$ be an additive category with kernels and cokernels. A morphism $f \in \mathcal{A}$ is *strict* if its coimage and image coincide canonically. We call $\mathcal{A}$ a *quasi-abelian category* if every pullback of a strict epimorphism is a strict epimorphism and every pushout of a strict monomorphism is a strict monomorphism.

Theorem A.43]. It contains all K-Banach spaces, and is generated under colimits by these, which are flat objects for the solid tensor product [10, Corollary A.61]. It contains all K-Fréchet spaces, which can be written as filtered colimits of K-Banach spaces [10, Proposition A.64].

Furthermore, on the derived level, one has a full stable ∞-subcategory of solid-nuclear complexes $\mathrm{Nuc}_K \subset \mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$. This subcategory is closed under taking direct summands, derived tensor product $\otimes_K^{L\blacksquare}$, finite limits, countable products and all colimits; it is generated under shifts and colimits by the objects $\underline{\mathrm{Hom}}_K(K[S], K)$ for varying profinite sets S. Moreover, any object $M \in \mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$ lies in Nuc_K if and only if $H^i(M)[0] \in \mathrm{Nuc}_K$, or equivalently $H^i(M) \in \mathrm{Mod}_K^{\mathrm{nuc}}$. See [11, Theorem A.17] for a detailed account.

(iii) Let $(V_n)_{n\in\mathbf{N}}$ be a countable projective system of solid-nuclear K-vector spaces, and let W be a K-Fréchet space; then by [10, Corollary A.67 (i)], we have

$$\varprojlim_n (V_n \otimes_K^{\blacksquare} W) \simeq (\varprojlim_n V_n) \otimes_K^{\blacksquare} W.$$

This can be refined into the following lemma (1.1.8.1). Besides, as a corollary [10, Corollary A.67 (ii)], if $(V_n)_{n\in\mathbf{N}}$ is a countable projective system of objects in $\mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$ such that each V_n is represented by a complex of solid-nuclear K-vector spaces, and if $W \in \mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$ is represented by a bounded above complex of K-Fréchet spaces, then

$$R\varprojlim_n (V_n \otimes_K^{L\blacksquare} W) \simeq (R\varprojlim_n V_n) \otimes_K^{L\blacksquare} W.$$

1.1.8.1 - Lemma. *Let $(V_n)_{n\in\mathbf{N}}$ be a countable projective system in $\mathrm{Mod}_K^{\blacksquare}$. Consider the full subcategory $\mathcal{C} \subset \mathrm{Mod}_K^{\blacksquare}$ consisting of W such that*

$$\varprojlim_n (V_n \otimes_K^{\blacksquare} W) \simeq (\varprojlim_n V_n) \otimes_K^{\blacksquare} W.$$

(i) *$\mathcal{C}$ is closed under finite colimits and direct summands.*

(ii) *If V_n and $\varprojlim_n V_n$ are all flat for $\otimes_K^{\blacksquare}$, then $\mathcal{C}$ is closed under finite limits.*

(iii) *If V_n are all solid-nuclear objects of $\mathrm{Mod}_K^{\blacksquare}$, then $\mathcal{C}$ contains all K-Fréchet spaces.*

Proof. (i) For taking direct summands, we remark that the vanishing of kernel and cokernel of the natural map between two sides is stable under taking direct summands of W. As for finite colimits, this is because $\otimes_K^{\blacksquare}$ commutes with all colimits in each variable and $\varprojlim_n$ commutes with finite colimits.

(ii) By flatness, $\otimes_K^{\blacksquare}$ preserves finite limits; and $\varprojlim_n$ commutes with all limits.

(iii) This is [10, Corollary A.67]. □

1.1.8.2 - Example. Here is an elementary example of Lemma (1.1.8.1). Let $V_1 \to V_2$ be a morphism between two K-Banach spaces with cokernel N in $\mathrm{Mod}_K^{\blacksquare}$. Let I be a countable set. Let $K\langle I\rangle$ denote the K-Banach space generated by I.

(i) The natural map $N \otimes_K^{\blacksquare} K\langle I\rangle \to \prod_I N$ is injective.

(ii) The limit $\varprojlim_J N \otimes_K^{\blacksquare} K\langle J\rangle$ vanishes, where J runs over all cofinite subsets of I.

Proof. (i) We have an injection $N \otimes_L^{\blacksquare} L\langle I\rangle \hookrightarrow N \otimes_L^{\blacksquare} \prod_I L$ by the injectivity of $L\langle I\rangle \to \prod_I L$ and the flatness of N for $\otimes_L^{\blacksquare}$ [10, Corollary A.61]. Consider the following commutative diagram

$$\begin{tikzcd}
V_1 \otimes_L^{\blacksquare} \prod_I L \arrow[r] \arrow[d, "i_1"', "\simeq"] & V_2 \otimes_L^{\blacksquare} \prod_I L \arrow[r] \arrow[d, "i_2"', "\simeq"] & N \otimes_L^{\blacksquare} \prod_I L \arrow[r] \arrow[d, "i"'] & 0 \\
\prod_I V_1 \arrow[r] & \prod_I V_2 \arrow[r] & \prod_I N \arrow[r] & 0
\end{tikzcd}$$

where the maps i_1 and i_2 are isomorphisms by [10, Corollary A.59] since I is countable, the first row is exact by flatness of $\prod_I L$ for $-\otimes_L^{\blacksquare}$ [10, Lemma A.58], the second row is exact by exactness of countable products (AB4*) [15, Theorem 2.2]. From this, one deduces that i is also an isomorphism. Thus, (i) follows immediately.

(ii) Consider the short exact sequence of projective systems indexed by cofinite subsets $J \subset I$

$$0 \to N \otimes_L^{\blacksquare} L\langle J \rangle \to N \otimes_L^{\blacksquare} L\langle I \rangle \to N \otimes_L^{\blacksquare} L\langle I \backslash J \rangle \to 0$$

where again we are using flatness of N. Using left exactness of limit, we obtain a left exact sequence

$$0 \to \varprojlim_J (N \otimes_L^{\blacksquare} L\langle J \rangle) \to N \otimes_L^{\blacksquare} L\langle I \rangle \to \varprojlim_J (N \otimes_L^{\blacksquare} L\langle I \backslash J \rangle)$$

whose rightmost term is identified with $\prod_I N$. Then (ii) follows from (i). □

1.2 Condensed group actions

We benefit from nice categorical properties of condensed modules, at the cost of our objects becoming more complicated since we have to keep track of their topology by evaluating them on *all* profinite sets rather than only on the singleton. Therefore, the treatment of condensed group actions and their cohomology, which substitutes for continuous group actions and continuous group cohomology, requires extra care. We shall recall the condensed theory of group cohomology and some computable examples, as well as the counterpart concerning smooth group actions.

1.2.1. Condensed group algebra. Let G be a condensed group. Let ρ be a G-action on a condensed ring R. We define the R-algebra $R[G]_\rho$, called the *skew group algebra of G over R*[12], to be the $R[G] \in \mathrm{CondAb}$ with the multpilication law $[g] \cdot [g'] = [gg']$ and $[g] \cdot x = g(x) \cdot [g]$ for $g \in G$ and $x \in R$. We define the category of R-modules with semilinear G-action by $\mathrm{Mod}^{\mathrm{cond}}_{R[G]_\rho}$, and the derived ∞-category of R-modules with semilinear G-action by $\mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R)^G := \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{R[G]_\rho})$; similarly for solid objects by replacing $\mathrm{Mod}^{\mathrm{cond}}$ with $\mathrm{Mod}^{\blacksquare}$, or, more generally, for analytic rings. We may simply denote $R[G] := R[G]_\rho$ with ρ understood.

1.2.2. Let G be a condensed group acting on a condensed ring R.

(i) The forgetful functor $\mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{R[G]}) \to \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R)$ induced by the structural ring homomorphism $R \to R[G]$ (i.e., forgetting the G-action) is conservative and commutes with all limits and colimits. It admits as a right adjoint the *orbit* functor $R\,\underline{\mathrm{Hom}}_R(R[G]_\rho, -)$, and as a left adjoint $- \otimes_R R[G]_\rho$.

(ii) Assume the G-action on R to be trivial. Then there is a natural trivial action functor $(-)^{\mathrm{triv}} \colon \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R) \to \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R)^G$ induced by the R-algebra homomorphism $R[G] \to R, [g] \mapsto 1$ (i.e., putting trivial G-action on complexes), which is fully faithful and commutes with all limits and colimits. It admits as right adjoint the *derived G-fixed points* functor $(-)^G \colon \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R)^G \to \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R)$, which is computed by $M^G \simeq R\Gamma(G, M) := R\,\underline{\mathrm{Hom}}_{R[G]}(R, M) \simeq R\,\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z}, M) \in \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_R)$, namely, the *condensed group cohomology of G with coefficients in M*. On the other hand, $(-)^{\mathrm{triv}}$ admits as left adjoint the *G-coinvariants functor* $- \otimes_{R[G]} R$.

A standard way to compute the condensed group cohomology of profinite group is to consider the condensed cochain complex, cf. (1.2.4.1).

1.2.3 - Lemma. *For any condensed group G and any $M \in \mathscr{D}(\mathrm{CondAb})^G$, we have an isomorphism in $\mathscr{D}(\mathrm{CondAb})$:*

$$R\Gamma(G, M) \simeq \underline{\mathrm{RHom}}((\cdots \to \mathbf{Z}[G \times G] \to \mathbf{Z}[G] \to \mathbf{Z} \to 0), M).$$

Proof. The key is the standard bar resolution $\mathbf{Z}[G^{\bullet+1}] \to \mathbf{Z} \to 0$ in $\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, and the natural equivalence $R\,\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z}[G^{\bullet+1}], -) \simeq R\,\underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], -)$ killing the first component G; for the latter, we have $\mathbf{Z}[G^{\bullet+1}] \simeq \mathbf{Z}[G] \otimes_{\mathbf{Z}} \mathbf{Z}[G^{\bullet}] \simeq \mathbf{Z}[G] \otimes^L_{\mathbf{Z}} \mathbf{Z}[G^{\bullet}]$ where the last isomorphism follows from the flatness of $\mathbf{Z}[G]$ over $\mathbf{Z}$. □

1.2.4 - Proposition. *Let G be a profinite group. For any $M \in \mathscr{D}(\mathrm{Solid})^{\underline{G}} \subset \mathscr{D}(\mathrm{CondAb})^{\underline{G}}$, the condensed group cohomology is computed by the condensed cochain complex*

$$R\Gamma(\underline{G}, M) \simeq \big(M \to \underline{\mathrm{Hom}}(\mathbf{Z}[G], M) \to \underline{\mathrm{Hom}}(\mathbf{Z}[G \times G], M) \to \cdots\big)\,. \tag{1.2.4.1}$$

Moreover, $R\Gamma(\underline{G}, -)$ commutes with filtered colimits on $\mathscr{D}(\mathrm{Solid})^{\underline{G}}$.

[12]In some literature, it is denoted by $R\#G$ or $R \rtimes_\rho G$, etc., hence there is no standard notation.

Proof. This follows from the facts that we have a natural equivalence of functors $\underline{\mathrm{RHom}}(\mathbf{Z}[S],-) \simeq \underline{\mathrm{RHom}}_{\mathrm{Solid}}(\mathbf{Z}[S]^{\blacksquare},-)$ on $\mathscr{D}(\mathrm{Solid})$ and that the functor $\underline{\mathrm{Hom}}_{\mathrm{Solid}}(\mathbf{Z}[S]^{\blacksquare}-,)$ is an exact functor on Solid, with $\mathbf{Z}[S]^{\blacksquare}$ being an internally compact projective object in Solid for any profinite set S. □

The following lemma is a special case of condensed group cohomology with solid-nuclear coefficients.

1.2.5 - Lemma. *Let G be a profinite group. Let $E \subset K$ be a p-adic local subfield. For any $V \in \mathrm{Mod}_E^{\mathrm{nuc}}$ with trivial $\underline{G}$-action and any $B \in \mathrm{Mod}_E^{\mathrm{nuc}} \cap \mathrm{Mod}_{E[G]}^{\mathrm{cond}}$, there is a natural isomorphism*

$$V \otimes_E^{\blacksquare} R\Gamma(\underline{G}, B) \xrightarrow{\simeq} R\Gamma(\underline{G}, V \otimes_E^{\blacksquare} B).$$

Proof. Since V is flat for $\otimes_E^{\blacksquare}$, it suffices to see that the natural morphism

$$V \otimes_E^{\blacksquare} \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], B) \to \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], V \otimes_E^{\blacksquare} B)$$

is an isomorphism. According to [10, Theorem A.43 (i)], $V \otimes_E^{\blacksquare} W$ is still solid-nuclear, so by characterisation of nuclear objects as trace-class functors [10, Proposition A.55 (i)], we have the vertical isomorphisms in the following commutative diagram

$$\begin{tikzcd}
V \otimes_E^{\blacksquare} \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], B) \arrow[r] & \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], V \otimes_E^{\blacksquare} B) \\
V \otimes_E^{\blacksquare} \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], E) \otimes_E^{\blacksquare} B \arrow[u, "\simeq"] \arrow[r, "\simeq"] & \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\bullet}], E) \otimes_E^{\blacksquare} V \otimes_E^{\blacksquare} B. \arrow[u, "\simeq"]
\end{tikzcd}$$

Now, we can conclude by Proposition (1.2.4). □

1.2.6 - Remark. The above lemma remains true in a more genral setting: we still have this isomorphism for all solid-nuclear $V \in \mathscr{D}(\mathrm{Mod}_E^{\blacksquare})$, which are characterised by the property that $H^i(V) \in \mathrm{Mod}_E^{\mathrm{nuc}}$ for all $i \in \mathbf{Z}$, and for general $B \in \mathscr{D}(\mathrm{Mod}_{E[G]}^{\blacksquare})$ [14, Proposition 13.14]. Indeed, the proof of *loc. cit.* actually works for $S = \mathscr{G}_K$ by the compact projectivity of $\mathbf{Z}[\mathscr{G}_K]^{\bullet}$.

In the particular case of profinite groups $\iota \colon G \simeq \mathbf{Z}_p^n$ (often non canonically isomorphic), its condensed group cohomology can be computed alternatively by the Koszul complex, determined naturally though not canonically by the isomorphism ι.

1.2.7. Let $M \in \mathrm{CondAb}$ and $f_1, \ldots, f_n \in \mathrm{End}_{\mathrm{CondAb}}(M)$. Its *Koszul complex* is the complex

$$\mathrm{Kosz}_M(f_1, \ldots, f_n) := \mathrm{Tot}\left(M \otimes_{f, \mathbf{Z}[X_1, \ldots, X_n]} \bigotimes_{i=1}^{n} (\mathbf{Z}[X_1, \ldots, X_n] \xrightarrow{\cdot X_i} \mathbf{Z}[X_1, \ldots, X_n])\right)$$

with the complex $\mathbf{Z}[X_1, \ldots, X_n] \xrightarrow{\cdot X_i} \mathbf{Z}[X_1, \ldots, X_n]$ sits in (cohomological) degrees 0 and 1, where M is regarded as a $\mathbf{Z}[X_1, \ldots, X_n]$-module via the map $\mathbf{Z}[X_1, \ldots, X_n] \to \mathrm{End}_{\mathrm{CondAb}}(M)$ given by $X_i \mapsto f_i$.

1.2.8 - Proposition. *Let G be a profinite group with an isomorphism $\iota \colon G \simeq \mathbf{Z}_p^n$, and let $\gamma_1, \ldots, \gamma_n$ denote the associated (topological) generators of G, transported via i from the standard basis of $\mathbf{Z}_p^n$. For any $M \in \mathscr{D}(\mathrm{Mod}_{\mathbf{Z}_p}^{\blacksquare})^{\underline{G}}$, the condensed group cohomology is computed by the following Koszul complex*

$$R\Gamma(\underline{G}, M) \simeq \mathrm{Kosz}_M(\gamma_1 - 1, \ldots, \gamma_n - 1). \tag{1.2.8.1}$$

The isomorphism depends on ι, and is compatible with changing the isomorphism ι (which amounts to changing generators).

Proof. We have a projective resolution $\mathrm{Tot}\left(\bigotimes_{\mathbf{Z}_p}^{\blacksquare}{}_{i=1}^{n} (\mathbf{Z}_p[G]^{\blacksquare} \xrightarrow{\cdot[\gamma_i]-\mathrm{id}} \mathbf{Z}_p[G]^{\blacksquare})\right) \to \mathbf{Z}_p \to 0$ in $\mathrm{Mod}_{\mathbf{Z}_p}^{\blacksquare}$, where the first $\mathbf{Z}_p[G]^{\blacksquare}$ sits in the degree -1 and the second sits in the degree 0. Taking $R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathbf{Z}_p[G]}^{\blacksquare}}(-, M)$, using that $R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathbf{Z}_p[G]}^{\blacksquare}}(\mathbf{Z}_p[G]^{\blacksquare}, -) \simeq R\,\underline{\mathrm{Hom}}_{\mathbf{Z}_p[G]}(\mathbf{Z}_p[G], -) \simeq \mathrm{id}$ on $\mathrm{Mod}_{\mathbf{Z}_p[G]}^{\blacksquare}$, we obtain the statement. □

We have the following relation between the two identifications.

1.2.9 - Lemma. *Let $\Gamma \simeq \mathbf{Z}_p^n$ be a profinite group with the associated generators $\gamma_1, \ldots, \gamma_n$, and let $M \in \mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p[\Gamma]}$. Then the composite identification $H^1(\underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma^\bullet], M)) \simeq H^1(\underline{\Gamma}, M) \simeq H^1(\mathrm{Kosz}_M(\gamma_1 - 1, \ldots, \gamma_n - 1))$ agrees with the map $(\gamma_1^*, \ldots, \gamma_n^*)\colon \underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma], M) \to M^{\oplus n}$ restricted to cocycles, where γ_i^* is induced by the map $\mathbf{Z}[*] \to \mathbf{Z}[\Gamma]$ sending the point to γ_i. More precisely, evaluated at a point, the class of a cocycle $\bar{c}$ gets identified with the class of the element $m_{\bar{c}} := (\bar{c}(\gamma_1), \ldots, \bar{c}(\gamma_d))$.*

Proof. Since $M(S) \simeq \underline{\mathrm{Hom}}(\mathbf{Z}[S], M)(*)$ for any profinite set S, we only need to prove the statement evaluated at a point.

Denote by Γ' a copy of Γ for the Koszul construction, and F (read as "digamma") another copy of the continuous cochain complex construction. We have quasi-isomorphisms of genuine complexes

$$\begin{tikzcd}
\mathrm{Kosz}_{\mathbf{Z}_p[\Gamma']^{\blacksquare}}(\gamma_1 - 1, \ldots, \gamma_n - 1) \arrow[r, "\simeq"] & \mathbf{Z}_p \\
\mathrm{Tot}\left(\mathrm{Kosz}_{\mathbf{Z}_p[\Gamma' \times F^{\bullet+1}]^{\blacksquare}}(\gamma_1 - 1, \ldots, \gamma_n - 1)\right) \arrow[u, "\simeq"] \arrow[r, "\simeq"] & \mathbf{Z}[F^{\bullet+1}]^{\blacksquare} \arrow[u, "\simeq"']
\end{tikzcd}$$

in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p[\Gamma]}$, inducing after $R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p[\Gamma]}}(-, M)$ the quasi-isomorphisms of genuine complexes

$$\begin{tikzcd}
\mathrm{Kosz}_M(\gamma_1 - 1, \ldots, \gamma_n - 1) \arrow[d, "\alpha'"', "\simeq"] & & R\Gamma(\underline{\Gamma}, M) \arrow[ll, "\simeq"'] \arrow[d, "\simeq"] \\
\mathrm{Tot}\left(\mathrm{Kosz}_{\underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma' \times F^{\bullet+1}], M)}(\gamma_1' - 1, \ldots, \gamma_n' - 1)\right) & & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[F^{\bullet+1}], M) \arrow[ll, "\alpha''", "\simeq"']
\end{tikzcd} \tag{1.2.9.1}$$

in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p}$. We are going to show that $\alpha'(m_{\bar{c}}) - \alpha''(\bar{c})$ is a coboundary in the total complex.

Consider the following upper right pieces

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'], M)^{\oplus n} \arrow[d, "\alpha' := p'^*"] & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'], M) \arrow[l, "(\gamma_i^* - 1)_{i=1}^n"'] \arrow[d, "p'^*"] & \\
\underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma' \times F], M)^{\oplus n} & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma' \times F], M) \arrow[l, "d' := (\gamma_i'^* - 1)_{i=1}^n"'] \arrow[d, "d'' = p_1^* - p_0^*"] & \\
\iddots & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma' \times F^2], M) & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[F^2], M) \arrow[l, "\alpha'' := p''^*"'] \arrow[d, "\delta"] \\
& & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[F^3], M)
\end{tikzcd} \tag{1.2.9.2}$$

of (1.2.9.1), or, equivalently, the diagram

$$\begin{tikzcd}
M^{\oplus n} \arrow[d, "\alpha' := \mathrm{const}"] & M \arrow[l, "(\gamma_i - 1)_{i=1}^n"'] \arrow[d, "\mathrm{const}"] & \\
M(F)^{\oplus n} & M(F) \arrow[l, "d' := (\gamma_i \circ \gamma_i^{-1*} - 1)_{i=1}^n"'] \arrow[d, "d'' = p_1^* - p_0^*"] & \\
\iddots & M(F^2) & M(F) \arrow[l, "\alpha''(\bar{c}) = c"'] \arrow[d, "\delta"] \arrow[ul, dashed, bend right, "\mathrm{id}"'] \\
& & M(F^2),
\end{tikzcd} \tag{1.2.9.3}$$

where we have killed the Γ-action by reducing the respective first components to the unit $1 \in \Gamma$. We write with a bar $\bar{f} \in M(\Gamma^\bullet)$ for the restriction, under this reduction, of a Γ-equivariant function $f \in \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma^{\bullet+1}], M)$.

Consider the identity map of $M(F)$ fitting as the dashed arrow into the diagram (1.2.9.3). It is a direct computation to check that

$$\alpha'' = d'' + \delta$$

on $M(F)$, and that

$$d'\bar{f}(g) = (-\bar{f}(\gamma_i) + \delta\bar{f}(\gamma_i, \gamma_i^{-1}g))_{i=1}^n$$

for $\bar{f} \in M(F)$. In particular, for any cocycle $\bar{c} \in Z^1M(F^\bullet) = \ker\delta \subset M(F)$, if we view it in degree 0 (upper right corner) of the total complex and apply the differential $d = d' + d''$, we obtain

$$d\bar{c} = \left(-(\mathrm{const}_{\bar{c}(\gamma_i)})_{i=1}^n + (\delta\bar{c}(\gamma_i, \gamma_i^{-1}g))_{i=1}^n, \alpha''(\bar{c}) - \delta\bar{c}\right) = (-\alpha'(m_{\bar{c}}), \alpha''(\bar{c})) \in M(F)^{\oplus n} \oplus M(F^2),$$

thus proving the identification. □

1.2.10 - Remark. In fact, conversely, for any $m = (m_1, \ldots, m_n) \in M(*)^{\oplus n}$ such that $(\gamma_i - 1)m_j = (\gamma_j - 1)m_i$, there exists a cocycle $\bar{c}_m$ such that $\bar{c}(\gamma_i) = m_i$ for $i = 1, \ldots, n$. For this, using these conditions and the fact that $\{\gamma_i\}_{i=1}^n$ form a basis of Γ, one can write down an explicit formula for $\bar{c}$, and then check that it defines an element $\bar{c} \in M(\Gamma)$. For this, using compact projective generation of $\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p[G]}$, one can reduce to the case where M is of the form $M = \mathbf{Z}_p[\Gamma]^{\blacksquare} \otimes^{\blacksquare}_{\mathbf{Z}} \mathbf{Z}[S]^{\blacksquare} \simeq \mathbf{Z}_p[\Gamma \times S]^{\blacksquare}$, in which the verification is direct. Alternatively, admitting the identification of the lemma, one can argue that there exists an element $m_0 \in M(*)$ such that $m + dm_0$ comes from a cocycle $\bar{c}$, i.e., such that $m_i + \gamma_i m_0 - m_0 = \bar{c}(\gamma_i)$ for $i = 1, \ldots, d$, but now $m_i = (\bar{c} - \delta m_0)(\gamma_i)$ where δ is the zeroth differential of the cochain complex $M(\Gamma^\bullet)$.

1.2.11. Galois cohomology. Now we study examples of condensed Galois cohomology, whose computation has essentially been done in the classical continuous cohomology context. We record these results and interpretate them in the condensed language.

Let $M \in \mathscr{D}(\mathrm{Mod}_K^{\mathrm{cond}})$. Consider $C = \widehat{\overline{K}}$, with its natural continuous $\mathscr{G}_K$-action, as an object of $\mathscr{D}(\mathrm{Mod}_K^{\mathrm{cond}})^{\underline{\mathscr{G}_K}}$ concentrated in degree 0. We have a canonical morphism $M \to M \otimes_K^L C$ in $\mathscr{D}(\mathrm{Mod}_K^{\mathrm{cond}})^{\underline{\mathscr{G}_K}}$, inducing the following morphism :

$$M \to R\Gamma(\underline{\mathscr{G}_K}, M \otimes_K^L C) = (M \otimes_K^L C)^{\underline{\mathscr{G}_K}},$$

and this construction is functorial in M. We may also replace C by other objects in $\mathscr{D}(\mathrm{Mod}_K^{\mathrm{cond}})^{\underline{\mathscr{G}_K}}$.

Here is a solid variant. As above, we have a natural transformation

$$D(\mathrm{Mod}_K^{\blacksquare}) \to D(\mathrm{Mod}_K^{\blacksquare}), \quad - \to (- \otimes_K^{L\blacksquare} C)^{\underline{\mathscr{G}_K}}.$$

The advantage of working with the solid category $\mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$ is that C is a flat object for $\otimes_K^{\blacksquare}$, whence the natural isomorphism $M \otimes_K^{L\blacksquare} C \simeq M \otimes_K^{\blacksquare} C$. Thus this natural transformation becomes

$$D(\mathrm{Mod}_K^{\blacksquare}) \to D(\mathrm{Mod}_K^{\blacksquare}), \quad - \to (- \otimes_K^{\blacksquare} C)^{\underline{\mathscr{G}_K}}.$$

Here is a noncompleted version. The object $\overline{K}$ is flat already for the non-solid tensor product $\otimes_K$, hence $M \otimes_K^L \overline{K} \simeq M \otimes_K \overline{K}$ and we get the natural transformation

$$D(\mathrm{Mod}_K^{\mathrm{cond}}) \to D(\mathrm{Mod}_K^{\mathrm{cond}}), \quad (-) \to (- \otimes_K \overline{K})^{\underline{\mathscr{G}_K}}.$$

1.2.12 - Example. Let us review some examples of Galois cohomology. Let $i \in \mathbf{N}, j \in \mathbf{Z}$.

(i) Tate calculated that

$$H^i_{\mathrm{cont}}(\mathscr{G}_K, C(j)) = \begin{cases} K, & j = 0, i = 0, \\ K \cdot \log\chi_{\mathrm{cyc}}, & j = 0, i = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2.12.1}$$

from which we deduce [35, Proposition 6.35] that

$$H^i_{\mathrm{cont}}(\mathscr{G}_K, t^j B^+_{\mathrm{dR}}) = \begin{cases} K, & j \leq 0, i = 0, \\ K \cdot \log \chi_{\mathrm{cyc}}, & j \leq 0, i = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.12.2}$$

These upgrade, by the condensification functor (1.1.7.1), to the computation of corresponding condensed Galois cohomology groups, thanks to [19, Lemma 3.3 (1)]. Furthermore, by (1.2.5), we can even allow nuclear coefficients: for any $W \in \mathrm{Nuc}_K$ (for example for W concentrated in degree 0 and coming from a K-Banach space, or, more generally, a classical nuclear Fréchet K-vector space or a K-vector space of compact type) endowed with the trivial $\mathscr{G}_K$-action, we have

$$H^i(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_K C(j)) = \begin{cases} W, & j = 0, i = 0, \\ W \cdot \log \chi_{\mathrm{cyc}}, & j = 0, i = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2.12.3}$$

$$H^i(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_K t^j B^+_{\mathrm{dR}}) = \begin{cases} W, & j \leq 0, i = 0, \\ W \cdot \log \chi_{\mathrm{cyc}}, & j \leq 0, i = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2.12.4}$$

Indeed, when $W = K[0]$ is trivial and concentrated in degree 0, the key computation has been done in (1.2.12.1) and (1.2.12.2); then one obtains the statement for general $W \in \mathrm{Nuc}_K$ by applying (1.2.3) and [1, Proposition 5.35] (see also [14, Proposition 13.14]).

(ii) By adding to C and B^+_{dR} the polynomial variable $\log t$ together with the semiliear $\mathscr{G}_K$-action $g(\log t) = \log t + \log(\chi_{\mathrm{cyc}}(\sigma))$, we define the $\underline{\mathscr{G}_K}$-modules $C[\log t]$ and $B^+_{\mathrm{dR}}[\log t]$. We have the computation [30]

$$H^i(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_K C[\log t](j)) = \begin{cases} W, & j = 0, i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

for $W \in \mathrm{Nuc}_K$ equipped with the trivial $\mathscr{G}_K$-action; similarly, we have

$$H^i(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_K t^j B^+_{\mathrm{dR}}[\log t]) = \begin{cases} W, & j \leq 0, i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$H^i(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_K B_{\mathrm{dR}}[\log t]) = \begin{cases} W, & i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

for $W \in \mathrm{Nuc}_K$. Indeed, these are reduced, by applying (1.2.3) and [1, Proposition 5.35]), to the computation for $W = K[0]$ trivial and concentrated in degree 0, in which case the detailed proof is recorded below in Proposition (1.2.13) for the convenience of the reader.

We denote

$$B^+_{\mathrm{pdR}} := B^+_{\mathrm{dR}}[\log t], \quad B_{\mathrm{pdR}} := B_{\mathrm{dR}}[\log t] \simeq B^+_{\mathrm{pdR}} \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$$

with $g(\log t) = \log t + \log \chi_{\mathrm{cyc}}(g)$.

1.2.13 - Proposition (Bosco–Dospinescu–Nizioł, [30]). *Let $C[\log t]$ (resp. $B^+_{\mathrm{dR}}[\log t]$) be the polynomial ring on C (resp. on B^+_{dR}) with one formal variable $\log t$, equipped with the semilinear $\mathscr{G}_K$-action given by $g(\log t) = \log t + \log(\chi_{\mathrm{cyc}}(\sigma))$. For $j \in \mathbf{Z}$, let $C[\log t](j)$ (resp. $B^+_{\mathrm{dR}}[\log t]$) be the corresponding Tate twist of $C[\log t]$ (resp. of $B^+_{\mathrm{dR}}[\log t]$).*

(i) *We have isomorphisms in $\mathrm{Mod}^{\blacksquare}_K$:*

$$H^i(\underline{\mathscr{G}_K}, C[\log t](j)) = \begin{cases} W, & j = 0, i = 0, \\ 0, & \textit{otherwise,} \end{cases}$$

(ii) *We have isomorphisms in $\mathrm{Mod}^{\blacksquare}_K$:*

$$H^i(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_K t^j B^+_{\mathrm{dR}}[\log t]) = \begin{cases} W, & j \leq 0, i = 0, \\ 0, & \textit{otherwise,} \end{cases}$$

and

$$H^i(\underline{\mathscr{G}_K}, W \otimes_K^{\blacksquare} B_{\mathrm{dR}}[\log t]) = \begin{cases} W, & i = 0, \\ 0, & \textit{otherwise,} \end{cases}$$

Proof. We include the full proof here, since the cited literature is not yet publicly available.

(i) Set

$$A := C[\log t], \quad A_m := C[\log t]^{\leq m} := \textstyle\sum_{l=0}^m C(\log t)^l, \quad m \in \mathbf{N}.$$

We have inclusion of $\underline{\mathscr{G}_K}$-modules $A_m \subset A_{m+1}$. Since $\mathbf{Z}[\underline{\mathscr{G}_K}]^{\blacksquare}$ is a compact object of $\mathrm{Solid} = \mathrm{Mod}_{\mathbf{Z}}^{\blacksquare}$, we have

$$H^i(\underline{\mathscr{G}_K}, A(j)) = \varinjlim_m H^i(\underline{\mathscr{G}_K}, A_m(j)). \tag{1.2.13.1}$$

For all $m \in \mathbf{N}$ and $j \in \mathbf{Z}$, we have a short exact sequence

$$0 \to A_m(j) \to A_{m+1}(j) \to C(j) \to 0$$

of $\underline{\mathscr{G}_K}$-modules induced by the inclusion $A_m \subset Am + 1$. Taking the condensed Galois cohomology, we obtain the associated long exact sequence

$$\begin{aligned} 0 &\to H^0(\underline{\mathscr{G}_K}, A_m(j)) \to H^0(\underline{\mathscr{G}_K}, A_{m+1}(j)) \to H^0(\underline{\mathscr{G}_K}, C(j)) \\ &\to H^1(\underline{\mathscr{G}_K}, A_m(j)) \to H^1(\underline{\mathscr{G}_K}, A_{m+1}(j)) \to H^1(\underline{\mathscr{G}_K}, C(j)) \\ &\to H^2(\underline{\mathscr{G}_K}, A_m(j)) \to H^2(\underline{\mathscr{G}_K}, A_{m+1}(j)) \to H^2(\underline{\mathscr{G}_K}, C(j)) \to \cdots . \end{aligned} \tag{1.2.13.2}$$

If $j \neq 0$ or $i \geq 2$, the computation (1.2.12.3) yields, by induction on $m \in \mathbf{N}$, the vanishing of all $H^i(\underline{\mathscr{G}_K}, A_m(j))$, so $H^i(\underline{\mathscr{G}_K}, A(j)) = 0$ by taking the colimit over $m \in \mathbf{N}$ (1.2.13.1). Now, let $j = 0$, then the long exact sequence (1.2.13.2) reduces to

(1.2.13.3)

$$0 \to H^0(\underline{\mathscr{G}_K}, A_m) \to H^0(\underline{\mathscr{G}_K}, A_{m+1}) \xrightarrow{\beta_m} H^0(\underline{\mathscr{G}_K}, C) \to H^1(\underline{\mathscr{G}_K}, A_m) \xrightarrow{\gamma_m} H^1(\underline{\mathscr{G}_K}, A_{m+1}) \to H^1(\underline{\mathscr{G}_K}, C) \to 0.$$

Let us show that $\beta_m = 0$. For this, let $P = \sum_{l=0}^{m+1} a_l (\log t)^l \in H^0(\underline{\mathscr{G}_K}, A_{m+1})$, where $a_l \in C$. By definition, we have $\beta_m(P) = a_{m+1}$, and for any $g \in \mathscr{G}_K$, we have $g(P) = P$. Identifying the coefficients of $(\log t)^{m+1}$ in the equality $g(P) = P$ yields $a_{m+1} \in H^0(\underline{\mathscr{G}_K}, C) = K$, and identifying the coefficients of $(\log t)^m$ yields $-(m+1)a_{m+1} \log \chi_{\mathrm{cyc}}(g) = (g-1)a_m$. The computation $H^1(\underline{\mathscr{G}_K}, C) = K \cdot \log \chi_{\mathrm{cyc}}$ (1.2.12.3) then forces $a_{m+1} = 0$.

Having established $\beta_m = 0$, an induction on $m \in \mathbf{N}$ shows that:

- the natural map $K \to H^0(\underline{\mathscr{G}_K}, A_m)$ is an isomorphism;
- the condensed K-module $H^1(\underline{\mathscr{G}_K}, A_m)$ comes from a genuine one-dimensional K-vector space for all $m \in \mathbf{N}$, thus $\gamma_m = 0$.

Therefore, we conclude the statement (i) by taking the colimit over $m \in \mathbf{N}$, using (1.2.13.1).

(ii) We are going to proceed as in the proof of (i). Set $M := B_{\mathrm{dR}}[\log t]$. For $j \in \mathbf{Z}$, set

$$M_j := t^j B_{\mathrm{dR}}^+[\log t], \quad M_{j,m} := t^j B_{\mathrm{dR}}^+[\log t]^{\leq m} := \textstyle\sum_{l=0}^m t^j B_{\mathrm{dR}}^+(\log t)^l, \quad m \in \mathbf{N}.$$

The subspaces $M_{j,m} \subset M_j \subset M$ are all $\underline{\mathscr{G}_K}$-stable, and we have short exact sequences

$$0 \to M_{j,m} \to M_{j,m+1} \to t^j B_{\mathrm{dR}}^+ \to 0, \quad j \in \mathbf{Z}, m \in \mathbf{N}.$$

Again by compactness of $\mathbf{Z}[\mathscr{G}_K]^{\blacksquare} \in \mathrm{Solid}$, we have

$$H^i(\underline{\mathscr{G}_K}, M_j) = \varinjlim_m H^i(\underline{\mathscr{G}_K}, M_{j,m}), \quad H^i(\underline{\mathscr{G}_K}, M) = \varinjlim_j H^i(\underline{\mathscr{G}_K}, M_j). \tag{1.2.13.4}$$

Consider the long exact sequence

$$\begin{aligned} 0 &\to H^0(\underline{\mathscr{G}_K}, M_{j,m}) \to H^0(\underline{\mathscr{G}_K}, M_{j,m+1}) \to H^0(\underline{\mathscr{G}_K}, t^j B_{\mathrm{dR}}^+) \\ &\to H^1(\underline{\mathscr{G}_K}, M_{j,m}) \to H^1(\underline{\mathscr{G}_K}, M_{j,m+1}) \to H^1(\underline{\mathscr{G}_K}, t^j B_{\mathrm{dR}}^+) \\ &\to H^2(\underline{\mathscr{G}_K}, M_{j,m}) \to H^2(\underline{\mathscr{G}_K}, M_{j,m+1}) \to H^2(\underline{\mathscr{G}_K}, t^j B_{\mathrm{dR}}^+) \to \cdots . \end{aligned} \tag{1.2.13.5}$$

This time, using the description of Galois cohomology of M_j given by (1.2.12.4) and an induction on m, we deduce that:

- if $j > 0$ or $i \geq 2$, then $H^i(\underline{\mathscr{G}_K}, M_{j,m}) = 0$;
- if $j \leq 0$, then the natural map $K \to H^0(\underline{\mathscr{G}_K}, M_{j,m})$ is an isomorphism, and the map $H^1(\underline{\mathscr{G}_K}, M_{j,m}) \to H^1(\underline{\mathscr{G}_K}, M_{j,m+1})$ is zero for all $m \in \mathbf{N}$.

Then (ii) follows by taking the colimit over $m \in \mathbf{N}$ and subsequently letting $j \to -\infty$, using (1.2.13.4). □

1.2.14. Discrete condensed objects. For $M \in \mathrm{Cond(Set)}$, we set $M^\delta := \underline{M(*)_{\mathrm{disc}}}$, or, equivalently, for profinite sets S, we set

$$M^\delta(S) := \varinjlim_{S \twoheadrightarrow S_i} M(S_i).$$

There is a natural continuous map $M(*)_{\mathrm{disc}} \to M(*)_{\mathrm{top}}$ underlying the morphism $M^\delta \to M$. We say that M is *discrete* if $M^\delta \to M$ is an isomorphism. The functor $(-)^\delta \colon \mathrm{Cond(Set)} \to \mathrm{Cond(Set)}$ preserves all colimits. Similarly, one can define M^δ for M in Cond(Ab), $\mathrm{Cond}(\mathscr{D}(\mathbf{Z}))$, etc. by the same formula, and we say that M is *discrete* if $M^\delta \to M$ is an isomorphism.

On the other hand, given an object N of Set (*resp.* of Ab, of $\mathscr{D}(\mathbf{Z})$, etc.), we can endow N with the *discrete condensed structure* N_{disc}, that is, for profinite sets S, we set

$$N_{\mathrm{disc}}(S) := \varinjlim_{S \twoheadrightarrow S_i} \prod_{S_i} N.$$

Then we obtain an object N_{disc} of Cond(Set) (*resp.* of Cond(Ab), of $\mathrm{Cond}(\mathscr{D}(\mathbf{Z})) \simeq \mathscr{D}(\mathrm{CondAb})$, etc.)

1.2.15 - Definition. Let S be a profinite set with a pro-finite presentation $S = \varprojlim_i S_i$ (where S_i are finite sets). Let $M \in \mathrm{CondAb}$. We define

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M) = \underline{\mathrm{Hom}}(S, \mathbf{Z}) \otimes_{\mathbf{Z}} M \in \mathrm{CondAb} \tag{1.2.15.1}$$

as the *condensed group of locally constant functions from S to M*. We denote $\mathrm{Hom}^{\mathrm{sm}}(S, M) := \underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M)(*)$.

1.2.16 - Lemma. *Let S be a profinite set with a pro-finite presentation $S = \varprojlim_i S_i$ and let $M \in$* CondAb.

(i) *There is a natural map $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M) \to \underline{\mathrm{Hom}}(S, M)$.*

(ii) *We have $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M) = \varinjlim_i M^{S_i}$, in particular for any extremally disconnected set T, we have $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M)(T) = \varinjlim_i \mathrm{Hom}(S_i, M(T))$; in other words, we have $\mathrm{Hom}^{\mathrm{sm}}(S, M) = M^\delta(S)$, or, more generally, $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M)(T) = \underline{\mathrm{Hom}}(T, M)^\delta(S)$. The map in* (i) *corresponds to the colimit of the natural maps $\mathrm{Hom}(S_i, M(T)) = M(S_i \times T) \to M(S \times T)$; in particular, it is injective.*

(iii) *If M is discrete, then $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(S, M)$.*

Proof. (i) The natural map is clear by remarking that $M = \underline{\mathrm{Hom}}(\mathbf{Z}, M)$.

(ii) For extremally disconnected sets T with pro-finite presentation $T = \varprojlim_j T_j$, we have $\underline{\mathrm{Hom}}(S, \mathbf{Z})(T) = \mathbf{Z}(S \times T) = \varinjlim_{i,j} \mathbf{Z}(S_i \times T_j)$, hence

$$\underline{\mathrm{Hom}}(S, \mathbf{Z}) = \varinjlim_i \mathbf{Z}^{S_i}$$

and by tensoring with M, we obtain

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M) = \varinjlim_i \mathbf{Z}^{S_i} \otimes_{\mathbf{Z}} M = \varinjlim_i M^{S_i}.$$

Evaluated on extremally totally disconnected sets T, this is

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M)(T) = \varinjlim_i M^{S_i}(T) = \varinjlim_i \mathrm{Hom}(S_i, M(T)).$$

We know that $\underline{\mathrm{Hom}}(S, M)(T) = M(S \times T)$, hence there are maps $M(S_i \times T) \to M(S \times T)$, and the above map is identified as its colimit.

(iii) If M is discrete, then $M(S \times T) = \varinjlim_{i,j} M(S_i \times T_j) = \varinjlim_i M(S_i \times T)$. □

1.2.17. Smooth group actions. Let G be a profinite group. Let M be an object of $\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$. We say that the $\underline{G}$-action on M is *smooth* if the map $M \to \underline{\mathrm{Hom}}(G,M)$ factors through

$$M \to \underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M) \hookrightarrow \underline{\mathrm{Hom}}(G,M) \tag{1.2.17.1}$$

where the injectivity is due to (1.2.16, ii). We denote by $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ the full subcategory of $\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$ consisting of condensed abelian groups with a smooth $\underline{G}$-action and $\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) := \mathscr{D}(\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]})$. Here is the solid situation: we denote $\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]} := \mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]} \cap \mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ and $\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}) := \mathscr{D}(\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]})$. The categories $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ and $\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]}$ are Grothendieck abelian categories with generators $\mathbf{Z}[S\times(G/H)]$ for profinite sets S and open subgroups $H \le G$; indeed, that these elements indeed form a family of generators follows from the observation that every element in $M(S)$ for any smooth condensed G-representation M is fixed by some H.

By (1.2.17.1) and (1.2.16, ii), $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]} \subset \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$ is stable under all colimits and solidification. Deriving it, we obtain a strictly commutative diagram of functors

$$\begin{tikzcd} \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \arrow[r] \arrow[d, "(-)^{L\blacksquare}"] & \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \arrow[d, "(-)^{L\blacksquare}"] \\ \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}) \arrow[r] & \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}). \end{tikzcd} \tag{1.2.17.2}$$

1.2.18 - Lemma. *Let G be a profinite group. The fully faithful embedding functor $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]} \subset \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$ has a right adjoint, which is given by*

$$M \mapsto M^{G\text{-sm}} := M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M). \tag{1.2.18.1}$$

More concretely, for any profinite set S, the set $M^{G\text{-sm}}(S)$ consists of $m \in M(S)$ such that its orbit $\mathrm{orb}_m \in M(G\times S)$ comes from the subset $M(G_i \times S)$ for some finite quotient G_i of G.

Proof. The natural projection map $\iota\colon M^{G\text{-sm}} \to M$ is injective by (1.2.16, ii). The right translation action $r\colon G\times G \to G, (g_1,g_2) \mapsto g_2 g_1$ induces a $\underline{G}$-action on $\underline{\mathrm{Hom}}(G,M)$ and on $\underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M)$ compatible with the G-action on M; indeed, for the latter, we need to check that the subobject $\underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M) \subset \underline{\mathrm{Hom}}(G,M)$ is stable under this G-action, but this follows from the expression (1.2.15.1). As a result, there is a natural induced G-action on $M^{G\text{-sm}}$ making ι a $\underline{G}$-equivariant map. Now let us show that the $\underline{G}$-action on M is smooth. For this, the coaction map $M^{G\text{-sm}} \to \underline{\mathrm{Hom}}(G, M^{G\text{-sm}})$ lands in $\underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M) \cap \underline{\mathrm{Hom}}(G,M^{G\text{-sm}}) = \underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M^{G\text{-sm}})$.

Finally, for the adjunction statement: for any $N_1 \in \mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ and $N_2 \in \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, any $\underline{G}$-equivariant map $N_1 \to N_2$ factors uniquely as $N_1 \to N_2^{G\text{-sm}} \subset N_2$ by factorisation property in the definition (1.2.17). □

1.2.19 - Lemma. *Let G be a profinite group. For any $M \in \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, we have*

$$M^{G\text{-sm}} \simeq \varinjlim_{H} H^0(\underline{H}, M)$$

where H runs over the filtered system of all open normal subgroups of G.

Proof. Let H be an open normal subgroup of G acting on M via restriction $\rho_H\colon M \to \underline{\mathrm{Hom}}(H,M)$. We have $\underline{\mathrm{Hom}}(G/H,M) \simeq \ker(m^* - \mathrm{pr}_1^*\colon \underline{\mathrm{Hom}}(G,M) \to \underline{\mathrm{Hom}}(G\times H,M))$ over $\underline{\mathrm{Hom}}(G,M)$, induced from the colimit diagram $\mathrm{colim}(m,\mathrm{pr}_1\colon G\times H \rightrightarrows G) \xrightarrow{\simeq} G/H$ with compatible maps from G (e.g. $(\mathrm{id}, e_G)\colon G \to G\times H$), hence

$$M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}(G/H,M) \simeq \ker(\rho_H - \mathrm{const}\colon M \to \underline{\mathrm{Hom}}(H,M)) \simeq H^0(\underline{H},M).$$

Here, we have used that

$$m^*, \mathrm{pr}_1^*\colon M \simeq M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}(G,M) \to M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}(G\times H,M) \simeq \underline{\mathrm{Hom}}(\frac{G\times H}{G\times\{e_G\}}, M),$$

where $\frac{G\times H}{G\times\{e_G\}}$ is the quotient topological space, and that they both factor through the subobject $\underline{\mathrm{Hom}}(H,M)$. We conclude by taking colimits over H and applying (1.2.18.1). □

1.2.20. Let $(-)^{RG\text{-sm}}\colon \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \to \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]})$ denote the right derived functor of $(-)^{G\text{-sm}}$. Then (1.2.19) implies that

$$M^{RG\text{-sm}} \simeq \varinjlim_{H} R\Gamma(\underline{H}, M)$$

for $M \in \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]})$, where H runs over the filtered system of all open normal subgroups of G.

Taking right adjoints of the commutative diagram (1.2.17.2), we find that $(-)^{G\text{-sm}}$ and $(-)^{RG\text{-sm}}$ restrict to functors on corresponding solid objects.

1.2.21 - Lemma. *Let G be a profinite group and $M \in \mathrm{CondAb}$. Let G act trivially on M and by right translation on G. Then we have a natural isomorphism*

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \simeq \underline{\mathrm{Hom}}(G, M)^{G\text{-sm}}.$$

Proof. For profinite set T, the map $\underline{\mathrm{Hom}}(G, M) \to \underline{\mathrm{Hom}}(G, \underline{\mathrm{Hom}}(G, M)) \simeq \underline{\mathrm{Hom}}(G \times G, M)$ is induced by the multiplication map $m\colon G^{(1)} \times G^{(2)} \to G^{(2)}, (g_1, g_2) \mapsto g_2 g_1$, where $G^{(1)}$ and $G^{(2)}$ are two copies of G. For extremally disconnected T, we have

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(G^{(1)}, \underline{\mathrm{Hom}}(G^{(2)}, M))(T) = \varinjlim_{i} \mathrm{Hom}(G_i^{(1)}, \underline{\mathrm{Hom}}(G^{(2)}, M)(T)) = \varinjlim_{i} M(G_i^{(1)} \times G^{(2)} \times T),$$

so

$$\begin{aligned} \underline{\mathrm{Hom}}(G, M)^{G\text{-sm}}(T) &= M(G^{(2)} \times T) \times_{M(G^{(1)} \times G^{(2)} \times T)} \varinjlim_{i} M(G_i^{(1)} \times G^{(2)} \times T) \\ &\simeq \varinjlim_{i} M(G_i^{(2)} \times T) \\ &= \underline{\mathrm{Hom}}^{\mathrm{sm}}(G^{(2)}, M)(T). \end{aligned}$$

Here, for the second to last identification, we used the following commutative diagram

$$\begin{tikzcd} M(G^{(2)} \times_{G_i} G^{(2)} \times T) \arrow[r, hook, "m^*"] & M(G^{(2)} \times (G^{(1)} \times_{G_i} G^{(1)}) \times T) \\ M(G^{(2)} \times T) \arrow[u, "\mathrm{pr}_1^* - \mathrm{pr}_2^*"] \arrow[r, hook, "m^*"] & M(G^{(2)} \times G^{(1)} \times T) \arrow[u, "\mathrm{pr}_1^* - \mathrm{pr}_2^*"'] \\ M(G_i^{(2)} \times T) \arrow[u, hook] \arrow[r, hook, "m^*"] & M(G^{(2)} \times G_i^{(1)} \times T) \arrow[u, hook] \end{tikzcd}$$

with exact columns to get a unique map $M(G^{(2)} \times T) \times_{M(G^{(1)} \times G^{(2)} \times T)} M(G_i^{(1)} \times G^{(2)} \times T) \to M(G_i^{(2)} \times T)$ making the diagram commutative, whence this map is an isomorphism. □

1.2.22. Smooth group cohomology. Let G be a profinite group. We define the *smooth group cohomology* functor of G as the right derived functor $R\Gamma_{\mathrm{sm}}(G, -)\colon \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}_{\mathbf{Z}[G]}) \to \mathscr{D}(\mathrm{CondAb})$ of $\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z}, M)$, which is right adjoint to the trivial action functor; similarly in the solid situation, we define, by abuse of notation, $R\Gamma_{\mathrm{sm}}(G, -)\colon \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}) \to \mathscr{D}(\mathrm{Solid})$.

For $M \in \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]})$ with image M' in $\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]})$, there is no ambiguity, since $R\Gamma_{\mathrm{sm}}(G, M)$ has image $R\Gamma_{\mathrm{sm}}(G, M')$ in $\mathscr{D}(\mathrm{CondAb})$. Indeed, this is because we have a strict commutative diagram of functors

$$\begin{tikzcd} \mathscr{D}(\mathrm{CondAb}) \arrow[r, "(-)^{\mathrm{triv}}"] \arrow[d, "(-)^{L\blacksquare}"] & \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \arrow[d, "(-)^{L\blacksquare}"] \\ \mathscr{D}(\mathrm{Solid}) \arrow[r, "(-)^{\mathrm{triv}}"] & \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}), \end{tikzcd}$$

where the right solidification functor is well-defined by (1.2.17).

1.2.23. Let G be a profinite group. Consider the sequence of morphisms $\mathrm{CondAb} \overset{(-)^{\mathrm{triv}}}{\to} \mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]} \hookrightarrow \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, which preserve colimits and finite limits. Taking their right adjoints, which thus preserve injective objects, we obtain

$$R\Gamma(\underline{G}, -) \simeq R\Gamma_{\mathrm{sm}}(\underline{G}, (-)^{RG\text{-sm}}).$$

1.2.24 - Lemma. *Let G be a profinite group. For any $M \in \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]})$ and any profinite set S, we have $R\Gamma_{\mathrm{sm}}(G, M)(S) \simeq R\Gamma(G, M(S))$ where the latter computes the usual profinite group cohomology of the smooth G-representation $M(S)$.*

In particular, if M has a $\underline{\mathbf{Q}}$-linear structure, then $R\Gamma_{\mathrm{sm}}(G, M)(S)$ is represented by $(M^\bullet_S)^G$ for any $\mathbf{Q}$-linear complex $M^\bullet_S$ representing $M(S)$.

Proof. The first statement is [50, Lemma 3.4.15] (cf. [21, Remark 4.28] for a classical but more restrictive explanation) and the last is due to vanishing of higher cohomology groups of profinite group cohomology of smooth representations over $\mathbf{Q}$. □

1.2.25 - Proposition. *Let G be a profinite group. For any $M \in \mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]}$, the smooth group cohomology is computed by the smooth cochain complex*

$$R\Gamma_{\mathrm{sm}}(\underline{G}, M) \simeq \left(M \to \underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \to \underline{\mathrm{Hom}}^{\mathrm{sm}}(G \times G, M) \to \cdots\right).$$

Proof. This is [50, Corollary 3.4.17]. □

1.2.26 - Corollary. *For $M \in \mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]}$, we have*

$$H^0_{\mathrm{sm}}(\underline{G}, M) = H^0(\underline{G}, M).$$

Proof. This is due to (1.2.25) and the factorisation $M \to \underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \hookrightarrow \underline{\mathrm{Hom}}(G, M)$ for $M \in \mathrm{CondAb}$ (1.2.17). □

The following lemma explains the relation between the classical continuous actions and the condensed group actions.

1.2.27 - Lemma. *Let G be a profinite group and let $M \in \mathscr{D}(\mathrm{CondAb})$ be a discrete object.*

(i) *Any condensed $\underline{G}$-action on M is smooth and is uniquely determined by the underlying G-action on $M(*)$, which is smooth.*

(ii) *Conversely, any smooth G-action on $M(*)$ extends to a condensed $\underline{G}$-action on M.*

Proof. (i) The smoothness of the $\underline{G}$-action on M is clear from (1.2.16, iii).

Now, we want to reconstruct the $\underline{G}$-action on M from the G-action on $M(*)$. It is the same as reconstructing corresponding structural maps

$$M(S) \to M(G \times S)$$

natural in S. The G-action on $M(*)$ can be interpretated as a map $M(*) \to M(G) = \varinjlim_n M(G_n)$; hence every $m \in M(*)$ has open stabliser in G, so that the G-action on $M(*)$ is smooth. We know that $M(S) = \varinjlim_i M(S_i)$ by discreteness. So the structural map

$$M(S) \to M(G \times S)$$

is identified as

$$\varinjlim_i M(S_i) \to \varinjlim_{i,n} M(G_n \times S_i),$$

which in turn agrees with

$$\varinjlim_i \left(M(S_i) \to \varinjlim_n M(G_n \times S_i)\right) = \varinjlim_i \left(M(*) \to \varinjlim_n M(G_n)\right)^{S_i}.$$

The map in the last parenthesis is the same as $M(*) \to M(G)$ by discreteness of M.

(ii) Conversely, if the action of G on $M(*)$ is smooth, then the action is described by a morphism $M(*) \to \varinjlim_n M(*)^{G_n} = \varinjlim_n M(G_n)$. Hence we may use the formula in the proof of (i) to define its (unique) extension to an $\underline{G}$-action on M. □

1.3 Condensed cohomology theories

As pointed out in (1.1.7), we need to put condensed structures on various cohomology theories in p-adic geometry. Let us start with the p-adic (pro)étale cohomology.

1.3.1. (Pro)étale cohomology. Let X be an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)$. We define the proétale site of X as $X_{\text{proét}} := X^{\diamond}_{\text{qproét}}$, the quasi-proétale site of the diamond associated to X, with canonical projection $\nu\colon X_{\text{qproét}} \to X_{\text{ét}}$ to the étale site, with the associated morphisms of topoi (ν^*,ν_*). We are interested in (pro)étale cohomology of (complexes of) sheaves of R-modules on X, where $R \in \{\mathbf{Z}/p^n,\mathbf{Z}_p,\mathbf{Q}_p\}$.

(i) For sheaves $\mathcal{F} \in \mathrm{Shv}(X_{\text{ét}},\mathrm{Ab})$, we have the natural equivalence $\mathcal{F} \xrightarrow{\simeq} R\nu_*\nu^*\mathcal{F}$ [65, Proposition 14.8], whence the natural equivalence $R\Gamma_{\text{ét}}(X,\mathcal{F}) \simeq R\Gamma_{\text{proét}}(X,\nu^*\mathcal{F})$. The proétale cohomology verifies proétale hyperdescent, hence so does the étale cohomology.

(ii) For any étale $\mathbf{Z}_p$-local system $\mathbf{L} = (\mathbf{L}/p^n)_{n\in\mathbf{N}}$ on X with completion $\widehat{\mathbf{L}} := \lim_n \nu^*(\mathbf{L}/p^n)$ on $X_{\text{proét}}$, we define

$$R\Gamma_{\text{ét}}(X,\mathbf{L}) := R\lim_n R\Gamma_{\text{ét}}(X,\mathbf{L}/p^n).$$

Then we have

$$\begin{aligned} R\Gamma_{\text{ét}}(X,\mathbf{L}) &\simeq R\lim_n R\Gamma_{\text{proét}}(X,\nu^*(\mathbf{L}/p^n)) \\ &\simeq R\Gamma_{\text{proét}}(X,R\lim_n \nu^*(\mathbf{L}/p^n)) \xleftarrow{\simeq} R\Gamma_{\text{proét}}(X,\widehat{\mathbf{L}}), \end{aligned} \tag{1.3.1.1}$$

where the second isomorphism follows from the fact that $R\lim_n$ and $R\Gamma_{\text{proét}}$ commute, the last isomorphism follows from the vanishing $R^i\lim_n \nu^*(\mathbf{L}/p^n) = 0$; indeed, by [64, Lemma 3.18], we may reduce it to checking that for totally disconnected perfectoid spaces U over X, we have $R^1\lim_n H^0(U,\mathbf{Z}/p^n) = 0$ and $H^i(U,\mathbf{Z}/p^n) = 0$ for $i > 0$, which is clear.

1.3.2. Two condensed structures on proétale cohomology. Let X be an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)$ with the canonical projection morphism $f_{\text{proét}}\colon X_{\text{proét}} \to *_{\text{proét}} = \mathrm{ProFin}$ to the proétale site of a point.

We have two versions of condensed proétale cohomology on X:

(i) For $\mathcal{F} \in \mathscr{D}(\mathrm{Shv}(X_{\text{proét}},\mathrm{Ab}))$, we may define its (derived) pushforward to the proétale of a point as

$$\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F}) := Rf_{\text{proét}*}\mathcal{F} \in \mathscr{D}(\mathrm{Shv}(*_{\text{proét}},\mathrm{Ab})) \simeq \mathscr{D}(\mathrm{CondAb}) \simeq \mathrm{Shv}^{\mathrm{hyp}}(*_{\text{proét}},\mathscr{D}(\mathrm{Ab}));$$

more precisely, we have

$$\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F})(S) = R\Gamma(X_{\text{proét}/X\times\underline{S}},\mathcal{F}) \simeq R\Gamma_{\text{proét}}(X\times\underline{S},\mathcal{F})$$

for any profinite set S.

(ii) We have a functor

$$\mathrm{Shv}(X_{\text{proét}},\mathrm{Ab}) \to \mathrm{Shv}(X_{\text{proét}},\mathrm{CondAb}) \simeq \mathrm{Shv}(X_{\text{proét}}\times *_{\text{proét}},\mathrm{Ab})$$

sending $\mathcal{F}$ to $\underline{\mathcal{F}}\colon U \mapsto (S \mapsto F(U\times\underline{S}))$, which by [26, Lemma A.20] (cf. [71, B.1.8, (i)]) is further identified with the pushforward $\mu_*\mathcal{F}$ along the morphism of sites

$$\mu\colon X_{\text{proét}} \to X_{\text{proét}}\times *_{\text{proét}}.$$

Then, for $\mathcal{F} \in \mathrm{Shv}(X_{\text{proét}},\mathrm{Ab})$, we define the object

$$R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}}) \in \mathscr{D}(\mathrm{CondAb}) \simeq \mathrm{Shv}^{\mathrm{hyp}}(*_{\text{proét}},\mathscr{D}(\mathrm{Ab}))$$

as the global section with condensed coefficients; more precisely, we have

$$R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}})(S) \simeq R\Gamma((X_{\text{proét}}\times *_{\text{proét}})_{/X\times\underline{S}},\mu_*\mathcal{F})$$

for any profinite set S.

Though not equivalent to each other, the two points of view are related for $\mathcal{F} \in \mathrm{Shv}(X_{\text{proét}}, \mathrm{Ab})$. Since the composite morphism of sites $X_{\text{proét}} \xrightarrow{\mu} X_{\text{proét}} \times *_{\text{proét}} \xrightarrow{\mathrm{pr}_1} *_{\text{proét}}$ agrees with $f_{\text{proét}}$, we have

$$R\Gamma((X_{\text{proét}} \times *_{\text{proét}})_{/X\times \underline{S}}, \mu_* \mathcal{F}) \simeq \underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}),$$

whence an evident natural map $R\Gamma_{\text{proét}}(X, \underline{\mathcal{F}}) \to \underline{R\Gamma}_{\text{proét}}(X, \mathcal{F})$, whose obstruction of being an isomorphism lies in the higher direct images $R^i\mu_*\mathcal{F}$ for $i > 0$.

1.3.3. Disambiguation of two condensed structures. Let X be an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. We give instances of $\mathcal{F} \in \mathrm{Shv}(X_{\text{proét}}, \mathrm{Ab})$ such that $R^i\mu_*\mathcal{F} = 0$ for $i > 0$, so that by (1.3.2), there is a natural equivalence

$$R\Gamma_{\text{proét}}(X, \underline{\mathcal{F}}) \xrightarrow{\simeq} \underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \tag{1.3.3.1}$$

in $\mathscr{D}(\mathrm{CondAb})$. In the following examples, we will actually prove that

$$H^i_{\text{proét}}(U \times \underline{S}, \mathcal{F}) = 0, \quad i > 0 \tag{1.3.3.2}$$

for sufficiently "small" strictly totally disconnected perfectoid spaces U over X and all extremally disconnected sets S; then $U \times \underline{S}$ is also a strictly totally disconnected perfectoid space [65, Lemma 7.19], i.e., a quasi-compact perfectoid space all of whose étale covers split.

(i) Let $\mathcal{F} = \nu^* F$ be a proétale sheaf on X pulled back from an étale sheaf F on X. Then (1.3.3.2) holds as we have $H^i_{\text{proét}}(U \times \underline{S}, \mathcal{F}) \simeq H^i_{\text{ét}}(U, F) = 0$ if $i > 0$ by splitting of étale covers of U.

(ii) Let $\mathcal{F}$ be a proétale $\mathbf{Z}_p$-local system on X. Then (1.3.3.2) holds for sufficiently "small" strictly totally disconnected U, in the sense that $\mathcal{F}|_U \simeq \mathbf{Z}_p^n$ is trivialised. We may assume that $\mathcal{F} = \widehat{\mathbf{Z}}_p := \lim_n \nu^*(\mathbf{Z}/p^n)$. Now $H^i_{\text{proét}}(U \times \underline{S}, \nu^*(\mathbf{Z}/p^n))$ vanishes if $i > 0$, and is, if $i = 0$, equal to $\mathrm{Cont}(|U \times \underline{S}|, \mathbf{Z}/p^n)$, the set of locally constant functions on the underlying topological space $U \times \underline{S}$ with values in $\mathbf{Z}/p^n$. So $(H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Z}/p^n))_{n\in\mathbf{N}}$ forms a Mittag–Leffler system. Therefore,

$$H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Z}_p) \simeq \lim_n H^i_{\text{proét}}(U \times \underline{S}, \nu^*(\mathbf{Z}/p^n)) \simeq \begin{cases} \mathrm{Cont}(|U \times S|, \mathbf{Z}_p), & i = 0, \\ 0, & i > 0, \end{cases} \tag{1.3.3.3}$$

where $\mathbf{Z}_p$ is endowed with its natural p-adic topology.

(iii) Let $\mathcal{F}$ be a proétale $\mathbf{Q}_p$-local system on X. Then (1.3.3.2) holds for sufficiently "small" strictly totally disconnected U, in the sense that $\mathcal{F}|_U \simeq \mathbf{Q}_p^n$ is trivialised. Recall that $\mathbf{Q}_p := \widehat{\mathbf{Z}}_p[\frac{1}{p}] = \mathrm{colim}_{p\cdot} \widehat{\mathbf{Z}}_p$. Then $H^i_{\text{proét}}(U \times S, \mathbf{Q}_p) \simeq \mathrm{colim}_{p\cdot} H^i_{\text{proét}}(U \times \underline{S}, \widehat{\mathbf{Z}}_p)$ since proétale cohomology on qcqs spaces commutes with filtered colimits (by coherence of the topoi $(U \times \underline{S})^\sim_{\text{proét}} \simeq (U \times \underline{S})^{\diamond,\sim}_{\text{qproét,qcqs}}$). We conclude by (1.3.3.3) that

$$H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Q}_p) \simeq \begin{cases} \mathrm{Cont}(|U \times \underline{S}|, \mathbf{Q}_p), & i = 0, \\ 0, & i > 0. \end{cases} \tag{1.3.3.4}$$

(iv) Let $\mathcal{F} = \widehat{\mathcal{O}}_X$ or $\mathcal{F} \in \{\mathbf{B}_I, \mathbf{B}, \mathbf{B}_{\log}, \mathbf{B}_{\log}[\frac{1}{t}], \mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m, \mathbf{B}^+_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}\}$ be a proétale period sheaf. Then by virtue of [10, Proposition 4.7] and [11, Proposition 2.37], the vanishing (1.3.3.2) holds for sufficiently "small" U, in the sense that U is an affinoid perfectoid space over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$.

In conclusion, let $\mathcal{F} \in \mathrm{Shv}(X_{\text{proét}}, \mathrm{Ab})$ be the pullback of an étale sheaf on X, be a proétale $\mathbf{Z}_p$- or $\mathbf{Q}_p$-local system, or belong to $\{\widehat{\mathcal{O}}_X, \mathbf{B}_I, \mathbf{B}, \mathbf{B}_{\log}, \mathbf{B}_{\log}[\frac{1}{t}], \mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m, \mathbf{B}^+_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}\}$, then:

(v) The equivalence (1.3.3.1) holds.

(vi) We have $\underline{R\Gamma}_{\text{proét}}(U, \mathcal{F}) \in \mathscr{D}(\mathrm{Solid})$ for sufficiently "small" U, even that $\underline{R\Gamma}_{\text{proét}}(U, \mathcal{F})$ is concentrated in degree 0 and is a $\mathbf{Q}_p$-Banach space if $\mathcal{F} \in \{\mathbf{Q}_p^n, \widehat{\mathcal{O}}_X, \mathbf{B}_I, \mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m\}$ and is a $\mathbf{Q}_p$-Fréchet space if $\mathcal{F} \in \{\mathbf{B}, \mathbf{B}^+_{\mathrm{dR}}\}$.

(vii) By descent and stability of solidness under all limits and colimits, we obtain $\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \in \mathscr{D}(\mathrm{Solid})$. In fact, if X is quasi-compact (and quasi-separated), then $\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F})$ is represented by a complex of $\mathbf{Q}_p$-Banach spaces if $\mathcal{F}$ is a $\mathbf{Q}_p$-local system or if $\mathcal{F} \in \{\mathbf{Q}_p^n, \mathbf{B}_I, \mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m\}$, and is represented by a complex of $\mathbf{Q}_p$-Fréchet spaces if $\mathcal{F} \in \{\mathbf{B}, \mathbf{B}^+_{\mathrm{dR}}\}$, whence is represented by a complex of solid-nuclear $\mathbf{Q}_p$-vector spaces in both cases.

1.3.4. The étale cohomology of qcqs objects is discrete. Let $\mathcal{F} = \nu^* F$ be a proétale sheaf on X pulled back from an étale sheaf F on X. Then by [65, Proposition 11.23, Proposition 14.9], for any profinite set $S = \varprojlim_i S_i$, we have $R\Gamma_{\text{proét}}(X \times \underline{S}, \nu^* F) \simeq \varinjlim_i R\Gamma_{\text{proét}}(X \times \underline{S_i}, \nu^* F) \simeq \varinjlim_i R\Gamma_{\text{proét}}(X, \nu^* F)^{S_i} \simeq \varinjlim_i R\Gamma_{\text{ét}}(X, F)^{S_i}$, hence

$$\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \simeq \underline{R\Gamma_{\text{ét}}(X, F)_{\text{disc}}}$$

and it is classic and discrete, i.e. is the condensification of $R\Gamma_{\text{ét}}(X, F) \in \mathscr{D}(\text{Ab})$ endowed with the discrete condensed structure (1.2.14). We define

$$\underline{R\Gamma}_{\text{ét}}(X, F) := \underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}).$$

1.3.5 - Proposition (Hochschild–Serre spectral sequence). *Let G be a profinite group and $\widetilde{X} \to X$ a proétale $\underline{G}$-torsor of analytic adic spaces over* $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. *Let $\mathcal{F} \in \text{Shv}(X_{\text{proét}}, \mathbf{Q}_p)$ together with a $\underline{G}$-equivariant structure on $\mathcal{F}|_{\widetilde{X}}$. Then we have a natural equivalence*

$$\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \simeq R\Gamma(\underline{G}, \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F}))$$

in $\mathscr{D}(\text{Solid})$.

Proof. First, we have $\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \simeq \lim_{\Delta} \underline{R\Gamma}_{\text{proét}}(\widetilde{X} \times \underline{G}^{\times \bullet}, \mathcal{F})$ by the proétale descent along the Čech hypercovering of the $\underline{G}$-torsor and $\underline{G}$-equivariance of $\mathcal{F}|_{\widetilde{X}}$.

There is an isomorphism

$$\underline{R\Gamma}_{\text{proét}}(\widetilde{X} \times \underline{G}^{\times n}, \mathcal{F}) \simeq R\,\underline{\text{Hom}}(\mathbf{Z}[G^{\times n}], \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F})).$$

Indeed, for an $n \in \mathbf{N}$ and a profinite set S, there are isomorphisms

$$\begin{aligned} \underline{R\Gamma}_{\text{proét}}(\widetilde{X} \times \underline{G}^{\times n}, \mathcal{F})(S) &= R\Gamma_{\text{proét}}(\widetilde{X} \times \underline{G}^{\times n} \times \underline{S}, \mathcal{F}) \\ &\simeq \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F})(G^{\times n} \times S) \\ &\simeq R\,\text{Hom}(\mathbf{Z}[G^{\times n} \times S], \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F})) \\ &\simeq R\,\text{Hom}(\mathbf{Z}[S], R\,\underline{\text{Hom}}(\mathbf{Z}[G^{\times n}], \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F}))), \end{aligned}$$

where, for the last isomorphism, we have used that $\mathbf{Z}[G^{\times n} \times S] \simeq \mathbf{Z}[G^{\times n}] \otimes_{\mathbf{Z}} \mathbf{Z}[S] \simeq \mathbf{Z}[G^{\times n}] \otimes^L_{\mathbf{Z}} \mathbf{Z}[S]$ is concentrated in degree 0 (the last isomorphism follows from the flatness of $\mathbf{Z}[S]$ in CondAb, and the second to last follows from the definition of $\mathbf{Z}[-]$[13]).

Next, recall that $\underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F}) \in \mathscr{D}(\text{Solid})$ (1.3.3, vii), and that $\mathbf{Z}[S]^{L\blacksquare} \simeq Z[S]^{\blacksquare}[0]$ is compact projective in $\mathscr{D}(\text{Solid})$ for any profinite set S (not only the extremally disconnected ones). But we have

$$R\,\underline{\text{Hom}}(\mathbf{Z}[G^{\times n}], M) \simeq R\,\underline{\text{Hom}}_{\mathscr{D}(\text{Solid})}(\mathbf{Z}[G^{\times n}]^{L\blacksquare}, M) \simeq \underline{\text{Hom}}_{\mathscr{D}(\text{Solid})}(\mathbf{Z}[G^{\times n}]^{\bullet}, M) \simeq \underline{\text{Hom}}(\mathbf{Z}[G^{\times n}], M)$$

for $M \in \mathscr{D}(\text{Solid})$, whence

$$R\,\underline{\text{Hom}}(\mathbf{Z}[G^{\times n}], \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F})) \simeq \underline{\text{Hom}}(\mathbf{Z}[G^{\times n}], \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F})).$$

Finally, the differentials in the condensed group cohomology are identified with those in the total complex of the Čech cosimplicial nerve. □

1.3.6. Notation. For simplicity, we will often remove the underline in the notation $\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F})$ if the context permits, while keeping that in $R\Gamma_{\text{proét}}(X, \underline{\mathcal{F}})$. Since in most cases that we will encounter the two versions even agree, there will be no harm in doing this.

[13] Let $M^\bullet$ be a complex in $\mathscr{D}(\text{CondAb})$. The corresponding condensed object $M \in \text{Shv}^{\text{hyp}}(*_{\text{proét}}, \mathscr{D}(\text{Ab}))$ is given by $M(S) := R\Gamma(S, M^\bullet) \in \mathscr{D}(\text{Ab})$, which is represented by $M^\bullet(S)$ if S is extremally disconnected. By the definition of free objects $\mathbf{Z}[-]$, we have $R\,\text{Hom}(\mathbf{Z}[S], M^\bullet) \simeq R\,\text{Hom}(\mathbf{Z}[S], I^\bullet) = I^\bullet(S)$ for any injective resolution $M^\bullet \xrightarrow{\simeq} I^\bullet$ in $\mathscr{D}(\text{CondAb})$; but this is also the definition of $R\Gamma(S, M^\bullet)$. Therefore, we obtain $R\,\text{Hom}(\mathbf{Z}[S], M^\bullet) \simeq M(S)$.

Next, we pass to log-crystalline cohomology, keeping in mind the intuition that étale cohomology of (p^n-torsion) sheaves ought to take discrete values on qcqs objects (1.3.4).

1.3.7. Condensed log-crystalline cohomology. Let us recall some basics on log-crystalline cohomology; for details, cf. [6, §1].

First, let $S^\sharp = (S, \mathcal{L}, I, \gamma)$ be a log pd-scheme with $p \in \mathcal{O}_S$ is nilpotent; more precisely, this consists of a log-scheme $(S, \mathcal{L})$ together with a pd-ideal (I, γ) of the structure sheaf $\mathcal{O}_S$. Assume that $S^\sharp$ is *quasi-coherent*, i.e., that the underlying log-scheme $(S, \mathcal{L})$ is quasi-coherent in the sense of Kato [45, 2.1]. Let $\mathscr{X}$ be an *integral* [45, 2.1] and *quasi-coherent* [45, 2.1] log-scheme over $S^\sharp$. Let $((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}$ be the log-crystalline site of $(\mathscr{X}, M_{\mathscr{X}})$ over $S^\sharp$, whose objects are $(\mathscr{U}, T)$ with $\mathscr{U}$ is an étale scheme over $\mathscr{X}$ with pullback log-structure $M_{\mathscr{U}}$, and $T = (T, M_T)$ is a pd-$S^\sharp$-thickening of $(\mathscr{U}, M_{\mathscr{U}})$ with defining pd-ideal $\mathcal{I}_T \subset \mathcal{O}_T$, and whose coverings are the étale ones. We may simplify the notation to $(\mathscr{X}/S^\sharp)_{\mathrm{cris}}$ or even $(\mathscr{X}/S)_{\mathrm{cris}}$ if the context permits. The site $((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}$ has a structure sheaf $\mathcal{O}_{\mathscr{X}/S^\sharp}\colon (\mathscr{U}, T) \mapsto \Gamma(T, \mathcal{O}_T)$, with pd-ideal sheaf $\mathcal{I}_{\mathscr{X}/S^\sharp}\colon (\mathscr{U}, T) \mapsto \Gamma(T, \mathcal{J}_T)$; it has also has a sheaf $\mathbf{G}_m^\natural\colon (\mathscr{U}, T) \mapsto \Gamma(T, (1+\mathcal{I}_T, \times))$ (note that $\mathcal{I}_T \subset \mathcal{O}_T$ is a nil-ideal), and a monoid sheaf $M_{\mathscr{X}/S^\sharp}\colon (\mathscr{U}, T) \mapsto \Gamma(T, M_T)$. These sheaves take values, on quasi-compact quasi-separated objects, in the category of abelian groups (*resp.* commutative monoids), which we will regard as condensed abelian groups (*resp.* condensed commutative monoids) with the discrete topology (1.2.14). Thanks to the local discreteness of sheaves, many operations on them (such as taking global sections, cohomology or internal Hom objects) could be enhanced to the condensed setting. For example, the condensed global sections of $\mathcal{O}_{\mathscr{X}/S^\sharp}$ is

$$R\Gamma_{\mathrm{cris}}(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}, \mathcal{O}_{\mathscr{X}/S^\sharp}) := \varprojlim_{(\mathscr{U}, T)} \underline{\Gamma(T, \mathcal{O}_T)_{\mathrm{disc}}} \in \mathscr{D}(\mathrm{CondAb}),$$

and we define the condensed log-crystalline cohomology as

$$R\Gamma_{\mathrm{cris}}((\mathscr{X}, M_{\mathscr{X}})/S^\sharp) := R\Gamma_{\mathrm{cris}}(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}, \mathcal{O}_{\mathscr{X}/S^\sharp}) \in \mathscr{D}(\mathrm{CondAb}).$$

The cocontinuous functor $((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}} \to \mathscr{X}_{\mathrm{\acute{e}t}}, (\mathscr{U}, T) \mapsto \mathscr{U}$ induces a canonical morphism of topoi $u^{\log}_{\mathscr{X}/S^\sharp}\colon ((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)^{\log,\sim}_{\mathrm{cris}} \to \mathscr{X}^\sim_{\mathrm{\acute{e}t}}$ such that

$$(u^{\log *}_{\mathscr{X}/S^\sharp}\mathcal{G})(\mathscr{U}, T) = \mathcal{G}(\mathscr{U}), \quad (u^{\log}_{\mathscr{X}/S^\sharp *}\mathcal{F})(\mathscr{U}) := \Gamma((\mathscr{U}/S^\sharp)_{\mathrm{cris}}, \mathcal{F}).$$

We may shorten the notation $u^{\log}_{\mathscr{X}/S^\sharp}$ to u if there is no confusion.

Now we turn to the p-adic setting. Let $S^\sharp = (S, \mathcal{L}, I, \gamma)$ be an integral and quasi-coherent p-adic formal log pd-scheme, that is, a sequence of exact closed embeddings of integral and quasi-coherent log pd-schemes $(S^\sharp_n)_{n \in \mathbf{N}}$ such that $\mathcal{O}_{S_n} = \mathcal{O}_{S_{n+1}}/p^n$ and $\mathcal{I}_n = \mathcal{I}_{n+1}\mathcal{O}_{S_n}$. For an integral and quasi-coherent *log-$S^\sharp$-scheme* $(\mathscr{X}, M_{\mathscr{X}})$, i.e., an integral and quasi-coherent log-scheme over $S^\sharp_n$ for a suitable $n \in \mathbf{N}$, we define $((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}$ as the union of fully faithful embeddings of $(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp_n)_{\mathrm{cris}})_{n \in \mathbf{N}}$. We again have sheaves $\mathcal{O}_{\mathscr{X}/S^\sharp}$, $\mathcal{I}_{\mathscr{X}/S^\sharp}$, $\mathbf{G}_m^\natural$ and $M_{\mathscr{X}/S^\sharp}$ valued in condensed abelian groups (*resp.* condensed commutative monoids) with discrete topology. We define the condensed (p-adic) log-crystalline cohomology as

$$R\Gamma_{\mathrm{cris}}((\mathscr{X}, M_{\mathscr{X}})/S^\sharp) := \lim_{n \gg 0} R\Gamma_{\mathrm{cris}}(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp_n)_{\mathrm{cris}}, \mathcal{O}_{\mathscr{X}/S^\sharp_n}) \in \mathscr{D}(\mathrm{CondAb}).$$

Again, we have a canonical morphism of topoi $u^{\log}_{\mathscr{X}/S^\sharp}\colon ((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)^{\log,\sim}_{\mathrm{cris}} \to \mathscr{X}^\sim_{\mathrm{\acute{e}t}}$.

1.3.8. Condensed infinitesimal cohomology. We are going to put natural condensed structures on the infinitesimal cohomologies defined in [41].

Let us first consider the arithmetic case. Let Z be a rigid space over K and consider the infinitesimal site $(Z/K)_{\mathrm{inf}}$ whose objects are pairs (U, T) such that U is an open subset of Z with a closed nil-immersion $U \hookrightarrow T$ of rigid spaces over K, and whose coverings are open coverings. It has a structure sheaf $\mathcal{O}_{Z/K}\colon (U, T) \mapsto \Gamma(T, \mathcal{O}_T)$ with an ideal sheaf $\mathcal{I}_{Z/K}$. They are naturally promoted to sheaves valued in C_K, the category of locally convex topological K-vector spaces (1.1.7) (which lands furthermore in the subcategory of K-Fréchet spaces), and we further regard them as sheaves valued in $\mathrm{Mod}_K^{\blacksquare}$ via the condensification functor (1.1.7.1). We define the arithmetic condensed infinitesimal cohomology as

$$R\Gamma_{\mathrm{inf}}(Z/K) := R\Gamma((Z/K)_{\mathrm{inf}}, \mathcal{O}_{X/K}) \in \mathscr{D}(\mathrm{Mod}_K^{\blacksquare}).$$

It agrees with the éh-de Rham cohomology, which we denote by

$$R\Gamma_{\mathrm{dR}}(Z/K) := R\Gamma(Z_{\mathrm{\acute{e}h}}, \Omega^{\bullet}_{Z/K,\mathrm{\acute{e}h}}) \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{K}).$$

For qcqs Z, it is represented by a bounded complex of $\mathbf{Q}_p$-Fréchet spaces (*resp.*, of $\mathbf{Q}_p$-Banach spaces if Z is smooth).

Similarly in the geometric B^+_{dR}-case, for any rigid space Z over C, we have the infinitesimal site $(X/B^+_{\mathrm{dR}})_{\inf} = \bigcup_m (X/B^+_{\mathrm{dR},m})_{\inf}$, equipped with a structure sheaf $\mathcal{O}_{X/B^+_{\mathrm{dR}}}$ together with an ideal sheaf $\mathcal{I}_{X/B^+_{\mathrm{dR}}}$. They are naturally made to take values in $\mathrm{Mod}^{\blacksquare}_{B^+_{\mathrm{dR}}}$, so we can define the geometric condensed infinitesimal cohomology as

$$R\Gamma_{\inf}(X/B^+_{\mathrm{dR},m}) := R\Gamma((X/B^+_{\mathrm{dR},m})_{\inf}, \mathcal{O}_{X/B^+_{\mathrm{dR}}}) \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{B^+_{\mathrm{dR},m}})$$

and

$$R\Gamma_{\inf}(X/B^+_{\mathrm{dR}}) := \lim_n R\Gamma_{\inf}(X/B^+_{\mathrm{dR},m}).$$

Most results in *op. cit.* can be upgraded to condensed statements.

2 Arithmetic syntomic cohomology

It is unclear whether the original arithmetic Hyodo–Kato morphism [21, §4.2] for rigid-analytic varieties is compatible with the geometric Hyodo–Kato morphism [23, §2]. Here, we prefer to make alternatively an *ad hoc* definition of the arithmetic Hyodo–Kato morphism so that it satisfies this compatibility by definition. The key tool is the Galois cohomology computation (1.2.12, ii).

2.0.1. Semistable formal schemes.

For p-adic admissible formal schemes over $\mathcal{O}_K$ or $\mathcal{O}_C$, we denote by $(-)_\eta$ their rigid generic fibre in the sense of adic spaces.

A (p-adic) formal scheme $\mathfrak{Z}$ over $\mathcal{O}_K$ is called *strictly semistable* if, locally for the Zariski topology, it admits an étale morphism to a formal scheme of the form

$$\mathrm{Spf}\left(\mathcal{O}_K\langle X_0,\dots,X_m\rangle / (X_0\dots X_l - \varpi)\right), \quad 0 \le l \le m$$

for some uniformiser ϖ of $\mathcal{O}_K$. Let $L_{\mathfrak{Z}}$ be the integral closure of K in $\Gamma(\mathfrak{Z}_\eta, \mathcal{O}_{\mathfrak{Z}_\eta})$, called the *splitting field* of $\mathfrak{Z}_\eta$. If $\mathfrak{Z}$ is connected, then $L_{\mathfrak{Z}}$ is a field, and $\mathcal{O}_{L_{\mathfrak{Z}}}$ is the integral closure of $\mathcal{O}_K$ in $\Gamma(\mathfrak{Z}_\eta, \mathcal{O}_{\mathfrak{Z}_\eta})$, and $\mathfrak{Z}$ is strictly semistable over $\mathcal{O}_{L_{\mathfrak{Z}}}$; in general, $\mathfrak{Z}$ is locally connected and we regard $L_{\mathfrak{Z}}$ as attaching the splitting field to each connected component. Let $\mathcal{M}^{\mathrm{ss}}_K$ be the full subcategory of formal schemes over $\mathcal{O}_K$ that are strictly semistable over $\mathcal{O}_L$ for some finite extension L of K.

A (p-adic) formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$ is called *strictly semistable* if, locally for the Zariski topology, it admits an étale morphism to a formal scheme of the form

$$\mathrm{Spf}\left(\mathcal{O}_C\langle X_0,\dots,X_m\rangle / (X_0\dots X_l - \varpi)\right), \quad 0 \le l \le m$$

for some $\varpi \in \mathcal{O}_C\backslash\{0\}$. We denote by $\mathcal{M}^{\mathrm{ss}}_C$ the full subcategory of strictly semistable formal schemes over $\mathcal{O}_C$. Let $\mathcal{M}^{\mathrm{ss},b}_C$ be the full subcategory of formal schemes over $\mathcal{O}_C$ that are base change from a strictly semistable formal scheme over $\mathcal{O}_L$ for some finite extension L of K.

The arithmetic and the geometric setting are compatible by the base change functors

(2.0.1.1)
$$\begin{tikzcd} \mathcal{M}^{\mathrm{ss}}_K \arrow[r, "(-)_\eta"] \arrow[d, "(-)^{\mathrm{ss}}_{\mathcal{O}_C}"] & \mathrm{RigSm}_K \arrow[d, "-\otimes_K C"] \\ \mathcal{M}^{\mathrm{ss},b}_C \arrow[r, "(-)_\eta"] & \mathrm{RigSm}_C \end{tikzcd}$$

where the left vertical map is defined as[14]

$$\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C} := \coprod_{\sigma \in \mathrm{Hom}_K(L_{\mathfrak{Z}}, C)} \mathfrak{Z} \otimes_{\mathcal{O}_{L_{\mathfrak{Z}}}, \sigma} \mathcal{O}_C.$$

There is a natural $\mathscr{G}_K$-action on $\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C}$ by permuting the components $\sigma \mapsto g^{-1}\sigma$ and meanwhile acting on the coefficients $\mathcal{O}_C$; this is compatible with the $\mathscr{G}_K$-action on Z_C. One may replace $L_{\mathfrak{Z}}$ by any finite extension L/K such that $\mathfrak{Z}$ is semistable over L and obtain the same object.

Exceptionally, from now on in this article, we will abbreviate the term "strictly semistable" to simply "*semistable*".

2.0.2. Recall that the *éh-topology* [42, Definition 2.4.1] on the category Rig_K is the Grothendieck topology whose covering families are generated by étale coverings, universal homeomorphisms, and coverings associated to blow-ups $\mathrm{Bl}_Z(X) \amalg Z \to X$, where $X \in \mathrm{Rig}_K$ and Z is a closed analytic subset of X.

Recall also that a *base* (in the sense of Beilinson [5, 2.1]) for an essentially small site $\mathcal{V}$ is a pair $(\mathcal{B}, \phi)$, where $\mathcal{B}$ is an essentially small category and $\phi\colon \mathcal{B} \to \mathcal{V}$ is a *faithful* functor such that: for any $V \in \mathcal{V}$ and a finite family of pairs (B_α, f_α) with $B_\alpha \in \mathcal{B}$ and $f_\alpha\colon V \to \phi(B_\alpha)$, there exists a set of objects $B'_\beta \in \mathcal{B}$ and a covering family $\{\phi(B'_\beta) \to V\}$ such that every composition $\phi(B'_\beta) \to V \to \phi(B_\alpha)$ lies in $\mathrm{Hom}(B'_\beta, B_\alpha) \subset \mathrm{Hom}(\phi(B'_\beta), \phi(B_\alpha))$. We will refer to a base in the sense of Beilinson as a *Beilinson base*. We can unfold a presheaf on the base $\mathcal{B}$ to a hypersheaf on $\mathcal{V}$ by hyperdescent.

By Temkin's altered local uniformization [67], we see that the pair $(\mathcal{M}^{\mathrm{ss}}_K, (-)_\eta)$ is a Beilinson base for the site $\mathrm{Rig}_{K,\text{éh}}$, and $(\mathcal{M}^{\mathrm{ss},b}_C, (-)_\eta)$ is a Beilinson base for the site $\mathrm{Rig}_{C,\text{éh}}$ [11, Proposition 3.10]; indeed, by [21, Proposition 2.8] this is true for the étale topology on smooth rigid spaces, but éh-locally we have smoothness [42, Corollary 2.4.8]. We can thus unfold a presheaf on $\mathcal{M}^{\mathrm{ss}}_K$ (*resp.* on $\mathcal{M}^{\mathrm{ss},b}_C$) to a hypersheaf on $\mathrm{Rig}_{K,\text{éh}}$ (*resp.* on $\mathrm{Rig}_{C,\text{éh}}$) by hyperdescent; we shall refer to this process as the *η-éh-hypersheafification.* In order to obtain a reasonable sheaf, the presheaf should satisfy hyperdescent for sufficiently refined hypercoverings.

2.0.3 - Remark. When applying Temkin's results in [67], one should note that Temkin's definition of semistability is slightly more general than the usual semistability. A formal scheme over $\mathcal{O}_K$ is called *strictly semistable* (*resp. semistable*) *à la Temkin* if locally for the Zariski (*resp.* étale) topology, it admits an étale map to the standard semistable formal scheme

$$\mathfrak{S}_{n,r} := \mathrm{Spf}\left(\mathcal{O}_K\langle X_0, \ldots, X_n\rangle / (X_0 \cdots X_r - \varpi^n)\right), \quad 1 \le r \le n,$$

over $\mathcal{O}_K$, where ϖ is a uniformizer of K and $n \in \mathbf{N}$. In contrast, the usual (strict) semistability requires $n = 1$. These are different notions because the above standard semistable formal scheme is not regular if $n \ge 2$.

Nevertheless, when we work locally for the *η-étale topology* (i.e. whose coverings are those families that become étale coverings on the generic fibres), we can always assume $n = 1$ by further localisation. Indeed, we may proceed as in the proof of [67, Lemma 2.4.1], in particular the step 2 there: consider $f_i\colon \mathfrak{S}_{n,r} \to \mathfrak{S}_{n-1,r}$ induced by

$$X_j \mapsto \begin{cases} \pi X_i, & j = i, \\ X_j, & j \neq i. \end{cases}$$

Temkin's argument *loc. cit.* allows to show that $\{f_0, \ldots, f_n\}$ form an η-étale covering.

2.0.4. Dagger varieties. Roughly speaking, a dagger variety (or dagger rigid space) over a non-archimedean field L is a rigid space over L together with an overconvergent structure sheaf, namely $X = (\widehat{X}, \mathcal{O}^\dagger_X)$; cf. [38] for basic definitions and properties. A presentation of a dagger affinoid rigid space U is a prosystem $(U_h)_{h\in\mathbf{N}}$ with U_h affinoid rigid spaces such that U and U_h are rational subspaces of U_0, that $U \subset^\dagger U_{h+1} \subset^\dagger U_h$. This system is coinitial among all rational subspaces of U_0 strictly containing X. The set of such presentations is non-empty and cofiltered. For any presentation $(U_h)_{h\in\mathbf{N}}$ of a dagger affinoid U, we have $\Gamma(U, \mathcal{O}^\dagger_U) = \varinjlim_h \Gamma(U_h, \mathcal{O}_{U_h})$, which is a countable filtered colimit of Banach spaces.

2.0.5. Condensed (φ, N)-modules. Let L be the absolutely unramified p-adic local field with perfect residue

[14] Beware that $\mathfrak{Z}_{\mathcal{O}_C} := \mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{O}_C$ is a formal model of Z_C, but is not semistable over $\mathcal{O}_C$ in general, hence $R\Gamma_{\mathrm{HK}}(Z_C)$ is not isomorphic to $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\mathcal{O}_C,1}/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p}$.

field l, i.e. $L = W(l)[\frac{1}{p}]$, which is canonically equipped with a Frobenius action σ_L. We are going to recall the definition of condensed (φ, N)-modules and their derived cousins, following the presentation in [11, §3.3.1].

A *condensed* (φ, N)*-module* over L is a tripe (V, φ, N) where $V \in \mathrm{Mod}_L^{\mathrm{cond}}$ is equipped with a σ_L-semilinear automorphism φ, called *Frobenius*, and N, called *monodromy operator*, is an L-linear endomorphism of V satisfying $N\varphi = p\varphi N$. The condensed (φ, N)-modules over L form an abelian category, whose derived ∞-category will be denoted by $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_L^{\mathrm{cond}})$.

Alternatively, the condensed (φ, N)-modules over L can be described as condensed left $L\{\varphi^{\pm 1}, N\}$-modules. And objects in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_L^{\mathrm{cond}})$ can be regarded as objects $M \in \mathscr{D}(\mathrm{Mod}_L^{\mathrm{cond}})$ endowed with a σ_L-semilinear automorphism φ, called *Frobenius*, and with an L-linear endomorphism N, called *monodromy operator*, satisfying the relation $N\varphi = p\varphi N$.

For example, for $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ semistable over $\mathcal{O}_L$, we have $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \in \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{F_L}^{\mathrm{cond}})$, where $\mathfrak{Z}_1^0$ is the pullback of $\mathfrak{Z}$ endowed with the canonical log-structure along $\mathcal{O}_L^\times \to \mathcal{O}_{L,1}^\times \to \mathcal{O}_{F_L}^0$; for $\mathfrak{X} \in \mathcal{M}_C^{\mathrm{ss},b}$, we have $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}_{\breve{F}}^0)_{\mathbf{Q}_p} \in \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{\breve{F}}^{\mathrm{cond}})$, where $\mathfrak{X}_1^0$ is the pullback of $\mathfrak{X}$ endowed with the canonical log-structure along $\mathcal{O}_C^\times \to \mathcal{O}_{C,1}^\times \to \mathcal{O}_{\breve{F}}^0$. These (φ, N)-structure are naturally defined [44, 6].

By restricting ourselves to solid underlying objects, one obtains the derived ∞-category $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_L^{\blacksquare})$. Similarly, one may also consider only the φ-action or the N-action.

2.0.6. Let $M \in \mathscr{D}(\mathrm{CondAb})$ with an N-action. We define

$$M[N^{-1}] := \mathrm{colim}(M \xrightarrow{N} M \xrightarrow{N} \cdots), \quad M^{N\text{-nilp}} := \mathrm{fib}(M \to M[N^{-1}]).$$

In particular, if M is N-nilpotent with finite nilpotency order, then we have $M[N^{-1}] = 0$ and the natural map $M^{N\text{-nilp}} \to M$ becomes a quasi-isomorphism.

2.1 Arithmetic and geometric Hyodo–Kato morphisms

We shall address the compatibility between arithmetic and geometric Hyodo–Kato morphisms previous defined by Colmez and Nizioł [21, 23]. We shall (re)define the arithmetic Hyodo–Kato morphism for rigid-analytic varieties Z over K. By hyperdescent strategy, we are reduced to the case of affine Z with semistable model, where the arithmetic Hyodo–Kato morphism is going to be defined via Galois descent from the geometric Hyodo–Kato morphism; the latter is defined as in [23].

2.1.1. Condensed Galois action on log-crystalline cohomology. Consider a log-scheme $\mathscr{X}^0$ over $\mathrm{Spec}\, \mathcal{O}_{\breve{F},1}^0$ (*resp.* $\mathscr{X}$ over a $\mathrm{Spec}\, \mathcal{O}_{C,1}^\times$) which descends to a log-smooth integral map of fine log-schemes $\mathscr{Z}^0 \to \mathcal{O}_{F_L,1}^0$ (*resp.* $\mathscr{Z} \to \mathcal{O}_{L,1}^\times$). By functoriality of log-crystalline cohomology, the abstract absolute Galois group $\mathscr{G}_L$ acts on $R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\overline{S}_1^0 \hookrightarrow \overline{S}^0)$ and on $R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris}}^\times))$. On the other hand, by base change along $\mathcal{O}_{F_L}^0 \to \mathcal{O}_{\breve{F}}^0$, thanks to the assumption of $\mathscr{Z}^0 \to \mathcal{O}_{F_L,1}^0$ being log-smooth integral [6, 1.11, Theorem, (i)], we have canonical isomorphisms

$$R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_{\breve{F}}^0) \simeq R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{\mathcal{O}_{F_L}} \mathcal{O}_{\breve{F}}, \quad R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris}}^\times) \simeq R\Gamma_{\mathrm{cris}}(\mathscr{Z}^\times/\mathcal{O}_L^\times) \otimes^{\blacksquare}_{\mathcal{O}_L} A_{\mathrm{cris}}.$$

Actually, the $\mathscr{G}_L$-action comes from that on the coefficients $\mathcal{O}_{\breve{F}}$ (*resp.* A_{cris}), which is continuous for their natural topologies, hence it is upgraded to a condensed group action.

Alternatively, we may have started by defining the $\underline{\mathscr{G}_L}$-action on log-crystalline cohomology of qcqs $\mathscr{X}$ and $\mathscr{X}^0$ as above. These $\mathscr{G}_L$-actions come from smooth $\underline{\mathscr{G}_L}$-actions respectively on discrete $R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_{\breve{F},n}^0)$ and $R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris},n}^\times)$ for $n \geq 1$, which are uniquely upgraded to condensed actions by (1.2.27). Taking limits and globalising, we can pass to the p-adic setting and drop the qcqs condition. This way, we can also define condensed Galois action for more general base pd log-schemes than $\mathcal{O}_{\breve{F}}^0$ and $\mathcal{O}_C^\times$ (for example, A_{cris} and $\widehat{A}_{l,\mathrm{st}}$, see later for the notation).

2.1.2. For fine log-schemes $\mathscr{U}$ log-smooth and locally of finite type over $\mathcal{O}_{F,1}^0$, the (condensed) log-crystalline cohomology $R\Gamma_{\mathrm{cris}}(\mathscr{U}/\mathcal{O}_F^0)$ is equipped with a natural monodromy operator N by [44, (3.6)].

2.1.2.1 - Lemma. *Let $\mathscr{Z}$ be a quasi-separated, fs (fine and saturated) log-scheme log-smooth and locally of finite type over $\mathcal{O}_K^\times$ of relative dimension d. Then the monodromy operator N on $R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}_F^0)$ is nilpotent with order bounded above by a function depending only on d.*

Proof. This is [11, Lemma 3.3], whose proof we include here for completeness. By hypothesis, the fs log-scheme $\mathscr{X}$ is log-regular, so by a logarithmic resolution of singularities [58, Theorem 5.10] implies that there exists a log-blow-up map $\mathscr{Y} \to \mathscr{X}$ over $\mathcal{O}_K^\times$ from a classically regular log-scheme $\mathscr{Y}$ that induces an isomorphism over the regular locus of $\mathscr{X}$. According to (the proof of) [59, Proposition 2.3], we have a natural quasi-isomorphism $R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_F^0) \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathscr{Y}^0/\mathcal{O}_F^0)$ compatible with the monodromy N. Then we conclude by applying Mokrane's nilpotency result [54, corollaire 3.19] to $R\Gamma_{\mathrm{cris}}(\mathscr{Y}^0/\mathcal{O}_F^0)$. ◻

2.1.3 - Theorem (Beilinson, Colmez–Nizioł, Bosco). *Let $\mathscr{X}$ be a qcqs fine log-smooth log-scheme over $\mathcal{O}_{L,1}^\times$ of Cartier type for some finite extension L/K. Let $\mathcal{X} = \mathscr{X} \otimes_{\mathcal{O}_{L,1}^\times} \mathcal{O}_{C,1}^\times$ and $\mathcal{X}^0 = \mathcal{X} \otimes_{\mathcal{O}_{C,1}^\times} \mathcal{O}_{\breve{F},1}^0$.*

(i) *In $\mathscr{D}(\mathrm{Mod}_{B_{\mathrm{st}}}^{\blacksquare})$, there exists a natural equivalence*

$$\varepsilon_{\mathrm{HK}}^{\mathrm{st}}\colon R\Gamma_{\mathrm{cris}}(\mathcal{X}^0/\mathcal{O}_{\breve{F}}^0) \otimes_{\mathcal{O}_{\breve{F}}}^{\blacksquare} B_{\mathrm{st}}^+ \simeq R\Gamma_{\mathrm{cris}}(\mathcal{X}/A_{\mathrm{cris}}^\times) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} B_{\mathrm{st}}^+,$$

which is independent of the choice of the descent $\mathscr{X}$ of $\mathcal{X}$, and is compatible with the $\underline{\mathscr{G}_L}$-action, the Frobenius φ and the monodromy operators N.

(ii) *In $\mathscr{D}(\mathrm{Mod}_C^{\blacksquare})$, there exists a natural equivalence*

$$\varepsilon_{\mathrm{dR}}^{\mathrm{st}}\colon R\Gamma_{\mathrm{cris}}(\mathcal{X}^0/\mathcal{O}_{\breve{F}}^0) \otimes_{\mathcal{O}_{\breve{F}}}^{\blacksquare} C \simeq R\Gamma_{\mathrm{cris}}(\mathcal{X}/\mathcal{O}_C^\times)_{\mathbf{Q}_p},$$

which is independent of the choice of the descent $\mathscr{X}$ of $\mathcal{X}$, compatible with the $\underline{\mathscr{G}_L}$-action and the Frobenius φ, and compatible with the previous equivalence via Fontaine's map $\theta\colon A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C$.

Proof. It has been treated in [23, Theorem 2.22, Corollary 2.31] and [11, Theorem 3.2]. Only the condensed $\underline{\mathscr{G}_L}$-equivariance needs an explanation. For this, we need to show that the natural maps in the construction of $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$ are $\underline{\mathscr{G}_L}$-equivariant.

Let us briefly explain the construction of $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$. For sufficiently large n, we have factorisations of Frob^n:

$$\mathrm{Frob}^n\colon \mathscr{X} \xrightarrow{F_n} \mathscr{X}^0 \hookrightarrow \mathscr{X}$$

$$\mathrm{Frob}^n\colon \mathscr{X}^0 \hookrightarrow \mathscr{X} \xrightarrow{F_n} \mathscr{X}^0.$$

Consider the commutative diagram of log-schemes

$$\begin{tikzcd} \mathcal{X}^0 \arrow[r, hook] \arrow[d, "\theta^0_{\mathscr{X}}"] & \mathcal{X}_1 \arrow[r] \arrow[d, "F_n\theta_{\mathscr{X}}"] & \overline{S}_1^\times \arrow[d, "F_{L,n}\theta_L"] \\ \mathscr{X}^0 \arrow[r, "\mathrm{Frob}^n"] & \mathscr{X}^0 \arrow[r] & S_{L,1}^0 \end{tikzcd} \tag{2.1.3.1}$$

which we shall denote by π. These data π determine a lifting $i_{l_\pi}^*\colon r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}_{L,1}^\times$ as well as a log pd-thickening $\widehat{A}_{l_\pi,\mathrm{st}} \twoheadrightarrow \mathcal{O}_{C,1}^\times$. Consider the commutative diagram

(2.1.3.2)

$$\begin{tikzcd} R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_{F_L,n}^0) \arrow[d, "(\mathrm{Frob}^n)^*"] & R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/r_{L,n}^{\mathrm{PD},0}) \arrow[l, "p_0^*"'] \arrow[drr, "\pi^*"] & & \\ R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_{F_L,n}^0) & & & R\Gamma_{\mathrm{cris}}(\mathcal{X}/\widehat{A}_{l_\pi,\mathrm{st}}) & R\Gamma_{\mathrm{cris}}(\mathcal{X}/A_{\mathrm{cris}}^\times) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} \widehat{A}_{l_\pi,\mathrm{st}}. \arrow[l, "\kappa_{l_\pi}^*"', "\simeq"] \end{tikzcd}$$

After taking limits and then passing to the isogeny category, $(\mathrm{Frob}^n)^*$ becomes invertible [23, Theorem 2.12 (1)], and p_0^* admits a unique natural (φ, N)-equivariant $\mathcal{O}_{F_L}$-linear section ι_0 in the isogeny category, called the *Hyodo–Kato section* [23, Theorem 2.12 (2), Proposition 2.14]; moreover, $(R\Gamma_{\mathrm{cris}}(\mathcal{X}/A_{\mathrm{cris}}^\times) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} \widehat{A}_{l_\pi,\mathrm{st}})_{\mathbf{Q}_p}^{N\text{-nilp}} \simeq$

$R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}}$. Recall also that $R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}^0_{F_L})$ is N-nilpotent (2.1.2). Altogether we get
(2.1.3.3)

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[d, "\simeq"', "(\mathrm{Frob}^n)^*"] \arrow[r, bend right, "\iota_0"'] & R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/r_L^{\mathrm{PD},0})^{N\text{-nilp}}_{\mathbf{Q}_p} \arrow[l, "p_0^*"'] \arrow[dr, "\pi^*"] & \\
R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} & & R\Gamma_{\mathrm{cris}}(\mathscr{X}/\widehat{A}_{l_\pi,\mathrm{st}})^{N\text{-nilp}}_{\mathbf{Q}_p} \xleftarrow[\simeq]{\kappa^*_{l_\pi}} R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}}.
\end{tikzcd}$$

The morphism $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$ is then defined as the B^+_{st}-linearisation of the composition from the lower left corner to the right end. This morphism is independent of the choice of sufficiently large n and eventually also independent of the choice of the descent $\mathscr{Z}$ of $\mathscr{X}$.

Regarding the condensed $\underline{\mathscr{G}_L}$-equivariance of $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$, one only needs to prove it for each morphism above. Since the action is nontrivial only on the cohomology of $\mathscr{X}$, we only need to verify the equivariance of π^* and $\kappa^{*,-1}_{l_\pi}$. But they are either the base change morphism along a $\mathscr{G}_L$-equivariant morphism or the inverse of one such, so we are done by the second description of the condensed $\underline{\mathscr{G}_L}$-action in (2.1.1). □

2.1.4. Local arithmetic Hyodo–Kato morphism. Let $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$ be a qcqs semistable formal scheme over $\mathcal{O}_L$ for some finite extension L/K. Let $\mathfrak{X}$ be the base change of $\mathfrak{Z}$ from $\mathrm{Spf}(\mathcal{O}_L)$ to $\mathrm{Spf}(\mathcal{O}_C)$. We have the following $\underline{\mathscr{G}_L}$-equivariant commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}}) \otimes^{\blacksquare}_{\mathcal{O}_{\breve{F}}} B^+_{\mathrm{st}} \arrow[d, "\simeq"', "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}"] \arrow[dddd, bend left=60, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \\
& R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[d] \\
& R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}} \arrow[d, "\simeq"] \\
& R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}}) \\
& R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{dR}} \arrow[u, "\simeq"] \\
& R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{dR}} \arrow[u, phantom, "\wr"] \arrow[d] \\
& R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}},
\end{tikzcd}$$

where $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})$ and $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L)$ are equipped with the trivial $\mathscr{G}_L$-action.

We define the *local arithmetic Hyodo–Kato morphism* as the composite map

$$\iota^{\mathrm{arith}}_{\mathrm{HK}} \colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \to (R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}})^{\underline{\mathscr{G}_L}} \xleftarrow{\simeq} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L),$$

where the last isomorphism follows from (1.2.12, ii), since $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L)$ is represented by a bounded complex of L-Banach spaces, By $\mathbf{Q}_p$-linearising the above map, we define, by abuse of notation and terminology, the *($\mathbf{Q}_p$-linearised) local arithmetic Hyodo–Kato morphism* as the map

$$\iota^{\mathrm{arith}}_{\mathrm{HK}} \colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L).$$

2.1.5 - Remark. There is a certain independence of $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ on the embedding $\sigma \colon L \hookrightarrow C$. For this, consider two embeddings $\sigma, \tau \colon L \hookrightarrow C$. There exists a $\gamma \in \mathscr{G}_K$ such that $\sigma = \gamma \circ \tau$. Consider $\mathfrak{X}_\sigma := \mathfrak{Z} \otimes_{L,\sigma} L$ and

$\mathfrak{X}_\tau := \mathfrak{Z} \otimes_{L,\tau} C$. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{X}_\sigma \arrow[r, "f_\gamma"] \arrow[d] & \mathfrak{X}_\tau \arrow[d] \\ \operatorname{Spf} \mathcal{O}_C \arrow[r, "\gamma"] & \operatorname{Spf} \mathcal{O}_C \end{tikzcd}$$

hence a commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\sigma"] \arrow[d, equal] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{F_L,\sigma} \breve{F} \arrow[r, "\sim"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\sigma,1}^0/\mathcal{O}_{\breve{F}^0}) \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_{\sigma,\eta}/B_{\mathrm{dR}}^+) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{pdR}}^+ \\ R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\tau"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{F_L,\sigma} \breve{F} \arrow[r, "\sim"] \arrow[u, "\mathrm{id}\otimes\gamma", "\simeq"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\tau,1}^0/\mathcal{O}_{\breve{F}^0}) \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \arrow[u, "f_\gamma^*", "\simeq"'] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_{\tau,\eta}/B_{\mathrm{dR}}^+) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{pdR}}^+. \arrow[u, "f_\gamma^*\otimes\mathrm{id}", "\simeq"'] \end{tikzcd}$$

which is $\underline{\mathscr{G}_L}$-equivariant (with respect to the $\mathscr{G}_{\sigma(L)}$-action on the top row and the $\mathscr{G}_{\tau(L)}$-action on the bottom row; noticing that $\mathscr{G}_{\sigma(L)} = \gamma\mathscr{G}_{\tau(L)}\gamma^{-1}$, so we have $\underline{\gamma\mathscr{G}_{\tau(L)} = \mathscr{G}_{\sigma(L)}\gamma}$ showing equivariance). Taking $\underline{\mathscr{G}_L}$-invariants respectively, we obtain a commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\iota_{\mathrm{HK},\sigma}^{\mathrm{arith}}"] \arrow[d, equal] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \\ R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\iota_{\mathrm{HK},\tau}^{\mathrm{arith}}"] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L). \arrow[u, "f_\gamma^*", "\simeq"'] \end{tikzcd}$$

which depends only on the restriction of γ on the Galois closure of the extension L/K.

2.1.6 - Lemma. *The local arithmetic Hyodo–Kato morphism $\iota_{\mathrm{HK}}^{\mathrm{arith}}$ is natural in $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$.*

Proof. One goes formally through the construction of $\iota_{\mathrm{HK}}^{\mathrm{arith}}$, eventually using (2.1.5). □

2.1.7 - Proposition. *For $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ a qcqs semistable formal scheme over $\mathcal{O}_L$ and $\mathfrak{X} := \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C$, there is a natural $\underline{\mathscr{G}_L}$-equivariant commutative diagram*

$$\begin{tikzcd} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{arith}}"] \arrow[d] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \arrow[d] \\ R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}_{\breve{F}}^0) \otimes^{\blacksquare}_{\mathcal{O}_{\breve{F}}} B_{\mathrm{st}}^+ \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+) \end{tikzcd}$$

exhibiting the compatibility between local arithmetic and geometric Hyodo–Kato morphisms.

Proof. This follows from the construction of the upper map (2.1.4). □

2.1.8 - Proposition. *If $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ be qcqs and semistable of relative dimension d over $\mathcal{O}_L$, then $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ lies in the subcategory $\mathrm{Nuc}_{F_L} \subset \mathscr{D}(\mathrm{Mod}_{F_L}^{\blacksquare})$, and lies in $\mathscr{D}^{[0,2d]}(\mathrm{Mod}_{F_L}^{\blacksquare})$. In particular, it is represented by a bounded below complex of solid-nuclear F_L-vector spaces, and for any $i \in \mathbf{Z}$, we have $H^i_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \in \mathrm{Mod}_{F_L}^{\mathrm{nuc}}$.*

Moreover, for any $i \in \mathbf{Z}$, the condensed F_L-vector space $H^i_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ is a direct summand of the cokernel of a morphism between F_L-Banach spaces.

Proof. It is essentially the same proof as in [11, Theorem 3.15 (ii)]. Let us sketch it. The isomorphism $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$ [15] induces by base change to B_{dR}^+ an $\underline{\mathscr{G}_L}$-equivariant isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{\mathcal{O}_{F_L}} B_{\mathrm{dR}}^+ \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B_{\mathrm{dR}}^+.$$

[15] Alternatively, we can also use the original arithmetic Hyodo–Kato isomorphism [21, 4.2.3 (i)] for the proof.

The right hand side is represented by a bounded complex of F_L-Banach spaces (as can be seen by analytic descent to affinoids) and lies in $\mathscr{D}^{[0,2d]}(\mathrm{Mod}^{\blacksquare}_{F_L})$ (as can be seen from the Hodge-to-de Rham spectral sequence in the éh-topology and the vanishing $H^i(X,\Omega^j)=0$ when $i>d$ or $j>d$ [42, Theorem 6.0.2]), in particular it lies in Nuc_{F_L}. On the left, we have a splitting via $B^+_{\mathrm{dR}}\simeq F_L\oplus M$ for some F_L-Banach space M, whence $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$, being a direct summand of the right hand side, also lies in Nuc_{F_L} by (1.1.8, ii) and belongs to $\mathscr{D}^{[0,2d]}(\mathrm{Mod}^{\blacksquare}_{F_L})$. Now, the second to last statement follows from the properties of nuclear complexes (1.1.8, ii), and the last statement is a consequence of the above splitting. □

2.1.9 - Proposition. *The local arithmetic Hyodo–Kato morphism $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ induces, after L-linearisation, an isomorphism*

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}\otimes_{F_L}L\xrightarrow{\simeq}R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L).$$

Proof. The isomorphism $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$ induces an $\underline{\mathscr{G}_L}$-equivariant isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}\otimes^{L\blacksquare}_{F_L}B^+_{\mathrm{pdR}}\simeq R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L)\otimes^{L\blacksquare}_{L}B^+_{\mathrm{pdR}}.$$

Since $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$ lies in Nuc_{F_L} (2.1.8), we may take $\underline{\mathscr{G}_L}$-invariants and apply (1.2.12) to conclude. □

2.1.10 - Remark. Notice that the scalar extension $-\otimes_{F_L}L$ in the proposition depends on the semistable model, and cannot be replaced by $-\otimes_F K$ which depends only on our given base field. More explicitly, for $\mathfrak{Z}\in\mathcal{M}^{\mathrm{ss}}_K$ semistable over $\mathcal{O}_L$, one has the following commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}\otimes_{F_L}L \arrow[r, "\simeq"] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}\otimes_{F}K \arrow[r] \arrow[u, hook, "\oplus"] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K). \arrow[u, "\simeq"']
\end{tikzcd}$$

The left vertical map is induced by $L_{0,K}:=F_L\otimes_F K\hookrightarrow F_L\otimes_{F_L}L$, which is the inclusion of the maximal unramified subextension of L/K into L; the latter admits a canonical retract being the normalised trace map $r^{\mathrm{Gal}}_{L/K}:=\overline{\mathrm{Tr}}_{L/L_{0,K}}:=\frac{1}{|\mathrm{Gal}(L/L_{0,K})|}\sum_{g\in\mathrm{Gal}(L/L_{0,K})}g$; hence it is a direct factor, and is an isomorphism if and only if L/K is an unramified extension.

2.1.11. Arithmetic Hyodo–Kato cohomology. We define $R\Gamma_{\mathrm{HK}}\colon\mathrm{Rig}_K\to\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_F)$ as the η-éh-hypersheafification (2.0.2) of the presheaf $R\Gamma^{\mathrm{pre}}_{\mathrm{HK}}\colon\mathfrak{Z}\mapsto R\Gamma(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_{L_{\mathfrak{Z}}}})_{\mathbf{Q}_p}$ on $\mathcal{M}^{\mathrm{ss}}_K$ valued in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_F)$.

2.1.12 - Proposition (Local-global compatibility). *For $\mathfrak{Z}\in\mathcal{M}^{\mathrm{ss}}_K$, the natural morphism*

$$R\Gamma_{\mathrm{HK}}(\mathfrak{Z}_\eta)\to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})$$

is an isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_F)$.

Proof. Let us recall the proof of [21, Proposition 4.11] (cf. [56, Proposition 3.18]). One needs to show that $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_{L_{\mathfrak{Z}}}})$ satisfies η-éh-hyperdescent for $\mathfrak{Z}\in\mathcal{M}^{\mathrm{ss}}_K$ (up to refinement of the hypercovering), admitting the fact that $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)$ does so. For this, we may assume $\mathfrak{Z}$ is qcqs with splitting field K, and suppose that $\mathfrak{Z}_\bullet$ is an affine η-éh-hypercovering of $\mathfrak{Z}$, with respective splitting field $L_\bullet$, all finite Galois over K.

Let $k\in\mathbf{N}$. Let E/K be a finite Galois extension containing all $L_{\bullet\le k+1}$, and let $G=\mathrm{Gal}(E/K)$. Consider $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_{\bullet\le k+1}}}_{\bullet\le k+1,\mathcal{O}_E}$; this is a $(k{+}1)$-truncated[16] η-éh-hypercovering of $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_K}_{\mathcal{O}_E}$, whose generic fibre is Z_E, with each member having splitting field $\mathcal{O}_E$; moreover there is a natural G-action on $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_{\bullet\le k+1}}}_{\bullet\le k+1,\mathcal{O}_E}$ making this hypercovering G-equivariant and compatible with the Galois action on Z_E. Here, we denoted

$$\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_L}_{\mathcal{O}_E}:=\coprod_{\sigma\in\mathrm{Hom}_K(L,E)}\mathfrak{Z}\otimes_{\mathcal{O}_L,\sigma}\mathcal{O}_E, \tag{2.1.12.1}$$

and let $g\in G$ act on it by permuting indices $\sigma\mapsto g^{-1}\circ\sigma$ and at the same time acting on the coefficients $\mathcal{O}_E$. It can be extended to an entire G-equivariant hypercovering $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_\bullet}}_{\bullet,\mathcal{O}_E}$. We have a bisimplicial object

[16] Here, being *n-truncated* means that it is defined on the finite subcategory Δ_n of Δ [28, §5.1].

$\mathfrak{Z}'_{\bullet,\circ} := \coprod_{G^{\times\circ}} \mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_\bullet}}_{\bullet,\mathcal{O}_E}$, the faces maps being evidently forgetting components of G except one being acting on $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_L}_{\mathcal{O}_E}$ via $\sigma \mapsto g^{-1}\circ\sigma$ on the index set (2.1.5)[17]. Its diagonal is an affine η-éh-hypercovering of $\mathfrak{Z}_\eta$ by semistables with splitting field $\mathcal{O}_E$ refining $\mathfrak{Z}_\bullet$. We have compatible isomorphisms

$$R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet\le k+1,\circ})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} \otimes_{F_E} E \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}((\mathfrak{Z}'_{\bullet\le k+1,\circ})_\eta/E)$$

by local arithmetic Hyodo–Kato isomorphism (2.1.9) since every formal schemes here are semistable over $\mathcal{O}_E$. Taking limit over the index $\bullet$, since $\mathfrak{Z}'_{\bullet,\circ}$ forms an η-éh-hypercovering of $\coprod_{G^{\times\circ}} \mathfrak{Z}\otimes_{\mathcal{O}_K}\mathcal{O}_E$, we obtain

$$\begin{aligned}\tau^{\le k}\lim_{[n]\in\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{[n],\circ})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p}\otimes_{F_E} E &\simeq \tau^{\le k}\lim_{[n]\in\Delta_{k+1}} R\Gamma_{\mathrm{dR}}((\mathfrak{Z}'_{[n],\circ})_\eta/E)\\ &\simeq \tau^{\le k}\prod_{G^{\times\circ}} R\Gamma_{\mathrm{dR}}((\mathfrak{Z}\otimes_{\mathcal{O}_K}\mathcal{O}_E)_\eta/E)\\ &\simeq \tau^{\le k}\prod_{G^{\times\circ}} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)\otimes_K E\\ &\simeq \tau^{\le k}\prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}\otimes_{\mathcal{O}_K}\mathcal{O}_E)^0_1/\mathcal{O}^0_E)_{\mathbf{Q}_p}\otimes_{F_E} E,\end{aligned}$$

inside which we identify the F_E-linear isomorphism

$$\begin{aligned}\tau^{\le k}\lim_{[n]\in\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{[n],\circ})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} &\simeq \tau^{\le k}\prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}\otimes_{\mathcal{O}_K}\mathcal{O}_E)^0_1/\mathcal{O}^0_E)_{\mathbf{Q}_p}\\ &\simeq \tau^{\le k}\prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_F)_{\mathbf{Q}_p}\otimes_F F_E\end{aligned}$$

compatible with the residual simplicial structures.

Next, since the diagonal of a bisimplicial set calculates the total realisation, we have

$$\begin{aligned}\tau^{\le k}\lim_{\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet\le k+1,\bullet\le k+1})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p}\otimes_F K &\xleftarrow{\simeq} \tau^{\le k}\lim_{\Delta_{k+1}\times\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet\le k+1,\circ\le k+1})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p}\otimes_F K\\ &\simeq \tau^{\le k}\lim_{\Delta_{k+1}}\prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p}\otimes_F F_E\otimes_F K\\ &\simeq \tau^{\le k} R\Gamma(G, R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p}\otimes_F F_E\otimes_F K)\\ &\xleftarrow{\simeq} \tau^{\le k} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p}\otimes_F K\end{aligned}$$

The last isomorphism follows from the observation that due to finiteness of G, the condensed group cohomology agrees with the smooth one, and can be computed pointwise; since these groups are $\mathbf{Q}$-vector spaces, this simplifies to pointwise and termwise genuine G-fixed points. Finally, we extract from it the isomorphism

$$\tau^{\le k} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p} \xrightarrow{\simeq} \tau^{\le k}\lim_{\Delta} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet,\bullet})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p}$$

showing the k-truncated η-éh-hyperdescent for the hypercovering $\mathfrak{Z}'_{\bullet,\bullet}$ that refines $\mathfrak{Z}_\bullet$. □

2.1.13. Global arithmetic Hyodo–Kato morphism. Let $Z\in\mathrm{Rig}_K$ and $\mathfrak{Z}_\bullet$ be an η-éh hypercovering of Z. We define the *global arithmetic Hyodo–Kato morphism* as

$$\iota^{\mathrm{arith}}_{\mathrm{HK}} := \lim_{\Delta}\iota^{\mathrm{arith}}_{\mathrm{HK},\bullet}\colon R\Gamma_{\mathrm{HK}}(Z)\simeq \lim_{\Delta} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\bullet,1}/\mathcal{O}^0_{F_{L_\bullet}})\to \lim_{\Delta} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_{\bullet,\eta}/L_\bullet)\xleftarrow{\simeq} R\Gamma_{\mathrm{dR}}(Z/K)$$

or by abuse of notation as the simplicial limit of $\mathbf{Q}_p$-linearised Hyodo–Kato morphisms. This can be made independent of the chosen η-étale hypercovering by taking colimit over all possible such coverings, which form a filtered system[18].

2.1.14 - Lemma. *For $Z\in\mathrm{Rig}_K$, the K-linearised Hyodo–Kato morphism $\iota^{\mathrm{arith}}_{\mathrm{HK}}\otimes\mathrm{id}$ has a canonoical natural K-linear retract*

$$r_Z\colon R\Gamma_{\mathrm{dR}}(Z/K)\to R\Gamma_{\mathrm{HK}}(Z)\otimes_F K.$$

[17] The intuition comes from the decomposition $E^{\otimes_K(m+1)}\simeq\prod_{G^m} E, x_0\otimes\cdots\otimes x_m\mapsto (x_0\cdot g_1(x_1)\ldots g_m(x_m))_{g\in G^m}$ and the interpretation of face maps.

[18] The category of such coverings up to homotopy is cofiltered. In fact, the ∞-category of such coverings is cofiltered.

It is an isomorphism if Z has a η-éh-hypercovering by semistable formal schemes over $\mathcal{O}_K$.

Proof. Assume first that Z has an η-éh-hypercovering by semistable formal schemes $\mathfrak{Z}_\bullet$ over $\mathcal{O}_L$ for some finite unramified extension L/K; we may assume $\mathfrak{Z}_\bullet$ to be qcqs (even affine). By éh-hyperdescent, local-global compatibility (2.1.12) and local Hyodo–Kato morphism (2.1.9)[19], we have isomorphisms

$$\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id}\colon R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \simeq \lim_{\Delta} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\bullet,1}/\mathcal{O}_F^0)_{\mathbf{Q}_p} \otimes_F K \xrightarrow{\simeq} \lim_{\Delta} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_{\bullet,\eta}/K) \xleftarrow{\simeq} R\Gamma_{\mathrm{dR}}(Z/K).$$

Hence there is a caonical retract $r_Z := (\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id})^{-1}$. Taking one step back, if $\mathfrak{Z}_\bullet$ are semistable over $\mathcal{O}_K$ only for degrees $\bullet \leq k+1$, then we get a canonical k-truncated retract $r_Z^{\leq k}$ of $\tau^{\leq k}(\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id})\colon \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \to \tau^{\leq k} R\Gamma_{\mathrm{dR}}(Z/K)$. This definition does not depend on the choice of the (truncated) éh-hypercovering, hence it is canonical; moreover, such retract is compatible with base change, i.e. for any finite Galois extension E/K, the rigid space $Z_E \in \mathrm{Rig}_E$ satisfying the same reduction condition as $Z \in \mathrm{Rig}_K$, there is a canonical $\mathrm{Gal}(E/K)$-equivariant commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \arrow[d] & R\Gamma_{\mathrm{dR}}(Z/K) \arrow[l, "r_Z"', "\simeq"] \arrow[d] \\ R\Gamma_{\mathrm{HK}}(Z_E) \otimes_{F_E} E & R\Gamma_{\mathrm{dR}}(Z_E/E). \arrow[l, "r_{Z_E}"', "\simeq"] \end{tikzcd}$$

There are also normalised trace maps retracting the vertical maps by étale descent along $Z_E \to Z$, giving again a canonical $\mathrm{Gal}(E/K)$-equivariant commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{HK}}(Z) \otimes_F K & R\Gamma_{\mathrm{dR}}(Z/K) \arrow[l, "r_Z"', "\simeq"] \\ R\Gamma_{\mathrm{HK}}(Z_E) \otimes_{F_E} E \arrow[u, "\overline{\mathrm{Tr}}_{E/K}"] & R\Gamma_{\mathrm{dR}}(Z_E/E). \arrow[l, "r_{Z_E}"', "\simeq"] \arrow[u, "\overline{\mathrm{Tr}}_{E/K}"'] \end{tikzcd}$$

The constructions are natural since they are inverse to $\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id}$, which is natural. Similarly we have a truncated analogue.

More generally, let $Z \in \mathrm{Rig}_K$ be qcqs. Let $\mathfrak{Z}_\bullet \in \mathcal{M}_K^{\mathrm{ss}}$ be an η-éh-hypercovering of Z by qcqs semistables splitting over $L_\bullet$. Similary as at the beginning of the proof of (2.1.12), there are increasing finite extensions E_k/L for $k \in \mathbf{N}$ such that E_k contains all $L_{\bullet\leq k+1}$, so that we obtain $(k+1)$-truncated η-éh-hypercoverings $\mathfrak{Z}^{(k)}_{\bullet\leq k+1}$ of Z_{E_k} in $\mathcal{M}_K^{\mathrm{ss}}$ which are compatible with base change between different k, compatible with the $\mathrm{Gal}(E_k/L)$-action, and such that $\mathfrak{Z}^{(k)}_{\bullet\leq k+1}$ has splitting field E_k. For $m \geq k$, we define the k-truncated retract as the composite

$$r_Z^{\leq k}\colon \tau^{\leq k} R\Gamma_{\mathrm{dR}}(Z/K) \to \tau^{\leq k} R\Gamma_{\mathrm{dR}}(Z_{E_m}/E_m) \xrightarrow{r^{\leq k}_{Z_{E_m}}} \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_{E_m}) \otimes_{F_{E_m}} E_m \xrightarrow{\overline{\mathrm{Tr}}_{E_m/K}} \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z) \otimes_F E,$$

which does not depend on m by the above commutative diagram, hence $r_Z^{\leq}$ is well-defined; it is indeed a retract of $\tau^{\leq k}(\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id})$ by diagram chasing. Taking filtered colimts, we obtain the retract $r_Z := \varinjlim_k r_Z^{\leq k}$ of $\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id}$. The naturality comes from that of the first special case.

Finally, the case for general Z follows formally from the qcqs case. □

2.1.15 - Proposition. *If $Z \in \mathrm{Rig}_K$ be qcqs, then $R\Gamma_{\mathrm{HK}}(Z)$ is represented by a complex of solid-nuclear F-vector spaces, and lies in $\mathscr{D}^{[0,2d]}(\mathrm{Mod}_F^{\blacksquare})$, and the monodromy operator N is nilpotent with order bounded above by a function depending only on d.*

Proof. By local-global compatibility (2.1.12), one reduces to the case where $Z = \mathfrak{Z}_\eta$ for some qcqs $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$. Then the solid-nuclear representability and the nilpotency are clear respectively by (2.1.8) and (2.1.2). The concentration statement is clear since it is true for $R\Gamma_{\mathrm{dR}}(Z/K)$, which retracts to $R\Gamma_{\mathrm{HK}}(Z) \otimes_F K$ by (2.1.14). □

[19] Alternatively, we can also use the original arithmetic Hyodo–Kato isomorphism [21, 4.2.3 (i)] for the proof.

2.1.16 - Remark. Bootstrapping boundedness into the proof of solid-nuclearity, we see that

$$R\Gamma_{\mathrm{HK}}(Z) \simeq \tau^{\leq 2d}\sigma^{\leq 2d+1}R\Gamma_{\mathrm{HK}}(Z) \simeq \tau^{\leq 2d} \lim_{\Delta_{2d+1}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\bullet,1}/\mathcal{O}^0_{F_{L_\bullet}})_{\mathbf{Q}_p}$$

for any η-éh-hypercovering $\mathfrak{Z}_\bullet$ of Z in $\mathcal{M}^{\mathrm{ss}}_K$. Therefore $R\Gamma_{\mathrm{HK}}(Z)$ can be represented by the truncation of a *bounded* complex of F_L-Banach spaces.

2.1.17. Geometric Hyodo–Kato cohomology. We have a completed version as well as a decompleted version of geometric Hyodo–Kato cohomology.

(i) Let the (completed) geometric Hyodo–Kato cohomology $R\Gamma_{\mathrm{HK}}\colon \mathrm{Rig}_C \to \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$ be the η-éh-hypersheafification (2.0.2) of the presheaf $\mathfrak{X} \mapsto R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p}$ on $\mathcal{M}^{\mathrm{ss},b}_C$.

(ii) Let the decompleted geometric Hyodo–Kato cohomology $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}\colon \mathrm{Rig}_C \to \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}})$ be the η-éh-hypersheafification of the $R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}\colon \mathfrak{X} \mapsto \mathrm{colim}_{\Sigma}\, R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$ on $\mathcal{M}^{\mathrm{ss},b}_C$, where Σ is the filtered system whose date are the reduction mod p of quadruples $(L,\mathfrak{Z},\sigma,\theta)$ such that $\sigma\colon L \hookrightarrow C$ is a finite extension of K, $\mathfrak{Z}$ is a semistable formal model over $\mathcal{O}_L$, and $\theta\colon \mathfrak{X} \xrightarrow{\simeq} \mathfrak{Z} \otimes_{\mathcal{O}_L,\sigma} \mathcal{O}_C$ is an isomorphism over $\mathcal{O}_C$, and whose morphisms are morphisms between the reduced objects $\mathfrak{Z}^0_1$ [21, 4.3.1].

One can globalise the geometric Hyodo–Kato morphism to obtain

$$\iota^{\mathrm{geom}}_{\mathrm{HK}}\colon R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \to R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}})$$

for $X \in \mathrm{Rig}_C$; reducing mod $\ker\theta$, we obtain the Hyodo–Kato isomorphism [11, Theorem 3.15]

$$\iota^{\mathrm{geom}}_{\mathrm{HK},C}\colon R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} C \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}(X/C) \tag{2.1.17.1}$$

compatible with the arithmetic Hyodo–Kato morphism (2.1.13).

Recall that for $\mathfrak{X} \in \mathcal{M}^{\mathrm{ss},b}_C$, any map of quadruples $(L,\mathfrak{Z},\sigma,\theta) \to (L',\mathfrak{Z}',\sigma',\theta')$ in Σ induces a canonical base change isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes_{F_L,\sigma} F_{L'} \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}'^0_1/\mathcal{O}^0_{F_{L'}})_{\mathbf{Q}_p} \tag{2.1.17.2}$$

in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F_L})$.

2.1.18 - Proposition (Local-global compatiblity). *For $\mathfrak{X} \in \mathcal{M}^{\mathrm{ss},b}_K$, the natural maps*

$$R\Gamma_{\mathrm{HK}}(\mathfrak{X}_\eta) \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \quad \text{in } \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$$

$$R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X}_\eta) \to R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X}) \quad \text{in } \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}})$$

are isomorphisms.

Proof. For $R\Gamma_{\mathrm{HK}}$, this is [11, Theorem 3.15 (i)]. For $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}$, this is done as in [21, Proposition 4.23 (1)] using the original Hyodo–Kato morphism via convergent cohomology of *op. cit.* (4.17). □

2.1.19 - Lemma. *For $Z \in \mathrm{Rig}_K$, there is canonically a natural $\underline{\mathscr{G}_K}$-action on $R\Gamma_{\mathrm{HK}}(Z_C)$, and a natural $\underline{\mathscr{G}_K}$-equivariant $\breve{F}$-linear morphism*

$$R\Gamma_{\mathrm{HK}}(Z) \otimes^{\blacksquare}_F \breve{F} \to R\Gamma_{\mathrm{HK}}(Z_C).$$

It is an isomorphism if Z has an η-éh-hypercovering by semistable formal schemes over $\mathcal{O}_K$. Moreoever, any $\breve{F}$-linear retract $r\colon C \to \breve{F}$ induces a natural $\breve{F}$-linear retract $r_{\mathrm{HK}}\colon R\Gamma_{\mathrm{HK}}(Z_C) \to R\Gamma_{\mathrm{HK}}(Z) \otimes^{\blacksquare}_F \breve{F}$.

Proof. By functoriality, $\mathscr{G}_K$ acts on $R\Gamma_{\mathrm{HK}}(Z_C)$. We want to make it condensed. By η-éh-hyperdescent, we may assume $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z}$ is a semistable formal scheme over $\mathcal{O}_L$ with splitting field a Galois extension L of K. Then $(\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C})_\eta \coloneqq \coprod_{\sigma\in\mathrm{Hom}_K(L,C)} \mathfrak{Z} \otimes_{\mathcal{O}_L,\sigma} \mathcal{O}_C$ (2.0.1.1) is a semistable formal model of Z_C, the natural $\mathscr{G}_K$-action on Z_C extends canonically to one on $\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C}$ by permuting the indices $\sigma \mapsto g^{-1}\sigma$ and meanwhile acting on the coefficients $\mathcal{O}_C$. By local-global compatibilities and base change, we have

$$R\Gamma_{\mathrm{HK}}(Z) \simeq R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$$

and
$$R\Gamma_{\mathrm{HK}}(Z_C) \simeq R\Gamma_{\mathrm{cris}}((\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C})^1_0/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \simeq \prod_{\sigma\in\mathrm{Hom}_K(L,C)} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}\otimes_{\mathcal{O}_L,\sigma}\mathcal{O}_C)^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p}$$
$$\simeq \prod_{\sigma\in\mathrm{Hom}_K(L,C)} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{F_L,\sigma}\breve{F}.$$

The natural $\mathscr{G}_K$-action on the right hand side is given by permuting components $\sigma\mapsto g\circ\sigma$ and acting on the coefficients $\breve{F}$, i.e. by letting $g\cdot(x_\sigma\otimes c_\sigma)_\sigma = (x_{g^{-1}\sigma}\otimes g(x_{g^{-1}\sigma}))$; this is refined canonically to be a condensed action. The natural map $R\Gamma_{\mathrm{HK}}(Z)\to R\Gamma_{\mathrm{HK}}(X)$ corresponds to the diagonal embedding; linearising over F, we obtain the desired $\underline{\mathscr{G}_K}$-equivariant $\breve{F}$-linear morphism, via the $\mathscr{G}_K$-equivariant morphism
$$F_L\otimes_F\breve{F}\xrightarrow{\simeq}\prod_{\sigma\in\mathrm{Hom}_F(F_L,\breve{F})} F_L\otimes_{F_L,\sigma}\breve{F}\hookrightarrow\prod_{\sigma\in\mathrm{Hom}_K(L,C)}F_L\otimes_{F_L,\sigma}\breve{F},$$
where the last map is induced by the restriction surjection $\mathrm{Hom}_K(L,C)\twoheadrightarrow\mathrm{Hom}_F(F_L,\breve{F})$ on indices. This becomes an isomorphism if L/K is unramified.

As for the retract, given a $\breve{F}$-linear splitting $r\colon C\to\breve{F}$, we define it as the composite
$$r_{\mathrm{HK}}\colon R\Gamma_{\mathrm{HK}}(Z_C)\xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}}R\Gamma_{\mathrm{dR}}(Z/C)\simeq R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K C\xrightarrow{r_Z\otimes r}R\Gamma_{\mathrm{HK}}(Z)\otimes^{\blacksquare}_F\breve{F}$$
using the arithmetic Hyodo–Kato retract r_Z (2.1.14), and check that it is indeed a retract. □

2.1.20 - Corollary. *For $Z\in\mathrm{Rig}_K$, there is a natural $\underline{\mathscr{G}_K}$-equivariant commutative diagram*
$$\begin{tikzcd}
R\Gamma_{\mathrm{HK}}(Z)\otimes^{\blacksquare}_F B^+_{\mathrm{st}} \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}\otimes\iota_p"] \arrow[d] & R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \arrow[d, "\simeq"] \\
R\Gamma_{\mathrm{HK}}(Z_C)\otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})
\end{tikzcd}$$
exhibiting the compatibility of arithmetic and geometric Hyodo–Kato morphisms.

Proof. The same proof as above applies, by reducing to semistable reduction case and using the compatibility between $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ and $\iota^{\mathrm{geom}}_{\mathrm{HK}}$ (2.1.7). □

2.1.21. As a consequence of (2.1.17.2), the $\mathscr{G}_L$-action on $R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X})$ is condensed and smooth, making $R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X})\in\mathscr{D}^{\mathrm{sm}}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}[\mathscr{G}_L]})$. By descent and truncation argument combined with (2.0.1.1) as in the proof of (2.1.14), for $Z\in\mathrm{Rig}_K$ qcqs, the $\mathscr{G}_K$-action on $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)$ is condensed and smooth, making $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)\in\mathscr{D}^{\mathrm{sm}}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}[\mathscr{G}_K]})$; moreover, for any $k\in\mathbf{N}$, there is a finite Galois extension E/K such that $\tau^{\leq k}R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)$ is defined over F_E, such that there exists an $(k+1)$-truncated η-éh-hypercovering $\mathfrak{Z}'_{\leq k+1}$ of Z_E compatible with $\mathrm{Gal}(E/K)$-action and such that $\mathfrak{Z}'_{\leq k+1}$ all have splitting field E; consequently, we have isomorphisms
$$\begin{aligned}\tau^{\leq k}R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)&\simeq\tau^{\leq k}\lim_{\Delta_{k+1}}\left(R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{F_E}F^{\mathrm{nr}}\right)\\&\simeq\left(\tau^{\leq k}\lim_{\Delta_{k+1}}R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E})_{\mathbf{Q}_p}\right)\otimes^{\blacksquare}_{F_E}F^{\mathrm{nr}}\simeq\left(\tau^{\leq k}R\Gamma_{\mathrm{HK}}(Z_E)\right)\otimes^{\blacksquare}_{F_E}F^{\mathrm{nr}}\end{aligned}$$
in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}[\mathscr{G}_K]})$. Similar arguments apply to the complete Hyodo–Kato cohomology, and show, with the help of (2.1.19), that for $Z\in\mathrm{Rig}_K$ qcqs and for any $k\in\mathbf{N}$, there exists a finite extension of E and an $(k+1)$-truncated η-éh-hypercovering $\mathfrak{Z}'_{\leq k+1}$ of Z_E compatible with $\mathrm{Gal}(E/K)$-action and with splitting field E, such that there are isomorphisms
$$\begin{aligned}\tau^{\leq k}R\Gamma_{\mathrm{HK}}(Z_C)&\simeq\tau^{\leq k}\lim_{\Delta_{k+1}}\left(R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{F_E}\breve{F}\right)\\&\simeq\left(\tau^{\leq k}\lim_{\Delta_{k+1}}R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E})_{\mathbf{Q}_p}\right)\otimes^{\blacksquare}_{F_E}\breve{F}\simeq\left(\tau^{\leq k}R\Gamma_{\mathrm{HK}}(Z_E)\right)\otimes^{\blacksquare}_{F_E}\breve{F}\end{aligned}$$
in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]})$. In particular, the natural map $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)\otimes^{\blacksquare}_{F^{\mathrm{nr}}}\breve{F}\xrightarrow{\simeq}R\Gamma_{\mathrm{HK}}(Z_C)$ in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]})$

is an isomorphism.

2.1.22 - Proposition. *Let $Z \in \mathrm{Rig}_K$ be qcqs. We have $H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z)$.*

Proof. By (2.1.21), there exists a finite Galois extension E/K such that $H^n_{\mathrm{HK}}(Z_C) \simeq H^n_{\mathrm{HK}}(Z_E) \otimes_{F_E} \breve{F}$ in $\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]}$, hence we are reduced to showing

$$H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z).$$

The computation $H^0(\underline{\mathscr{G}_K}, B_{\mathrm{dR}}) = K$ and the inclusion $\iota_p \colon B_{\log}[\frac{1}{t}] \hookrightarrow B_{\mathrm{dR}}$ implies

$$K \hookrightarrow H^0(\underline{\mathscr{G}_E}, B_{\log}[\frac{1}{t}]) \hookrightarrow H^0(\underline{\mathscr{G}_K}, B_{\mathrm{dR}}) = K,$$

hence they are all equal. On the other hand, since $H^n_{\mathrm{HK}}(Z_E)$ is a solid-nuclear F_E-vector space (2.1.15), and so is $B_{\log}[\frac{1}{t}]$, we have by (1.2.5)

$$H^0(\underline{\mathscr{G}_E}, H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} B_{\log}[\frac{1}{t}]) \simeq H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} H^0(\underline{\mathscr{G}_K}, B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z_E).$$

Moreover, since $\mathrm{Gal}(E/K)$ is a finite group, by smooth of its action, we obtain (1.2.24)

$$H^0(\underline{\mathrm{Gal}(E/K)}, H^n_{\mathrm{HK}}(Z_E)) \simeq H^n R\Gamma(\underline{\mathscr{G}_K}, R\Gamma_{\mathrm{HK}}(Z_E)) \simeq H^n_{\mathrm{HK}}(Z),$$

where the last isomorphism follows from the étale descent along $Z_E \to Z$. □

2.2 Overconvergent variants

Now, we consider dagger varieties instead of rigid-analytic varieties. After recalling two (essentially equivalent) approaches to Hyodo–Kato cohomology and de Rham cohomology for dagger varieties, we shall see that the previous discussions on Hyodo–Kato morphisms have natural overconvergent analogues (i.e. analogues for dagger varieties). These are more than purely analogues: compared with their rigid-analytic counterparts, they enjoy better finiteness properties, which will provide crucial controls in arguments for partially proper rigid spaces.

2.2.1. Cohomology for dagger varieties. There are two approaches to cohomology for dagger varieties [21, §5], yielding strictly quasi-isomorphic theories. On the one hand, one can consider the dagger analogue of Beilinson base $(\mathcal{M}^{\mathrm{ss},\dagger}_K, (-)_\eta)$ and $(\mathcal{M}^{\mathrm{ss},b,\dagger}_K, (-)_\eta)$ in order to exploit the finiteness property of rigid cohomology or de Rham cohomology. This way, we obtain the Hyodo–Kato cohomology à la Grosse-Klönne $R\Gamma^{\mathrm{GK}}_{\mathrm{HK}}$ as an éh-hypersheaf on $\mathrm{Rig}^\dagger_K$ unfolded from the rational log-rigid cohomology [17, 3.1.2] $\mathfrak{Z} \mapsto R\Gamma_{\mathrm{rig}}(\mathfrak{Z}^0_1/\mathcal{O}^0_F)$ on $\mathcal{M}^{\mathrm{ss},\dagger}_K$ and as an éh-hypersheaf on $\mathrm{Rig}^\dagger_C$ unfolded from $\mathfrak{X} \mapsto R\Gamma_{\mathrm{rig}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})$ on $\mathcal{M}^{\mathrm{ss},b,\dagger}_C$.

On the other hand, let L be K or C and let $\mathscr{V}$ be a presentable ∞-category of coefficients; there is a general procedure to produce éh-hypersheaves on $\mathrm{Rig}^\dagger_L$ from étale hypersheaves on RigSm_L: for any $\mathcal{F} \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}_{L,\text{ét}}, \mathscr{V})$, we define

$$\mathcal{F}^\dagger(U) := \varinjlim_h R\Gamma(U_h, \mathcal{F})$$

for any presentation $(U_h)_h$ of a smooth affinoid U; this is well-defined and functorial, and satisfies étale hyperdescent, thus giving $\mathcal{F}^\dagger \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{SmAffd}^\dagger_{L,\text{ét}}, \mathscr{V})$; then we éh-hypersheafify it to get

$$\mathcal{F}^\dagger \in \mathrm{Shv}^{\mathrm{hyp}}(\mathrm{Rig}^\dagger_{L,\text{éh}}, \mathscr{V});$$

this satisfies éh-local-global compatibility for smooth affinoids cf. [21, 3.2.3] and [11, 3.24][20]. One has a natural morphism $R\Gamma(X, \mathcal{F}^\dagger) \to R\Gamma(\widehat{X}, \mathcal{F})$ for $X \in \mathrm{Rig}^\dagger_L$, which is moreover an isomorphism if X is partially proper

[20]The construction 3.18 in *op. cit.* should be modified as follows: consider the fully faithful embedding $\iota \colon \mathrm{Rig}^\dagger_L \to \mathrm{pro}(\mathrm{Rig}_L)$ into the pro-system of rigid-analytic analytic spaces, preserving products and étale coverings, yielding a morphism of topoi $(\iota^*, \iota_*) \colon \mathrm{pro}(\mathrm{Rig}_L)^\sim_{\text{ét}} \to \mathrm{Rig}^{\dagger,\sim}_{L,\text{ét}}$; consider the functor $l \colon \mathrm{Rig}_L \to \mathrm{pro}(\mathrm{Rig}_L)$ sending a rigid space to its constant system, also yielding a morphism of topoi $(l^*, l_*) \colon \mathrm{pro}(\mathrm{Rig}_L)^\sim_{\text{ét}} \to \mathrm{Rig}^\sim_{L,\text{ét}}$; then we have $\mathcal{F}^\dagger$ is the éh-hypercompletion of the étale sheaf (*resp.* of the étale hypersheaf) $\iota_* l^* \mathcal{F} \in \mathrm{Rig}^{\dagger,\sim}_{L,\text{ét}}$.

[11, Proposition 3.26]. This approach applies to for example sheaves $\mathcal{F} \in \{R\Gamma_{\mathrm{HK}}, R\Gamma_{\mathrm{dR}}(-/L)\}$, as well as $\mathcal{F} = F^\bullet R\Gamma_{\mathrm{inf}}(-/B_{\mathrm{dR}}^+)$ if $L = C$, in the respective categories.

These two definitions are indeed compatible.

2.2.2 - Proposition. *Let $X \in \mathrm{Rig}_C^\dagger$.*

(i) *There is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{\breve{F}}^{\blacksquare})$:*

$$R\Gamma_{\mathrm{HK}}(X) \simeq R\Gamma_{\mathrm{HK}}^{\mathrm{GK}}(X).$$

(ii) *If $X = \mathfrak{X}_\eta$ for some $\mathfrak{X} \in \mathrm{Rig}_C^{\mathrm{ss},b,\dagger}$, then there is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{\breve{F}}^{\blacksquare})$:*

$$R\Gamma_{\mathrm{HK}}(X) \simeq R\Gamma_{\mathrm{rig}}(\mathfrak{X}_1^0/\mathcal{O}_{\breve{F}}^0).$$

(iii) *There is a natural Hyodo–Kato isomorphism in $\mathscr{D}(\mathrm{Mod}_C^{\blacksquare})$:*

$$\iota_{\mathrm{HK}}^{\mathrm{geom},\dagger} : R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} C \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}(X/C).$$

Proof. This is [11, Theorem 3.29]. Let us explain it briefly. The (i) is [23, 4.2.1 (iv), Lemma 4.17]. This together with local-global compatibility of $R\Gamma_{\mathrm{HK}}^{\mathrm{GK}}(X)$ of *loc. cit.* implies (ii). For (iii), we reduce by éh-hyperdescent and [10, Corollary A.67 (ii)] to the basic semistable reduction case, we only need need to prove the statements for smooth dagger affinoids with presentation; but then this follows from the rigid-analytic case (2.1.17.1) and taking filtered colimits as in the construction of $\mathcal{F}^\dagger$ (2.2.1). □

2.2.3 - Proposition. *Let $Z \in \mathrm{Rig}_K^\dagger$.*

(i) *There is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_F^{\blacksquare})$:*

$$R\Gamma_{\mathrm{HK}}(Z) \simeq R\Gamma_{\mathrm{HK}}^{\mathrm{GK}}(Z).$$

(ii) *If $Z = \mathfrak{Z}_\eta$ for some $\mathfrak{Z} \in \mathrm{Rig}_K^{\mathrm{ss},\dagger}$, then there is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_F^{\blacksquare})$:*

$$R\Gamma_{\mathrm{HK}}(Z) \simeq R\Gamma_{\mathrm{rig}}(\mathfrak{Z}_1^0/\mathcal{O}_F^0).$$

(iii) *There is a natural Hyodo–Kato morphism in $\mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$:*

$$\iota_{\mathrm{HK}}^{\mathrm{arith},\dagger} : R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \to R\Gamma_{\mathrm{dR}}(Z/K)$$

with a canonical natural retract $r_Z^\dagger$, which is an equivalence if Z has an η-éh-hypercovering by semistable formal schemes over $\mathcal{O}_K$.

Proof. The (i) is [23, 4.2.1 (ii), Lemma 4.14]. This together with local-global compatibility of $R\Gamma_{\mathrm{HK}}^{\mathrm{GK}}(Z)$ [21, Proposition 5.5] implies (ii). For (iii), we reduce as above to smooth dagger affinoids with presentation $Z = \mathfrak{Z}_\eta$ with $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss},\dagger}$; then we use (2.1.13) to construct $\iota_{\mathrm{HK}}^{\mathrm{arith},\dagger}$, and (2.1.14) to construct $r_Z^\dagger$, by passing to filtered colimits. □

2.2.4. Moreover, the identifications (2.2.2, ii) and (2.2.3, ii) are compatible with base change morphisms, as shown in their proofs. Then the same argument as in (2.1.21), now using base change properties of rational log-rigid cohomology instead of log-crystalline cohomology (2.1.17.2), we find that for $Z \in \mathrm{Rig}_K^\dagger$ qcqs and for any $k \in \mathbf{N}$, there exists a finite extension of E and an $(k+1)$-truncated η-éh-hypercovering $\mathfrak{Z}'_{\leq k+1}$ of Z_E compatible with $\mathrm{Gal}(E/K)$-action and with splitting field E, such that there are isomorphisms

$$\tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_C) \simeq \tau^{\leq k} \lim_{\Delta_{k+1}} R\Gamma_{\mathrm{rig}}((\mathfrak{Z}'_{\leq k+1})_1^0/\mathcal{O}_{F_E}^0) \otimes_{F_E}^{\blacksquare} \breve{F} \simeq \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_E) \otimes_{F_E}^{\blacksquare} \breve{F}$$

in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{\breve{F}[\mathscr{G}_K]}^{\blacksquare})$.

2.2.5 - Remark. The advantage of using dagger version is that, for qcqs dagger variety X over C (*resp.* qcqs dagger variety Z over K), the condensed cohomology groups $H^i_{\mathrm{HK}}(X)$ (*resp.* $H^i_{\mathrm{HK}}(Z)$, $H^i_{\mathrm{dR}}(X/C)$, $H^i_{\mathrm{dR}}(Z/K)$) are finite-dimensional condensed vector spaces over $\breve{F}$ (*resp.* F, C, K) [21, Proposition 5.12, Proposition 5.6, the first paragraph à la Grosse-Klönne just before 5.1.1].

2.2.6 - Proposition. *Let $Z \in \mathrm{Rig}^{\dagger}_K$ be qcqs. We have $H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z)$ for all $n \in \mathbf{N}$.*

Proof. The reasoning goes as in (2.1.22), but there is one step which relatively much easier than (2.1.22): for any finite Galois extension E/K, one obtains

$$H^0(\underline{\mathscr{G}_E}, H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z_E) \otimes_{F_E} H^0(\underline{\mathscr{G}_E}, B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z_E)$$

by classicality and finiteness of $H^n_{\mathrm{HK}}(Z_E)$ [21, Proposition 5.6 (1)] rather than using nuclearity. □

2.2.7 - Corollary. *Let $Z \in \mathrm{Rig}^{(\dagger)}_K$ be partially proper. We have $H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z)$.*

Proof. In the partially proper case, writing Z as the strictly increasing union of qcqs $U \in \mathrm{Rig}^{\dagger}_K$, we have $R\Gamma_{\mathrm{HK}}(Z_C) \simeq R\varprojlim_U R\Gamma_{\mathrm{HK}}(U_C)$, hence by [10, Corollary A.67 (i)]

$$R\Gamma_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N} \simeq R\varprojlim_U (R\Gamma_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N}), \quad n \in \mathbf{N},$$

where $t^{-N} B_{\log,\le N} \subset t^{-N} B_{\log} = t^{-N} B[U]$ denotes the subspace consisting of polynomials in U of degree $\le N$ with coeffcients in $t^{-N} B_{\log}$, which is a $\breve{F}$-Fréchet space, being a finite direct sum of the $\breve{F}$-Fréchet space B. Now for any $i \in \mathbf{N}$, the system $\{H^i_{\mathrm{HK}}(U_C)\}_U$ is Mittag–Leffler by finite-dimensionality, hence so is the system $\{H^i_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N}\}_U$ for any $i, N \in \mathbf{N}$. Therefore, their $R^1 \varprojlim_U$ vanish, so

$$H^n_{\mathrm{HK}}(Z_C) \simeq \varprojlim_U H^n_{\mathrm{HK}}(U_C)$$
$$H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N} \simeq \varprojlim_U (H^n_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N}).$$

Applying the proposition (2.2.6) to each U, and recalling that $H^0(\underline{\mathscr{G}_K}, -)$ commutes with limits, we obtain

$$H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N}) = \varprojlim_U H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\le N}) = \varprojlim_U H^n_{\mathrm{HK}}(U) = H^n_{\mathrm{HK}}(Z).$$

Taking the filtered colimit for $N \longrightarrow +\infty$, then we are done. □

2.3 Arithmetic syntomic cohomology

We shall define the arithmetic syntomic cohomology in the Bloch–Kato style as in [21], by replacing their Hyodo–Kato morphism with ours (we will also see that in the semistable case, these two Hyodo–Kato morphisms are homotopic (2.3.4), but possibly non canonically) and show that it is still isomorphic to the arithmetic Fontaine–Messing syntomic cohomology. Our proof of the isomorphism will require inevitably passing through the construction of the geometric Hyodo–Kato isomorphism and eventually many fastidious diagram chasing, since our definition of the arithmetic Hyodo–Kato morphism involves Galois descent.

Let us start by recalling a comparison result of [23] in the geometric situation, and then descend to the arithmetic case.

2.3.1. Geometric syntomic cohomology. For $X \in \mathrm{Rig}_C$ and $r \in \mathbf{N}$, one defines its *(Bloch–Kato) syntomic cohomology* as

$$R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X, r) := \left[(R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi = p^r, N=0} \xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}} R\Gamma_{\inf}(X/B^+_{\mathrm{dR}})/F^r \right] \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}), \tag{2.3.1.1}$$

where the bracket means taking the limit of the diagram, and where we take the *derived* φ- and N-eigenspaces. For $\mathfrak{X} \in \mathcal{M}^{\mathrm{ss}}_C$, one has the *geometric Fontaine–Messing syntomic cohomology*

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{X}, r) := \left[R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi = p^r}_{\mathbf{Q}_p} \xrightarrow{\mathrm{can}} R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p}/F^r \right] \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}). \tag{2.3.1.2}$$

In this case, the two syntomic cohomology complexes are actually naturally isomorphic

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{X}, r) \simeq R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\mathfrak{X}_\eta, r)$$

in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$ via the following natural commutative diagram [23, Proposition 5.3]

(2.3.1.3)
$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p}/F^r \arrow[d, "\simeq"', "\kappa\ {[23, \S 3.3]}"] \\
(R\Gamma_{\mathrm{cris}}(\mathfrak{X}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}})/F^r \arrow[d, "\simeq"', "{[23, \text{Proposition } 3.27\ (2)]}"] \\
(R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F},1})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}})/F^r.
\end{tikzcd}$$

Now we deal with the arithmetic case.

2.3.2. Arithmetic syntomic cohomology. For $Z \in \mathrm{Rig}_K$ and $r \in \mathbf{N}$, one defines its *arithmetic syntomic cohomology* as

$$R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(Z, r) := [R\Gamma_{\mathrm{HK}}(Z)^{\varphi=p^r, N=0} \xrightarrow{\iota^{\mathrm{arith}}_{\mathrm{HK}}} R\Gamma_{\mathrm{dR}}(Z/K)/F^r] \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}). \tag{2.3.2.1}$$

For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$, one has the *arithmetic Fontaine–Messing syntomic cohomology*

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{Z}, r) := \left[R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \xrightarrow{\mathrm{can}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}/F^r\right] \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}). \tag{2.3.2.2}$$

They are similarly defined for dagger varieties.

2.3.3 - Proposition. *For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$, one has a natural isomorphism in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$:*

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{Z}, r) \simeq R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\mathfrak{Z}_\eta, r).$$

Proof. Let L be the splitting field of $\mathfrak{Z}$ over $\mathcal{O}_K$ (or any other field L such that $\mathfrak{Z}$ actually has a semistable structural morphism to $\mathcal{O}_L$). Consider the following natural commutative diagram
(2.3.3.1)
$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{bch}"', "\simeq"] \arrow[dddrr, bend left, "c^{\urcorner}"] \arrow[dddrr, bend right, "c_{\llcorner}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[r, equal] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{bch}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}/F^r \arrow[d, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[d, "i^*"', "\simeq"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}^\times_L)_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[dd, "\mathrm{bch}"', "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}^\times_L)_{\mathbf{Q}_p}/F^r \arrow[dd, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r^{\mathrm{PD},0}_L)^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[d, "p^*_0"', "\simeq"] & & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L)/F^r
\end{tikzcd}$$

where the left upper vertical arrow is an isomorphism by a standard Frobenius argument [20, Proof of Lemma 5.9], the left middle vertical arrow i^* is an isomorphism by [46, Lemma 4.2], and the left lower vertical arrow p^*_0 is an isomorphism because Frobenius is highly nilpotent on T ; the right upper vertical arrow is an isomorphism by Beilinson's identification [6, Theorem 1.9.2], $\mathfrak{Z}$ endowed with the log-structure induced by its special fibre being smooth over $\mathcal{O}^\times_L$. Every commutativity except that of the middle eye-form cavity is clear.

It remains to show that there is a natural equivalence $c^{\urcorner} \simeq c_{\llcorner}$. Since everything here lies in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_L) \hookrightarrow \mathscr{D}(\mathrm{Mod}^{\blacksquare}_L)^{\mathscr{G}_L}$, by adjunction it amounts to proving the existence of natural equivalence between their postcompositions with $R\Gamma_{\mathrm{dR}}(Z/L) \to R\Gamma_{\mathrm{dR}}(Z/L) \otimes^{\blacksquare}_{L} B^+_{\mathrm{pdR}}$, i.e. that the diagram

(2.3.3.2)

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{bch}"', "\simeq"] \arrow[ddddrr, bend left, "c^{\urcorner}"] \arrow[ddddrr, bend right, "c_{\llcorner}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[r, equal] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[d, "\mathrm{bch}"'] & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[d, "i^*"', "\simeq"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}^{\times}_{L})_{\mathbf{Q}_p} \arrow[ddd, "\mathrm{bch}"', "\simeq"] & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r^{\mathrm{PD},0}_{L})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[d, "p_0^*"', "\simeq"] & & & \\
& & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_{\eta}/L) \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_{\eta}/L) \otimes^{\blacksquare}_{L} B^+_{\mathrm{pdR}}
\end{tikzcd}$$

commutes, where we put $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C$. We need to identify $c^{\urcorner}$ and $c_{\llcorner}$ respectively with some more explicit maps, then show the natural equivalence between them.

First, let us begin with identifying $c^{\urcorner}$. Consider the following diagram

(2.3.3.3)

$$\begin{tikzcd}
& R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \arrow[dd, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[ur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] \arrow[ddr, "c^{\urcorner}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[ur, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"'] & \\
& R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}^{\times}_{L})_{\mathbf{Q}_p} \arrow[d, "\mathrm{bch}"', "\simeq"] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_{\eta}/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B^+_{\mathrm{pdR}} \\
& R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_{\eta}/L) \arrow[ur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_{\eta}/L) \otimes^{\blacksquare}_{L} B^+_{\mathrm{pdR}} \arrow[u, "\mathrm{bch}"', "\simeq"]
\end{tikzcd}$$

where all blocks commute, the right diamond-shaped block commutes by crystalline-theoretic base change compatibility.

Next, we consider the diagram

(2.3.3.4)

$$\begin{tikzcd}
& R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \arrow[dd, bend left, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[ur, "\mathrm{bch}"] \arrow[ddr, "c^{\urcorner}"'] & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r,N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[d, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \\
& & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_{\eta}/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B^+_{\mathrm{pdR}} \\
& R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_{\eta}/L) \arrow[ur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_{\eta}/L) \otimes^{\blacksquare}_{L} B^+_{\mathrm{pdR}} \arrow[u, "\mathrm{bch}"', "\simeq"]
\end{tikzcd}$$

whose upper right block commutes by the definition of $\iota^{\mathrm{geom}}_{\mathrm{HK}}$.

Next, the diagram

(2.3.3.5)

$$\begin{tikzcd}[column sep=small]
 & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[urr, "\mathrm{bch}"] \arrow[ddrr, "c^{\urcorner}"'] & & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[d, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \\
 & & & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B^+_{\mathrm{pdR}} \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[uurr, "\mathrm{bch}"] & R\Gamma(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}"] \arrow[uurr, "\mathrm{bch}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[r, "\mathrm{can}"] \arrow[ur, "\mathrm{bch}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}} \arrow[u, "\mathrm{bch}", "\simeq"']
\end{tikzcd}$$

commutes.

Hence, to identify $c^{\urcorner}$ and $c_{\llcorner}$, we are left to show the commutativity of the diagram

(2.3.3.6)

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"', "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[d, "i^*"', "\simeq"] & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r^{\mathrm{PD},0}_L)^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[d, "p_0^*"', "\simeq"] & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[uur, "\mathrm{bch}"] & R\Gamma(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[uur, "\mathrm{bch}"] &
\end{tikzcd}$$

particularly that of the left block.

For sufficiently large n, we have factorisations of Frobenius $\mathrm{Frob}^n \colon \mathfrak{Z}_1 \xrightarrow{F_n} \mathfrak{Z}^0_1 \hookrightarrow \mathfrak{Z}_1$ and $\mathrm{Frob}^n \colon \mathfrak{Z}^0_1 \hookrightarrow \mathfrak{Z}_1 \xrightarrow{F_n} \mathfrak{Z}^0_1$. Consider the commutative diagram of log-schemes

(2.3.3.7)
$$\begin{tikzcd}
\mathfrak{X}^0_1 \arrow[r, hook] \arrow[d, "\theta^0_{\mathfrak{Z}}"] & \mathfrak{X}_1 \arrow[r] \arrow[d, "F_n\theta_{\mathfrak{Z}}"] & \overline{S}^{\times}_1 \arrow[d, "F_{L,n}\theta_L"] \\
\mathfrak{Z}^0_1 \arrow[r, "\mathrm{Frob}^n"] & \mathfrak{Z}^0_1 \arrow[r] & S^0_{L,1}
\end{tikzcd}$$

which we can denote by π. These data π determine a lifting $i^*_{l_\pi} \colon r^{\mathrm{PD}}_L \twoheadrightarrow \mathcal{O}^{\times}_{L,1}$.

Consider the diagram before taking Frobenius fixed points

$$
\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[rr, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"'] & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[dl, "\simeq"', "\kappa^*_{l_\pi}"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)_{\mathbf{Q}_p} \arrow[d, "i^*"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\pi,\mathrm{st}})^{N\text{-nilp}}_{\mathbf{Q}_p} & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p} \arrow[d, bend right, "p_0^*"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r_L^{\mathrm{PD},0})^{N\text{-nilp}}_{\mathbf{Q}_p} \arrow[l, "(\mathrm{Frob}^n)^*"'] \arrow[u, "\pi^*"] & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[u, bend right, "\iota_0"'] \arrow[uurr, bend right, "\mathrm{bch}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[l, "(\mathrm{Frob}^n)^*", "\simeq"'] \arrow[u, "\iota_0"] &
\end{tikzcd}
\tag{2.3.3.8}
$$

where ι_0 is the (unique) natural φ-equivariant F_L-linear Hyodo–Kato section; the lower right block commutes by the definition of $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$; the lower left circuit commutes (even becomes an isomorphism) after taking $(-)^{N=0}$; the upper left vertical base change morphism becomes an isomorphism after taking $(-)^{\varphi=p^r}$; and i^* becomes isomorphism after taking $(-)^{N=0}$ of the target.

For the commutativity of the $(-)^{\varphi=p^r}$ invariant of the upper left block of (2.3.3.8), we may look at the following commutative diagram

$$
\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}) \arrow[rrr, bend left, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}) \arrow[l, "(\mathrm{Frob}^n)^*"'] \arrow[d, "\mathrm{bch}"'] & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[ddl, bend left, "\kappa^*_{l_\pi}"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1) \arrow[d, "i^*"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1) \arrow[d, "i^*"'] & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r_L^{\mathrm{PD},0}) & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r_L^{\mathrm{PD},0}) \arrow[l, "(\mathrm{Frob}^n)^*"'] \arrow[r, "\pi^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\pi,\mathrm{st}})_{\mathbf{Q}_p} &
\end{tikzcd}
\tag{2.3.3.9}
$$

which is clear by base change maps.

Putting these altogether, we obtain the commutativity of (2.3.3.6), which is natural since it is insensitive to the $n \gg 0$ chosen, cf. [23, Proof of Theorem 2.22, Independence of the choice of π and ξ_0]. This completes the proof that $c^{\urcorner} \simeq c_{\llcorner}$ after $-\otimes^{\blacksquare}_L B^+_{\mathrm{pdR}}$, whence $c^{\urcorner} \simeq c_{\llcorner}$ by taking derived $\mathscr{G}_L$-fixed points. $\square$

2.3.4 - Remark. Given a $\mathcal{O}_{F_L}$-class $l \in (\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$ associated with $i_l^*\colon r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}^\times_{L,1}$ lifting $r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}^0_{F_L,1}$ along $\mathcal{O}^\times_{L,1} \twoheadrightarrow \mathcal{O}^0_{F_L,1}$, and assuming that $v(l) = \frac{1}{e}$, there is another approach providing an equivalence

$$\alpha_l\colon R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{Z}, r) \simeq R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\mathfrak{Z}_\eta, r),$$

a priori depending on the choice of l, which however will be of other interests. It consists of finding a $\underline{\mathscr{G}_L}$-equivariant morphism $\varepsilon^{\mathrm{st}}_{\mathrm{HK},l}\colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^\times_{\mathrm{cris}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}}$ equivalent to $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$. For the pair (π, n) associated with the data (2.3.3.7) such that $\frac{p^n}{e} \geq 1$, the $\varepsilon^{\mathrm{st}}_{\mathrm{HK},l}$ is defined as the first row of the following commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\iota_{0,l}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}^{N\text{-nilp}} \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l,\mathrm{st}})_{\mathbf{Q}_p}^{N\text{-nilp}} & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[l, "\simeq"'] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[u, "(\mathrm{Frob}^n)^*"] \arrow[r, "\iota_0"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}^{N\text{-nilp}} \arrow[u, "(\mathrm{Frob}^n)^*"] \arrow[r, "\pi^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\pi,\mathrm{st}})_{\mathbf{Q}_p}^{N\text{-nilp}} \arrow[u, "\mu_n^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[l, "\simeq"'] \arrow[u, equal]
\end{tikzcd}$$

where the third vertical arrow is induced by

$$\mu_n\colon \widehat{A}_{l_\pi,\mathrm{st}} \to \widehat{A}_{l,\mathrm{st}}, \quad t_{a^{p^n}} \mapsto (t_a)^{p^n}, \quad \forall a \in \tau_{\frac{1}{e}} := \left\{ a \in (\mathcal{O}_C\backslash\{0\})/(1+\mathfrak{m}_C) \middle| v(a) = \frac{1}{e} \right\}.$$

All but the left square commute by base change compatibility; that of the left follows from the invertibility of Frob^* on $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ and the uniqueness of φ-equivariant section $\iota_{0,l}$ [23, Theorem 2.12 (2)]. Thus it provides the homotopy between $\varepsilon^{\mathrm{st}}_{\mathrm{HK},l}$ and $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$.

As a byproduct, the following commutative diagram implies in particular that our arithmetic Hyodo–Kato-morphism defined by Galois descent is (non canonically) homotopic to the one defined by zig-zag in [21].

2.3.5 - Corollary. *For $Z \in \mathrm{Rig}_K$, one has a natural isomorphism in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$:*

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(Z,r) \simeq R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(Z,r).$$

Proof. The equivalence produced in the proposition being natural and not involving any special choices of elements of L, it glues to the desired global equivalence. □

2.3.6. Therefore, our new arithmetic syntomic cohomology agrees with Colmez–Nizioł's arithmetic syntomic cohomology by [21, Proposition 4.32], although we still do not know whether the (global) Hyodo–Kato morphism agree with theirs.

We will denote $R\Gamma_{\mathrm{syn}}(Z,r) := R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(Z,r)$ from now on.

3 Syntomic descent spectral sequence

The main goal of this section is to construct, for any rigid space Z over K, a syntomic descent spectral sequence abutting to its syntomic cohomology groups, which will be compatible with the Hochschild–Serre (Galois descent) spectral sequence converging to its proétale cohomology groups.

The morphism between their abutments will be the syntomic-proétale map constructed using the fundamental exact sequence of period sheaves in p-adic Hodge theory. And the morphism between their E_2-terms, in the case where Z is proper over K, can be identified with the natural morphism from its geometric Selmer groups to the Galois cohomology groups of its geometric étale cohomology groups.

The main ingredients of the construction are the compatibility between the arithmetic and the geometric Hyodo–Kato morphisms and the C_{st}-conjecture for partially proper varieties and qcqs dagger varieties (which is so far verified for proper varieties, Stein varieties and affinoid dagger varieties), thanks to a technical result of Nekovář–Nizioł on spectral sequences.

3.1 Syntomic-proétale period maps

We shall define the arithmetic and geometric syntomic-proétale period maps by using the fundamental exact sequence of period sheaves in p-adic Hodge theory, and compare them with the Fontaine–Messing period maps.

3.1.1. **Fundamental exact sequence in p-adic Hodge-theory.** For $r \in \mathbf{N}$ and $i \geq 0$, we have a (strict) exact sequence of $\mathbf{Q}_p$-Banach spaces [33, théorème 5.3.7, ii)] (see also [32, exemple 6.4.2])

$$0 \to \mathbf{Q}_p(r) \to (t^{-i}B_{\log})^{\varphi=p^r, N=0} \to t^{-i}B_{\mathrm{dR}}^+/t^r B_{\mathrm{dR}}^+ \to 0.$$

For any $X \in \mathrm{Rig}_C$, the above exact sequence upgrades to an exact sequence of sheaves on $X_{\text{proét}}$

$$0 \to \mathbf{Q}_p(r) \to (t^{-i}\mathbf{B}_{\log})^{\varphi=p^r, N=0} \to t^{-i}\mathbf{B}_{\mathrm{dR}}^+/t^r \mathbf{B}_{\mathrm{dR}}^+ \to 0$$

by twisting the corresponding exact sequence for $i = 0$ [10, Formula (4.10)], where $\mathbf{B}_{\log} = \mathbf{B}[U]$ [11, Definition 2.27] with the $\mathscr{G}_K$-action given by $g(U) = U + \log[\frac{g(p^\flat)}{p^\flat}]$, the Frobenius action $\varphi(U) = pU$ and the monodromy action $N = -\frac{d}{dU}$. This can be also written as a bicartesian square (or a fibre square in the derived category)

$$\begin{tikzcd}
\mathbf{Q}_p(r) \arrow[r] \arrow[d] & (t^{-i}\mathbf{B}_{\log})^{\varphi=p^r, N=0} \arrow[d] \\
t^r\mathbf{B}_{\mathrm{dR}}^+ \arrow[r] & t^{-i}\mathbf{B}_{\mathrm{dR}}^+.
\end{tikzcd}$$

Taking its cohomology and then $i \to +\infty$, one obtains a fibre square

$$\begin{tikzcd}
R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & R\Gamma_{\text{proét}}(X, \mathbf{B}_{\log})[\frac{1}{t}]^{\varphi=p^r, N=0} \arrow[d] \\
R\Gamma_{\text{proét}}(X, t^r\mathbf{B}_{\mathrm{dR}}^+) \arrow[r] & R\Gamma_{\text{proét}}(X, \mathbf{B}_{\mathrm{dR}}^+)[\frac{1}{t}],
\end{tikzcd} \tag{3.1.1.1}$$

refining the fibre square

$$\begin{tikzcd}
R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & R\Gamma_{\text{proét}}(X, \mathbf{B}_{\log}[\frac{1}{t}])^{\varphi=p^r, N=0} \arrow[d] \\
R\Gamma_{\text{proét}}(X, t^r\mathbf{B}_{\mathrm{dR}}^+) \arrow[r] & R\Gamma_{\text{proét}}(X, \mathbf{B}_{\mathrm{dR}})
\end{tikzcd}$$

which we obtain by first taking $i \to +\infty$ and then its cohomology. The two diagrams coincide if X is qcqs.

3.1.2. **Compatibilities with the geometric Hyodo–Kato morphism.** Assume that $X = Z_C$ is base changed from some $Z \in \mathrm{Rig}_K$. Then there is a natural isomorphism [11, Theorem 5.2]

$$R\Gamma_{\text{proét}}(X, t^\bullet\mathbf{B}_{\mathrm{dR}}^+) \simeq \mathrm{Fil}^\bullet(R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}})$$

compatible with filtrations and a natural φ-equivariant morphism [11, Theorem 4.1]

$$R\Gamma_{\text{proét}}(X, \mathbf{B}) \leftarrow R\Gamma_B(X) \simeq (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log})^{N=0}.$$

They are compatible with the geometric Hyodo–Kato morphism [11, Theorem 5.3] for $r \in \mathbf{N}$, i.e., there is a natural commutative diagram

$$\begin{tikzcd}
R\Gamma_{\text{proét}}(X, t^{-r}\mathbf{B}_{\mathrm{dR}}^+) & R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} t^{-r}B_{\mathrm{dR}}^+ \arrow[l] \\
R\Gamma_{\text{proét}}(X, \mathbf{B}) \arrow[u] & (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log})^{N=0}. \arrow[l] \arrow[u, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"']
\end{tikzcd} \tag{3.1.2.1}$$

If moreover X is qcqs, then we have identifications $R\Gamma_{\text{proét}}(X, \mathbf{B}_{\mathrm{dR}}) \simeq R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}}$ and $R\Gamma_{\text{proét}}(X, \mathbf{B}_{\log}[\frac{1}{t}])^{N=0} \xleftarrow{\simeq} R\Gamma(X, \mathbf{B}[\frac{1}{t}]) \xleftarrow{\simeq} (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log}[\frac{1}{t}])^{N=0}$; in general, by covering X with qcqs opens, we obtain natural equivalences and morphisms

$$\begin{aligned}
R\Gamma_{\text{proét}}(X,\mathbf{B}_{\text{dR}}) &\simeq \lim_{U\text{ qcqs}} R\Gamma_{\text{proét}}(U,\mathbf{B}_{\text{dR}}) \\
&\simeq \lim_{U\text{ qcqs}} (R\Gamma_{\text{dR}}(U/K)\otimes^{\blacksquare}_{K} B_{\text{dR}}) \quad \leftarrow \big(\lim_{U\text{ qcqs}} R\Gamma_{\text{dR}}(U/K)\big)\otimes^{\blacksquare}_{K} B_{\text{dR}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \simeq R\Gamma_{\text{dR}}(X)\otimes^{\blacksquare}_{K} B_{\text{dR}}
\end{aligned} \tag{3.1.2.2}$$

$$\begin{aligned}
R\Gamma_{\text{proét}}(X,\mathbf{B}_{\log}[\tfrac{1}{t}])^{N=0} &\simeq \lim_{U\text{ qcqs}} R\Gamma_{\text{proét}}(U,\mathbf{B}_{\log}[\tfrac{1}{t}])^{N=0} \\
&\simeq \lim_{U\text{ qcqs}} (R\Gamma_{\text{HK}}(U)\otimes^{\blacksquare}_{\breve F} B_{\log}[\tfrac{1}{t}])^{N=0} \quad \leftarrow \big((\lim_{U\text{ qcqs}} R\Gamma_{\text{HK}}(U))\otimes^{\blacksquare}_{\breve F} B_{\log}[\tfrac{1}{t}]\big)^{N=0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \simeq (R\Gamma_{\text{HK}}(X)\otimes^{\blacksquare}_{\breve F} B_{\log}[\tfrac{1}{t}])^{N=0}
\end{aligned} \tag{3.1.2.3}$$

compatible with geometric Hyodo–Kato morphism.

3.1.2.4 - Lemma. *There is a natural (φ,N)-equivariant morphism*

$$R\Gamma_{\text{HK}}(X)\otimes^{\blacksquare}_{\breve F} B_{\log}\to R\Gamma_{\text{proét}}(X,\mathbf{B}_{\log})$$

which induces, after inverting t and then taking $(-)^{N=0}$, the φ-equivariant morphisms in (3.1.2.3).

Proof. Working locally, we may assume $X=\mathfrak{X}_\eta$ where $\mathfrak{X}\in\mathcal{M}_C^{\text{ss},b}$ is base changed from a p-adic formal scheme $\mathfrak{Z}$ quasi-compact and semistable over $\mathcal{O}_L$ for some finite extension L of K. For compact intervals $I\subset(0,+\infty)$ with rational endpoints, let $\mathbf{B}_{\log,I}:=\mathbf{B}_I[U]$ with the Frobenius and the monodromy operator given by the same formulae as for $\mathbf{B}_{\log}$ in (3.1.1), let $\widehat{\mathbf{B}}_{\log,I}$ be the p-adic completion of $\mathbf{B}_{\log,I}$, and let $\widehat{\mathbf{B}}_{\log}:=R\varprojlim_I \widehat{\mathbf{B}}_{\log,I}$; we have $(\widehat{\mathbf{B}}_{\log})^{N\text{-nilp}}\simeq\mathbf{B}_{\log}$. We define similarly the (condensed) period rings $B_{\log,I}$, $\widehat{B}_{\log,I}$ and $\widehat{B}_{\log}$.

For compact intervals $I\subset[\frac{1}{p-1},+\infty)$ with rational endpoints, we have $A_{\text{cris}}\subset B_I$ and $B^+_{\text{st}}\subset B_{\log,I}$; so we can consider the (φ,N)-equivariant morphisms

$$\begin{aligned}
R\Gamma_{\text{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{\breve F} B_{\log,I} &\xrightarrow[\simeq]{\varepsilon^{\text{st}}_{\text{HK}}} R\Gamma_{\text{cris}}(\mathfrak{X}_1/A^{\times}_{\text{cris}})\otimes^{\blacksquare}_{A_{\text{cris}}} B_{\log,I} \\
&\simeq (R\Gamma_{\text{cris}}(\mathfrak{X}_1/A^{\times}_{\text{cris}})\otimes^{\blacksquare}_{A_{\text{cris}}} B_I)\otimes^{\blacksquare}_{B} B_{\log,I} \\
&\to R\Gamma_{\text{proét}}(X,\mathbf{B}_I)\otimes^{\blacksquare}_{B_I} B_{\log,I} \\
&\xrightarrow{\simeq} R\Gamma_{\text{proét}}(X,\mathbf{B}_{\log,I}),
\end{aligned} \tag{3.1.2.5}$$

where the third morphism is due to [11, Formulae (4.34), (4.35) in the proof of Theorem 4.3]; indeed, the presence of $L\eta_\mu$ *loc. cit.* serves only to ensure that the morphism $R\Gamma_{\text{cris}}(\mathfrak{X}_1/A^{\times}_{\text{cris}})\otimes^{\blacksquare}_{A_{\text{cris}}} B_I\to R\Gamma_{\text{proét}}(X,\mathbf{B}_I)$ thereby constructed becomes a quasi-isomorphism after applying $L\eta_\mu$. Since the Frobenius action φ is invertible on $R\Gamma_{\text{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}$ [23, Theorem 2.12 (1)], the above composed map extends to all compact intervals $I\subset(0,+\infty)$ with rational endpoints. By taking limit over all such $I\subset(0,+\infty)$, we obtain morphisms

$$R\Gamma_{\text{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{\breve F}\widehat{B}_{\log}\to R\lim_I \big(R\Gamma_{\text{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{\breve F}\widehat{B}_{\log,I}\big)\to R\lim_I R\Gamma_{\text{proét}}(X,\widehat{\mathbf{B}}_{\log,I})\xleftarrow{\simeq} R\Gamma_{\text{proét}}(X,\widehat{\mathbf{B}}_{\log}),$$

where the last isomorphism follows as in [10, Lemma 4.8][21]. We conclude the construction by taking $(-)^{N\text{-nilp}}$, using the N-nilpotency property of $R\Gamma_{\text{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}\simeq R\Gamma_{\text{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{F_L}\breve F$ (2.1.2). Finally, the compatibility with the given isomorphism (3.1.2.3) follows from the construction (see also [11, Formula (4.53) in the proof of Theorem 4.1]). ◻

[21] By virtue of [64, Lemma 3.18], it suffices to show that for $j>0$, we have $R^j\varprojlim_I \widehat{\mathbf{B}}_{\log,I}(W)=0$ for any affinoid perfectoid space $W=\mathrm{Spa}(R,R^+)$ over $\mathrm{Spa}(C,\mathcal{O}_C)$. For this, observe that for any intervals $I\subset J\subset(0,+\infty)$ with rational endpoints, the map $\mathbf{B}_J(R,R^+)\to\mathbf{B}_I(R,R^+)$ is a continuous morphism of $\mathbf{Q}_p$-Banach spaces with dense image, hence so is $\widehat{\mathbf{B}}_{\log,J}(R,R^+)\to\widehat{\mathbf{B}}_{\log,I}(R,R^+)$. It follows then from the topological Mittag-Leffler lemma [40, chapitre 0, remarque (13.2.4, i)] that $R^j\varprojlim_I\widehat{\mathbf{B}}_{\log,I}(R,R^+)=0$.

3.1.2.6 - Corollary. *For $r \in \mathbf{N}$, there is a natural commutative diagram*

$$\begin{tikzcd} R\Gamma_{\mathrm{pro\acute{e}t}}(X, t^{-r}\mathbf{B}_{\mathrm{dR}}^+) & R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} t^{-r}B_{\mathrm{dR}}^+ \arrow[l] \\ R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{log}}) \arrow[u] & R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\mathrm{log}}. \arrow[l] \arrow[u, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \end{tikzcd} \tag{3.1.2.7}$$

which is compatible with (3.1.2.1) *after taking* $(-)^{N=0}$. □

3.1.3. Syntomic-proétale period maps. By (3.1.2), the fundamental exact sequence (3.1.1) induces natural morphisms of fibre sequences

$$\begin{aligned} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) &\xrightarrow{\simeq} \left[\begin{tikzcd} & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{log}}[\frac{1}{t}])^{\varphi=p^r, N=0} \arrow[d, "\mathrm{can}"] \\ R\Gamma_{\mathrm{pro\acute{e}t}}(X, t^r\mathbf{B}_{\mathrm{dR}}^+) \arrow[r] & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{dR}}) \end{tikzcd}\right] \\ &\leftarrow \left[\begin{tikzcd} & (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\mathrm{log}}[\frac{1}{t}])^{\varphi=p^r, N=0} \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \\ \mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}}) \arrow[r] & R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}} \end{tikzcd}\right] \\ &\leftarrow \left[\begin{tikzcd} & (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\mathrm{st}}^+)^{\varphi=p^r, N=0} \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \\ \mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}}^+) \arrow[r] & R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}}^+ \end{tikzcd}\right] \\ &= R\Gamma_{\mathrm{syn}}(X, r) \end{aligned} \tag{3.1.3.1}$$

where the second morphism becomes an isomorphism if X is qcqs. The diagram (3.1.3.1) composes to a geometric syntomic-proétale comparison morphism $\rho_{\mathrm{syn}}^{\mathrm{geom}} \colon R\Gamma_{\mathrm{syn}}(X, r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ between the syntomic cohomology and the proétale cohomology, which induces, by taking $\underline{\mathscr{G}_K}$-invariants, an arithmetic syntomic-proétale comparison morphism $\rho_{\mathrm{syn}}^{\mathrm{arith}}$, which fits into the commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{syn}}(Z, r) \arrow[r, dashed, "\rho_{\mathrm{syn}}^{\mathrm{arith}}"] \arrow[d] & R\Gamma_{\mathrm{pro\acute{e}t}}(Z, \mathbf{Q}_p(r)) \arrow[d] \\ R\Gamma_{\mathrm{syn}}(X, r) \arrow[r, "\rho_{\mathrm{syn}}^{\mathrm{geom}}"] & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)). \end{tikzcd} \tag{3.1.3.2}$$

3.1.4 - Remark. By (3.1.2.7), the maps in the construction of the geometric syntomic-proétale comparison map (3.1.3.1) are actually induced by maps of data (A, B, ι, r), where $A \in \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{\mathbf{Q}_p}^{\mathrm{cond}})$, $B \in \mathscr{D}\mathscr{F}(\mathrm{Mod}_{\mathbf{Q}_p}^{\mathrm{cond}}) := \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathscr{D}(\mathrm{Mod}_{\mathbf{Q}_p}^{\mathrm{cond}}))$, $\iota = \iota_{\mathrm{HK}}^{\mathrm{geom}}$ and $r \in \mathbf{N}$.

3.1.5. Fontaine–Messing period maps. We would like to compare the above constructed period maps (3.1.3.2) with another collection of period maps, the *Fontaine–Messing period maps*

$$\alpha_r^{\mathrm{FM}} \colon R\Gamma_{\mathrm{syn}}(X, r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)),$$

which are defined by globalising the Fontaine–Messing period maps on semistable models $\alpha_r^{\mathrm{FM}} \colon R\Gamma_{\mathrm{syn}}(\mathfrak{X}, r) \to R\Gamma_{\mathrm{\acute{e}t}}(\mathfrak{X}, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}, \mathbf{Q}_p(r))$, both in the arithmetic and geometric cases [21, §7] (recall the definition of the Fontaine–Messing syntomic cohomology (2.3.1.2), (2.3.2.2)). They are constructed in a natural way so that they satisify Galois equivariance: if $X = \mathfrak{X}_\eta$ with $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C$ such that $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L, then $\alpha_r^{\mathrm{FM,geom}}$ are $\underline{\mathscr{G}_L}$-equivariant; in particular, if $X = Z \otimes_K C$, then $\alpha_r^{\mathrm{FM,geom}}$ is $\underline{\mathscr{G}_K}$-equivariant.

Whether these seemingly two types of period maps are homotopic, i.e., whether $\rho_{\mathrm{syn}} = \alpha^{\mathrm{FM}}$ in the underlying homotopy category (both in geometric and arithmetic cases), is closely related to the uniqueness of (geometric) p-adic period morphisms addressed by Nizioł [60, 61] in the algebraic setting and by Sally Gilles [37] in the formally algebraic setting. Although the latter have seemingly treated only the case of proper semistable models, their proofs contain a great amount of local constructions which we will employ to obtain the following proposition.

3.1.6 - Proposition. *The geometric syntomic-proétale period map $\rho_{\mathrm{syn}}^{\mathrm{geom}}$ is naturally homotopic to the geometric Fontaine–Messing period map $\alpha_r^{\mathrm{FM,geom}}$.*

Proof. By éh-hyperdescent, one may reduces to the case of $X = \mathfrak{X}_\eta$, where $\mathfrak{X} \in \mathcal{M}_C^{\mathrm{ss},b}$ is affine and descends to $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L. Essentially by construction, it suffices to proves the natural commutativity of the following diagram

$$\begin{tikzcd}
R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{B}_I) \arrow[r] & R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{B}_{\mathrm{dR}}) \\
R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(\mathfrak{X}, r) \arrow[u, "\alpha_r^{\mathrm{FM,geom}}"] \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \arrow[u, "\mathrm{can}"'] \arrow[ur, "\mathrm{can}"] &
\end{tikzcd}$$

where we have used the canonical isomorphism $R\Gamma(\mathfrak{X}) \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})$ to identify the absolute log-crystalline cohomology defining the Fontaine–Messing syntomic cohomology. Here, the vertical canonical morphism can is defined in [11, Proof of Theorem 4.3] (as was used in (3.1.2.5) for the third morphism), which actually factors as $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \xrightarrow{\gamma^{\mathrm{CK}}} R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{A}_{\mathrm{inf}}) \to R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{B}_I)$ where the first is induced by Česnavičius-Koshikawa's φ-equivariant comparison isomorphism [13, Theorem 5.4] and the latter is induced by the canonical map of proétale period sheaves $\mathbf{A}_{\mathrm{inf}} \to B_I$. Moreover, the slanted canonical morphism can is induced by the morphisms of sites $(\mathfrak{X}_\eta/B_{\mathrm{dR},m}^+)_{\mathrm{inf}} \to (\mathfrak{X}/A_{\mathrm{cris}}^{\times})_{\mathrm{cris}}$. The right triangle commutes by [11, Proposition 5.11], so it remains to show that the left square commutes.

The upper left arrow being induced by maps of proétale sheaves $\mathbf{Z}_p(r) \to \mathbf{A}_{\mathrm{inf}} \to B_I$, we are thus reduced to showing the natural commutativity of the following diagram

$$\begin{tikzcd}
R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{A}_{\mathrm{inf}}) \\
R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(\mathfrak{X}, r) \arrow[u, "\alpha_r^{\mathrm{FM,geom}}"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p}. \arrow[u, "\gamma^{\mathrm{CK}}"']
\end{tikzcd}$$

For this, we apply the aforementioned local constructions in Gilles's works [37]: she constructed in ***op. cit.***, §8 for such $\mathfrak{X}$ a natural Lazard type period map $\alpha_r^0 \colon R\Gamma_{\mathrm{syn}}(\mathfrak{X}, r) \to R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p}$ such that $\tau^{\leq r}\alpha_r^0$ is an equivalence[22], then she proved a natural identification $\alpha_r^0 \simeq \alpha_r^{\mathrm{FM,geom}}$ in ***op. cit.***, théorème 9.1[23] on the one

[22] In ***op. cit.***, §8, the morphism α_r^0, or more precisely its local and integral version $\alpha_{r,\Sigma,\lambda}$, was constructed directly as quasi-isomorphisms of $\tau^{\leq r}$*-truncated complexes* but not for untruncated complexes , see the formula just before ***op. cit.***, proposition 8.8. This formula was deduced from ***op. cit.***, proposition 8.6. But in fact, if we carefully look at the proof of this last proposition, we find that:

- Firsly, multiplication-by-$t^\bullet$ morphism is defined even before the truncation, though it becomes an quasi-isomorphism only after the truncation;
- Secondly, the multiplication-by-t^r morphism and the morphism β there are quasi-isomorphisms even before the truncation, cf. ***op. cit.***, lemme 5.3 and lemme 5.8 respectively.

Therefore, as it turns out, the ***op. cit.***, proposition 8.6 actually shows the existence of a forward morphism $\beta' \colon \mathrm{Kosz}(\varphi, \partial, F^r R_{\Sigma,\Lambda}^{[u,v]}) \to \mathrm{Kosz}(\varphi, \Gamma_{\Sigma,\Lambda}, R_{\Sigma,\Lambda}^{[u,v]}(r))$ which becomes a quasi-isomorphism after the truncation $\tau^{\leq r}$. Taking respective Frobenius eigenspaces, one obtains the untruncated map $\alpha_{r,\Sigma,\Lambda}^0$, which is then globalised to be the untruncated morphism α_r^0 in the our text.

[23] Again, her statement in ***op. cit.*** was for $\tau^{\leq r}$-truncated complexes, but what she proved was not that restrictive. In fact, in her proof:

- Firstly, the properness hypothesis was not used there;
- Secondly, the morphism ***op. cit.***, (33) there which rewrites the (truncated) syntomic cohomology as (truncated) Frobenius eigenspace of $\mathbf{A}_{\mathrm{cris}}(-)$ construction holds for untruncated complexes;
- Thirdly, the proof of identification reduces to that of ***op. cit.***, lemme 9.2, which says that $\tau^{\leq r}\alpha_{r,\Sigma,\Lambda}^{\mathrm{FM}} \simeq \tau^{\leq r}\alpha_{r,\Sigma,\Lambda}^0$, actually stays valid for untruncated complexes, see ***op. cit.***, proposition 7.5, or look at the original source of ideas [20, Theorem 4.16].

hand, and that on the other hand the analogue of the above commutative diagram

$$\begin{tikzcd} R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{A}_{\text{inf}}) \\ R\Gamma^{\text{FM}}_{\text{syn}}(\mathfrak{X}, r) \arrow[u, "\alpha_r^0"] \arrow[r, "\text{can}"] & R\Gamma_{\text{cris}}(\mathfrak{X}_1/A^{\times}_{\text{cris}})_{\mathbf{Q}_p}. \arrow[u, "\gamma^{\text{CK}}"'] \end{tikzcd}$$

commutes by the step (ii) in the proof of *op. cit.*, lemme 9.11 (this step was a purely local calculation)[24]; hence we are done. □

3.1.7 - Corollary. *The arithmetic syntomic-proétale period map $\rho^{\text{arith}}_{\text{syn}}$ is homotopic to the arithmetic Fontaine–Messing period map $\alpha_r^{\text{FM,arith}}$.*

Proof. This follows from the Galois descent construction of $\rho^{\text{arith}}_{\text{syn}}$ and (3.1.6), since when composed with the natural map $R\Gamma_{\text{proét}}(Z, \mathbf{Q}_p(r)) \to R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r))$, the period maps $\rho^{\text{arith}}_{\text{syn}}$ and $\alpha_r^{\text{FM,arith}}$ give rise to homotopic morphisms $\rho^{\text{geom}}_{\text{syn}} \simeq \alpha_r^{\text{FM,geom}}$ by (3.1.6). □

3.1.8 - Corollary. *Let $Z \in \mathrm{Rig}_K$ and $r \in \mathbf{N}$. The the syntomic-proétale comparison maps $\rho^{\text{arith}}_{\text{syn}}$ for Z and $\rho^{\text{geom}}_{\text{syn}}$ for Z_C become isomorphisms after truncation $\tau^{\leq r}$.*

Proof. The geometric case follows from (!) [11, Theorem 7.2]. But alternatively, the geometric and arithmetic cases follow respectively from Proposition (3.1.6) and Corollary (3.1.7), since we already know that the truncated isomorphisms holds for the Fontaine–Messing period maps [21, Corollary 7.3] for smooth varieties, which extends to singular cases by éh-hyperdescent. □

3.1.9 - Remark. Although one might argue that the corollary (3.1.6) can be proven directly by a geometric truncated quasi-isomorphism statement (say of Bosco [11]) plus Galois descent, without passing through the identification (3.1.7), we should stress that this Galois descent step might not work naively due to one important difference between the rigid-analytic setting and the algebraic setting: the Galois action on the rigid-analytic geometric syntomic cohomology not being smooth nor having vanishing higher continuous Galois cohomology groups, taking Galois fixed points is not an exact operation on the complex, which is contrary to the case of algebraic varieties.

3.2 Construction of morphisms of spectral sequences

We are going to construct the syntomic and étale (or Hochschild–Serre) spectral sequences for partially proper rigid spaces over K and a natural morphism between them compatible with the one on their abutments induced by period maps. For this, we will follow the strategy of Nekovář–Nizioł [56], which relies on a technical result on spectral sequences *op. cit.*. However, this result does not apply directly to these two spectral sequences, but rather applies to two intermediate spectral sequences which we shall also explain. The conditions of the technical result, which can be expressed in terms of bicartesianness of certain square diagram, are in particular verified in the partially proper case thanks to the known cases of the C_{st}-conjecture. Along the way in parallel, we will also provide the dagger analogue of this story where the C_{st}-conjecture is also verified.

Let us begin by recalling the aforementioned technical result of Nekovář–Nizioł.

3.2.1. Postnikov system on Bloch–Kato type diagrams. Let R be a condensed ring. Let $A \in \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\text{cond}}_R)$ and $B \in \mathscr{DF}(\mathrm{Mod}^{\text{cond}}_R) := \mathrm{Fun}(\mathbf{Z}^{\text{op}}, \mathscr{D}(\mathrm{Mod}^{\text{cond}}_R))$ and a morphism $\iota \colon A \to B$. Let $r \in \mathbf{N}$. Given such data (A, B, ι, r), we will consider a specific type of *Postnikov system* of $C_{\text{syn}}(r) := \mathrm{Syn}_{(A,B,\iota,r)} :=$

[24] Let us use the notation of *loc. cit.*. As mentioned in the footnote (22), the Lazard type period map α_r^0 well-defined even before the truncation, especially because the morphism β is a quasi-isomorphism even before the truncation. Therefore, in the diagram *loc. cit.*, (55), it is unnecessary to take the truncation $\tau^{\leq r}$ in order to obtain commutativity, the diagram itself untruncated is already commutative by the same proof, chiefly thanks to the commutativity of the diagram *loc. cit.*, (57), which was no more than a direct computation showing that Bhatt–Morrow–Scholze's $\widetilde{\beta}$ is compatible with the morphism β' of *loc. cit.*.

$\mathrm{fib}(A^{\varphi=p^r,N=0} \to B)$ (where the eigenspaces are taken in the derived sense) on the *Bloch–Kato type limit diagram*

$$\mathrm{Syn}_{(A,B,\iota,r)} \simeq \left[\begin{tikzcd} & \mathrm{Fil}^r B \arrow[d] \\ A \arrow[r, "(\varphi-p^r\text{,}\iota)"] \arrow[d, "N"] & A\oplus B \arrow[d, "(N\text{,}0)"] \\ A \arrow[r, "p\varphi-p^r"] & A \end{tikzcd}\right]$$

in the direction of homotopy limit; namely, it consists of a finite collection of adjacent triangles

$$\begin{tikzcd} \mathrm{gr}^0 C_{\mathrm{syn}}(r) \arrow[dr] & & \mathrm{gr}^1 C_{\mathrm{syn}}(r) \arrow[dr] & & \mathrm{gr}^2 C_{\mathrm{syn}}(r) \arrow[dr] & \\ C_{\mathrm{syn}}(r)^0 \arrow[u] & & C_{\mathrm{syn}}(r)^1 \arrow[u] \arrow[ll, "{[1]}"] & & C_{\mathrm{syn}}(r)^2 \arrow[u] \arrow[ll, "{[1]}"] & C_{\mathrm{syn}}(r)^3 \arrow[l, "{[1]}"] \end{tikzcd}$$

where

$$C_{\mathrm{syn}}(r)^0 \coloneqq \mathrm{Syn}_{(A,B,\iota,r)} \coloneqq \left[\begin{tikzcd} & \mathrm{Fil}^r B \arrow[d] \\ A \arrow[r, "(\varphi-p^r\text{,}\iota)"] \arrow[d, "N"] & A\oplus B \arrow[d, "(N\text{,}0)"] \\ A \arrow[r, "p\varphi-p^r"] & A \end{tikzcd}\right], \quad C_{\mathrm{syn}}(r)^1 \coloneqq \left[\begin{tikzcd} & A\oplus B \arrow[d, "(N\text{,}0)"] \\ A \arrow[r, "p\varphi-p^r"] & A \end{tikzcd}\right], \quad C_{\mathrm{syn}}(r)^2 \coloneqq [A], \quad C_{\mathrm{syn}}(r)^3 \coloneqq 0.$$

so that

$$\mathrm{gr}^0 C_{\mathrm{syn}}(r) = A \oplus \mathrm{Fil}^r B, \quad \mathrm{gr}^1 C_{\mathrm{syn}}(r) = A \oplus (A\oplus B), \quad \mathrm{gr}^2 C_{\mathrm{syn}}(r) = A.$$ [25]

This gives rise to a *Postnikov exact couple*

$$D_1^{i,j} = H^j(C_{\mathrm{syn}}(r)^i) \xrightarrow{g} H^j(C_{\mathrm{syn}}(r)^{i-1}[1]) = H^{j+1}(C_{\mathrm{syn}}(r)^{i-1}) = D^{i-1,j+1}$$

with associated spectral sequence which we call *Postnikov spectral sequence*

$$E_1^{i,j} = H^j(\mathrm{gr}^i C_{\mathrm{syn}}(r)) \Rightarrow \varinjlim(\cdots \to D_1^{i,j} \xrightarrow{g} D_1^{i-1,j+1} \to \cdots)$$

(under certain conditions). When the filtration is ***finitely exhaustive***, i.e. when $C_{\mathrm{syn}}(r)^i[-i] \xrightarrow{\simeq} C_{\mathrm{syn}}$ for $i \ll 0$, this colimit becomes

$$\varinjlim(\cdots \to H^{i+j}(C_{\mathrm{syn}}(r)^i[-i]) \xrightarrow{g} H^{i+j}(C_{\mathrm{syn}}(r)^{i-1}[-i+1]) \to \cdots) \xrightarrow{\simeq} H^{i+j}(C_{\mathrm{syn}}(r)).$$

3.2.2. Postnikov and hypercohomology spectral sequences. Let R and S be two condensed rings. Consider a left exact functor $F\colon \mathrm{Mod}_R^{\blacksquare} \to \mathrm{Mod}_S^{\blacksquare}$, e.g. $F = \underline{\mathrm{Hom}}_R(S,-)$ if S is an R-algebra. We denote by RF its right derived functor.

Let $A \in \mathscr{D}^+_{(\varphi,N)}(\mathrm{Mod}_R^{\blacksquare})$ and $B \in \mathscr{DF}^+(\mathrm{Mod}_R^{\blacksquare}) \coloneqq \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathscr{D}^+(\mathrm{Mod}_R^{\mathrm{cond}}))$ and a morphism $\iota\colon A \to B$, and $r \in \mathbf{N}$. Applying F to the above Posnikov system, one obtains a Postnikov system of $RF(C_{\mathrm{syn}}(r))$ with graded pieces $\mathrm{gr}^i = RF(\mathrm{gr}^i C_{\mathrm{syn}}(r))$, and whence the associated Postnikov spectral sequence

$${}^{\lim}E_1^{i,j} = R^jF(\mathrm{gr}^i C_{\mathrm{syn}}(r))) \Rightarrow R^{i+j}F(C_{\mathrm{syn}}(r)).$$

On the other hand, without any Postnikov datum, we still have the *hypercohomology exact couple*

$${}^{\mathrm{hyp}}D_2^{i,j} = R^{i+j}F(\tau^{\leq j-1}C_{\mathrm{syn}}(r)) \to R^{i+j}F(\tau^{\leq j}C_{\mathrm{syn}}(r)) =^{\mathrm{hyp}} D_2^{i-1,j+1}$$

[25]To extend to $C_{\mathrm{syn}}(r)^\bullet$ for any index $\bullet \in \mathbf{Z}$, one may set $C_{\mathrm{syn}}(r)^{\bullet \geq 3} \coloneqq 0$ and $C_{\mathrm{syn}}(r)^i \coloneqq C_{\mathrm{syn}}(r)[i]$ for $i \leq 0$.

and associated *hypercohomology spectral sequence*

$$^{\mathrm{hyp}}E_2^{i,j} = R^i F(H^j(C_{\mathrm{syn}}(r))) \Rightarrow R^{i+j} F(C_{\mathrm{syn}}(r)).$$

The main technical result that we need is the following theorem, which relates the two spectral sequences under very restrictive but still reasonable conditions.

3.2.3 - Proposition (Nekovář–Nizioł). *Assume that we are in the setting* (3.2.2). *If the sequence*

$$0 \to H^j(C_{\mathrm{syn}}(r)) \to H^j(\mathrm{gr}^0 C_{\mathrm{syn}}(r)) \to H^j(\mathrm{gr}^1 C_{\mathrm{syn}}(r)) \to H^j(\mathrm{gr}^2 C_{\mathrm{syn}}(r)) \to 0$$

is exact, or, equivalently, the natural morphism

$$H^j(C_{\mathrm{syn}}(r)) \to \left[\begin{tikzcd} & H^j(\mathrm{Fil}^r B) \arrow[d] \\ H^j(A) \arrow[r, "(\varphi - p^r, \iota)"] \arrow[d, "N"] & H^j(A) \oplus H^j(B) \arrow[d, "(N,0)"] \\ H^j(A) \arrow[r, "p\varphi - p^r"] & H^j(A) \end{tikzcd}\right] \tag{3.2.3.1}$$

in $\mathscr{D}(\mathrm{Mod}_R^{\blacksquare})$ is an isomorphism for any $j \in \mathbf{Z}$, then there exists a natural morphism of exact couples $({}^{\lim}D_2^{i,j}, {}^{\lim}E_2^{i,j}) \to ({}^{\mathrm{hyp}}D_2^{i,j}, {}^{\mathrm{hyp}}E_2^{i,j})$. Consequently, there is a natural morphism of spectral sequences ${}^{\lim}E_t^{i,j} \to {}^{\mathrm{hyp}}E_t^{i,j}$ which starts from the E_2-page with common abutment $R^{i+j}F(C_{\mathrm{syn}}(r))$, and for spectral filtrations we have ${}^{\lim}F \subset {}^{\mathrm{hyp}}F$.

Proof. This is a special case of Nekovář–Nizioł's result [56, Theorem 2.18]. □

Let us recollect some general results on (φ, N)-modules, before applying the above result to the fundamental diagram for $X \in \mathrm{Rig}_C$.

3.2.4 - Lemma. *Let (V, φ) be a finite φ-module over $\breve{F}$. Then, for any $m \in \mathbf{Z}$ greater than all the slopes of (V, φ), we have*

$$\varphi - 1\colon V \otimes_{\breve{F}} t^{-m} B \to V \otimes_{\breve{F}} t^{-m} B$$

is surjective. In particular, the map

$$\varphi - 1\colon V \otimes_{\breve{F}} B[\tfrac{1}{t}] \to V \otimes_{\breve{F}} B[\tfrac{1}{t}]$$

is surjective.

Proof. This is proved in [11, Lemma 7.5]. □

3.2.5 - Lemma. *Let $(V, \varphi, N) \in \mathrm{Mod}_{(\varphi,N)}^{\mathrm{cond}}$ be a condensed (φ, N)-module over $\breve{F}$ of bounded nilpotency index. Then the map*

$$N\colon V \otimes_{\breve{F}} B_{\log} \to V \otimes_{\breve{F}} B_{\log}$$

is surjective. Moreover, if one has V is the projective limit of a system $\{V_n\}_{n \in \mathbf{N}}$ of finite (φ, N)-modules whose slopes are uniformly bounded above by $\beta \in \mathbf{Z}$, then for any integer $r \geq \beta$, the derived fixed points $(V \otimes_{\breve{F}}^{\blacksquare} B_{\log})^{\varphi = p^r, N=0}$ is concentrated in degree 0; in particular, the derived fixed points $(V \otimes_{\breve{F}}^{\blacksquare} B_{\log}[\frac{1}{t}])^{\varphi = p^r, N=0}$ is concentrated in degree 0 for any $r \in \mathbf{Z}$.

Proof. The final sentence concerning $B_{\log}[\frac{1}{t}]$ follows from the corresponding one concerning $B_{\log}$, by writing $B_{\log}[\frac{1}{t}]$ as the filtered colimit of $\{t^{-j} B_{\log}\}_{j \in \mathbf{N}}$.

First we assume that V is finite-dimensional over $\breve{F}$. The first part is proved in [11, Lemma 7.6]. The second part follows by combining with the previous lemma (3.2.4). For general (V, N) with bounded nilpotency index, the proof of the first part is the identical as in *loc. cit.*.

Let us now assume that $V = \varprojlim_n V_n$ with V_n finite (φ, N)-modules over $\breve{F}$ with slopes uniformly bounded above by $\beta \in \mathbf{Z}$. We may assume that the maps $V \to V_n$ are surjective. For $n \in \mathbf{N}$, consider $\mathcal{E}(V_n)$, the vector bundle on the Fargues–Fontaine curve associated with the finite (φ, N)-module V_n. The derived fixed points $(V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0}$ computes the cohomology of the vector bundle $\mathcal{E}(V_n) \otimes_{\mathcal{O}} \mathcal{O}(r)$. If $r \geq \beta$, this vector bundle has non-negative Harder–Narasimhan slopes, so its first cohomology vanishes and $(V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0}$ is concentrated in degree 0; this is in fact the proof of the second part in the finite-dimensional case. To pass to the projective limit, one observes that there is a natural structure of *Banach–Colmez space* on $(V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0}$. Moreover, one can compute its *Dimension*

$$\operatorname{Dim}(V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0} = (\deg \mathcal{E}(V_n) \otimes_{\mathcal{O}} \mathcal{O}(r), \dim V_n) \in \mathbf{N} \times \mathbf{N} \subset \mathbf{N} \times \mathbf{Z}.$$

The first component is called the *dimension* and the second component the *height* of this Banach–Colmez space. Since its height is nonnegative, so is the height of any Banach–Colmez subspace of it; and the order of containment induces the decreasing of order of the Dimension in the lexicographic order of $\mathbf{N} \times \mathbf{N}$. In consequence, any chain of Banach–Colmez subspaces of it stabilises, hence we deduce that the projective system $\{(V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0}\}_{n \in \mathbf{N}}$ is Mittag-Leffler. Therefore, we obtain

$$R \lim_n (V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0} \simeq \lim_n (V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0},$$

which is concentrated in degree 0. It remains to show that

$$(V \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r, N=0} \simeq R \lim_n (V_n \otimes_{\breve{F}} B_{\log})^{\varphi=p^r, N=0}.$$

As N is nilpotent on V and one can write $B_{\log} = \widehat{B_{\log}}^{N=\text{nilp}}$ where $\widehat{B_{\log}} := \varprojlim_I B_I \langle U \rangle$ is a $\breve{F}$-Fréchet space, it is equivalent to showing that

$$(V \otimes^{\blacksquare}_{\breve{F}} \widehat{B_{\log}})^{\varphi=p^r, N=0} \simeq R \lim_n (V_n \otimes_{\breve{F}} \widehat{B_{\log}})^{\varphi=p^r, N=0}.$$

Since (derived) limits commute with (derived) fixed points and we have $R \lim_n V_n \simeq \lim_n V_n = V$ by Mittag-Leffler property of the projective system $\{V_n\}_{n \in \mathbf{N}}$ of finite-dimensional $\breve{F}$-vector spaces, one reduces further to checking that

$$(R \lim_n V_n) \otimes^{\blacksquare}_{\breve{F}} \widehat{B_{\log}} \simeq R \lim_n (V_n \otimes_{\breve{F}} \widehat{B_{\log}}).$$

But this follows by applying [10, Corollary A.67 (i)] since $\widehat{B_{\log}}$ is a $\breve{F}$-Fréchet space. □

This allows to establish an analogue of [17, Lemma 3.20, Lemma 3.28] by replacing the coefficient period ring B^+_{st} with $B_{\log}$.

3.2.6 - Corollary. *For partially proper $X \in \mathrm{Rig}_C$ or quasi-compact $X \in \mathrm{Rig}^{\dagger}_C$, we have isomorphisms in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}$:*

$$H^i((R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r, N=0} \simeq (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r, N=0} \quad 0 \leq i \leq r,$$

$$H^i((R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\tfrac{1}{t}])^{\varphi=p^r, N=0} \simeq (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\tfrac{1}{t}])^{\varphi=p^r, N=0} \quad i, r \geq 0,$$

where the fixed points are taken in the derived sense.

Proof. The proof goes *verbatim* as in [17, Lemma 3.20, Lemma 3.28] but replacing the equations *loc. cit.* (3.21) with the above lemma (3.2.5). Except from this, another key input is the fact that for $U \in \mathrm{Rig}^{\dagger}_C$, the finite φ-module $H^i_{\mathrm{HK}}(U)$ over $\breve{F}$ has slopes in $[0, i]$ by [11, Theorem 3.30]. □

Now we are ready to consider the fundamental diagram for $X \in \mathrm{Rig}_C$.

3.2.7. Fargues–Fontaine syntomic cohomology. In [11, Definition 6.1], Bosco defined the so-called *Fargues–Fontaine syntomic cohomology* $R\Gamma_{\mathrm{syn,FF}}(X, \mathbf{Q}_p(r))$ for $r \geq 0$ and rigid-analytic varieties X over C, which can be identified as follows

(3.2.7.1) $$R\Gamma_{\mathrm{syn,FF}}(X, \mathbf{Q}_p(r)) := \left[(R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r, N=0} \xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}} R\Gamma_{\mathrm{inf}}(X / B^+_{\mathrm{dR}}) / F^r \right] \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}),$$

where the geometric Hyodo–Kato morphism $\iota_{\mathrm{HK}}^{\mathrm{geom}}\colon R\Gamma_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve{F}} B_{\log} \to R\Gamma_{\inf}(X/B^+_{\mathrm{dR}})$ is obtained from the geometric Hyodo–Kato isomorphism (2.1.3) as in the B^+_{st}-coefficient case.

There is a natural comparison map between the Fargues–Fontaine syntomic cohomology and the (Bloch–Kato) syntomic cohomology defined previously in $R\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$ (2.3.1.1):

$$R\Gamma_{\mathrm{syn}}(X,r) \to R\Gamma_{\mathrm{syn,FF}}(X,\mathbf{Q}_p(r)), \tag{3.2.7.2}$$

given by changing coefficient period rings B^+_{st} to $B_{\log}$. This becomes a quasi-isomorphism after the truncation $\tau^{\leq r}$. Moreover, the syntomic proétale period maps and their construction (3.1.3) factor through the above comparison map.

3.2.8 - Remark. We warn the reader that the map (3.2.7.2) is not an isomorphism in degrees $i > r$ in general. The difference lies in that between the period rings $B_{\log}$ and B^+_{st}. For example, as shown in the example of the projective line $X = \mathbf{P}^1_C$ [11, Example 6.36], since $H^2_{\mathrm{HK}}(X) \simeq \breve{F}(-1)$ has Frobenius slope 1, the natural map

$$H^3_{\mathrm{syn}}(X,0) \to (\breve{F}(-1)\otimes_{\breve{F}} B)/(\varphi-1) \simeq H^3_{\mathrm{syn,FF}}(X,\mathbf{Q}_p(0))$$

can be identified with the natural map

$$B^+_{\mathrm{cris}}/(p\varphi-1) \to B/(p\varphi-1);$$

the latter group is isomorphic to $H^1(\mathrm{FF},\mathcal{O}(-1)) \simeq C/\mathbf{Q}_p$, which is highly nontrivial.

3.2.9. C_{st}**-conjecture and fundamental diagrams.** For $X \in \mathrm{Rig}_C$ partially proper or affinoid or $X \in \mathrm{Rig}^\dagger_C$ quasi-compact, we say that *the* C_{st}*-conjecture holds* for X if the following commutative diagram

$$(\mathrm{FD}^+_{i,r}) \qquad \begin{array}{ccc} H^i_{\mathrm{syn,FF}}(X,r) & \longrightarrow & H^i(\mathrm{Fil}^r R\Gamma_{\inf}(X/B^+_{\mathrm{dR}})) \\ \downarrow & & \downarrow \\ (H^i_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0} & \xrightarrow{\iota_{\mathrm{HK}}^{\mathrm{geom}}} & H^i_{\inf}(X/B^+_{\mathrm{dR}}), \end{array}$$

which we call the *fundamental diagram*, is bicartesian for any $i = r \geq 0$, or equivalently by (3.2.5) that

$$(\mathrm{FD}^+_{i,r}{}') \qquad H^i_{\mathrm{syn,FF}}(X,r) \xrightarrow{\simeq} \left[\begin{array}{ccc} & & H^i(\mathrm{Fil}^r R\Gamma_{\inf}(X/B^+_{\mathrm{dR}})) \\ & & \downarrow \\ H^i_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve{F}} B_{\log} & \xrightarrow{(\varphi-p^r,\iota)} & H^i_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve{F}} B_{\log}\oplus H^i_{\inf}(X/B^+_{\mathrm{dR}}) \\ \downarrow N & & \downarrow (N,0) \\ H^i_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve{F}} B_{\log} & \xrightarrow{p\varphi-p^r} & H^i_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve{F}} B_{\log} \end{array}\right]$$

is an equivalence for $i = r \geq 0$.

3.2.10 - Remark. The original conjecture [24] is stated with the period ring B^+_{st} in place of $B_{\log}$. The latter is more geometric from the point of view of the Fargues–Fontaine curve [32]. We will refer to the original conjecture as the B^+_{st}*-coefficient* C_{st}*-conjecture*. Both versions of the conjecture turn out to be true in many cases. Differences between results in the current literature are: on the one hand, Colmez and Nizioł have provided in [24] some extra instances of small varieties (3.2.18) that verify the B^+_{st}-coefficient C_{st}-conjecture, which has not yet been proved in the $B_{\log}$-coefficient situation; on the other hand, for smooth affinoid rigid spaces over C, only the $B_{\log}$-version is well formulated, and it has been proved in dimension 1 [11, Conjecture 7.11, Theorem 7.12]. Many of our statements with $B_{\log}$ can be transferred to those with B^+_{st} as long as we change the Fargues–Fontaine syntomic cohomology to the (Bloch–Kato) syntomic cohomology. Therefore, we will benefit from both versions of C_{st}-conjecture.

3.2.11 - Remark. Assume that $X = Z_C$ with qcqs $Z \in \mathrm{Rig}^\dagger_K$ or partially proper $Z \in \mathrm{Rig}_K$ (the latter is equivalent to a partially proper $Z \in \mathrm{Rig}^\dagger_K$, see footnote 3). *If the* C_{st}*-conjecture holds for* Z_C*, then* $(\mathrm{FD}^+_{i,r})$ *is* *bicartesian for* $0 \leq i \leq r$. Indeed, fix $i \geq 0$ and run induction on $r \geq i$. When $r = i$, this is the C_{st}-conjecture. Now for the induction step, consider the following commutative diagram

(3.2.11.1)

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^i_{\mathrm{syn,FF}}(X,r) \arrow[r] \arrow[d, "\simeq"', "t"] & (H^r_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0} \arrow[d, hook, "t"] & \oplus & H^i(F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})) \arrow[r] \arrow[d, "\simeq"', "t"] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \arrow[r] \arrow[d, hook, "t"] & 0 \\
0 \arrow[r] & H^i_{\mathrm{syn,FF}}(X,r+1) \arrow[r] \arrow[d] & (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^{r+1},N=0} \arrow[d] & \oplus & H^i(F^{r+1}(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})) \arrow[r] \arrow[d] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \arrow[r] \arrow[d, two heads] & 0 \\
 & 0 \arrow[r] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C & \oplus & 0 \arrow[r] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C \arrow[r] & 0
\end{tikzcd}$$

whose top row is exact by induction hypothesis, whose second vertical arrow is exact by [17, Lemma 3.39, and proof of Proposition 3.36] (cf. [11, Formula (7.15) and its following paragraph] for the case of $B_{\log}$ in place of B^+_{st}), and whose third vertical arrow is an isomorphism since $i \leq r$. Therefore, the bottom row is isomorphism, from which we deduce by diagram chasing that the middle row is also exact, thus showing the bicartesianness for $(i, r+1)$.

3.2.12 - Example. Many cases of the C_{st}-conjecture (*resp.* the B^+_{st}-coefficient C_{st}-conjecture) have been established.

(i) For X proper, this is done by Colmez–Nizioł in [24, Theorem 6.2, Corollary 6.15] and by Bosco in [11, Theorem 7.4] (plus the argument of [24, Corollary 6.15]).

(ii) For X smooth Stein, this is done by Colmez–Nizioł in [24, Theorem 6.14, Corollary 6.19] and by Bosco in [11, Theorem 7.7] (plus the argument of [24, Corollary 6.19]).

(iii) For X smooth dagger affinoid, this is done by Colmez–Nizioł in [24, Theorem 6.14, Corollary 6.19] and by Bosco in [11, Proof of Theorem 7.7] (plus the argument of [24, Corollary 6.19]).

3.2.13. Now, we invert t in the fundamental diagram ($\mathrm{FD}^+_{i,r}$), and define the complex

$$C_{\mathrm{syn}}(r) := \left[(R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\tfrac{1}{t}])^{\varphi=p^r,N=0} \xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}} (R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})/F^r \right] \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}).$$

Consider the commutative diagram

$$(\mathrm{FD}_{i,r}) \qquad \begin{tikzcd}
H^i(C_{\mathrm{syn}}(r)) \arrow[r] \arrow[d] & H^i(F^r(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})) \arrow[d] \\
(H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\tfrac{1}{t}])^{\varphi=p^r,N=0} \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] & H^i_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}
\end{tikzcd}$$

When X is qcqs, the natural map $C_{\mathrm{syn}}(r) \longrightarrow R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ is a quasi-isomorphism, see (3.1.3.1).

3.2.14 - Remark. (i) For $Z \in \mathrm{Rig}_K$, the natural morphism

$$\mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}) \longrightarrow \mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})$$

becomes an isomorphism after taking the canonical truncation $\tau^{\leq r}$. We may replace B_{dR} by $t^{-j}B^+_{\mathrm{dR}}$ for $j \in \mathbf{N}$ and prove the same statement, hence $t^{-j}B_{\mathrm{dR}}$ is a K-Banach space so that [10, Corollary A.67] is applicable. By éh-hyperdescent, we reduce to the case where Z is smooth affinoid, this is because the cofibre of the map $\mathrm{Fil}^r(\Omega^{\bullet}_{Z/K} \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}) \longrightarrow \mathrm{Fil}^r(\Omega^{\bullet}_{Z/K} \otimes^{\blacksquare}_K B_{\mathrm{dR}})$ is concentrated in degrees $> r$.

(ii) For $X \in \mathrm{Rig}_C$, the natural morphism

$$\mathrm{Fil}^r R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \longrightarrow \mathrm{Fil}^r(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})$$

becomes an isomorphism after taking the canonical truncation $\tau^{\leq r}$. Again, we may replace B_{dR} by $t^{-j}B^+_{\mathrm{dR}}$ for sufficiently large $j \in \mathbf{N}$. By éh-hyperdescent, we may assume X to be smooth affinoid, then using Elkik's algebraisation technique [31, Theorem 7, Remark 2], we may assume X descends to a smooth affinoid Z over some finite extension L/K. Then the statement follows from (i).

3.2.15 - Lemma. *For partially proper $X \in \mathrm{Rig}_C$ and quasi-compact $X \in \mathrm{Rig}_C^\dagger$, there is a natural exact sequence*

$$0 \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^i,N=0} \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\tfrac{1}{t}])^{\varphi=p^i,N=0} \to H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Proof. For X smooth affinoid dagger, this is [11, Formula (7.15)]. In general, we would like to take inverse limit; but to avoid the problem of exchanging countable inverse limit and tensor product $-\otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]$, we start by showing that [11, Formula (7.15)] remains true for partially proper spaces. Indeed, for qcqs $U \in \mathrm{Rig}_C^\dagger$, we have an exact sequence

$$0 \to \mathcal{E}(H^i_{\mathrm{HK}}(U)) \otimes_{\mathcal{O}} \mathcal{O}(i) \to \mathcal{E}(H^i_{\mathrm{HK}}(U)) \otimes_{\mathcal{O}} \mathcal{O}(i+j) \to \iota_*(H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}}) \to 0$$

of coherent sheaves on the Fargues–Fontaine curve, where, $\mathcal{E}(H^i_{\mathrm{HK}}(U))$ is the vector bundle over the Fargues–Fontaine curve attached to the finite (φ, N)-module $H^i_{\mathrm{HK}}(U)$ over $\breve{F}$. Taking global sections, we obtain an exact sequence of $\breve{F}$-vector spaces

$$(3.2.15.1)\quad 0 \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} B_{\log})^{\varphi=p^i,N=0} \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B_{\log})^{\varphi=p^i,N=0} \to H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

because the first cohomology of $\mathcal{E}(H^i_{\mathrm{HK}}(U)) \otimes_{\mathcal{O}} \mathcal{O}(i)$ vanishes by non-negative Harder–Narasimhan slopes considerations [11, Theorem 3.30 (ii)]. Letting $j \to +\infty$, one obtains the desired exact sequence for quasi-compact $X \in \mathrm{Rig}_C^\dagger$. On the other hand, as N is nilpotent on $H^i_{\mathrm{HK}}(U)$ and one can write $B_{\log} = \widehat{B_{\log}}^{N\text{-nilp}}$ where $\widehat{B_{\log}} := \varprojlim_I B_I\langle U \rangle$ is a $\breve{F}$-Fréchet space, the above exact sequence can be written as

$$0 \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} \widehat{B_{\log}})^{\varphi=p^i,N=0} \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}\widehat{B_{\log}})^{\varphi=p^i,N=0} \to H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Now $\widehat{B_{\log}}$ being a $\breve{F}$-Fréchet space, passing to limit with respect to a strictly increasing covering of any partially proper X by qcqs smooth dagger affinoid U, and using [10, Corollary A.67 (i)] and the vanishing $R^1 \varprojlim_U (H^i_{\mathrm{HK}}(U) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B_{\log})^{\varphi=p^i,N=0} = 0$ for $i, j \in \mathbf{N}$ (cf. proof of [11, Formula (7.17)]), one obtains the exact sequence

$$0 \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} \widehat{B_{\log}})^{\varphi=p^i,N=0} \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}\widehat{B_{\log}})^{\varphi=p^i,N=0} \to H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Again by N-nilpotency on $R\Gamma_{\mathrm{HK}}(X)$, this can be rewritten as the exact sequence

$$(3.2.15.2)\quad 0 \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^i,N=0} \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B_{\log})^{\varphi=p^i,N=0} \to H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Finally, letting $j \to +\infty$, one obtains the desired exact sequence for partially proper $X \in \mathrm{Rig}_C$. □

3.2.16 - Lemma. *For partially proper $X \in \mathrm{Rig}_C$ and quasi-compact $X \in \mathrm{Rig}_C^\dagger$, the natural morphisms*

$$R\Gamma_{\mathrm{syn}}(X, r) \to R\Gamma_{\mathrm{syn,FF}}(X, r) \to C_{\mathrm{syn}}(r)$$

in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$ become quasi-isomorphisms after taking the canonical truncation $\tau^{\leq r}$.

Proof. The first quasi-isomorphism has been recalled in (3.2.7). The second quasi-isomorphism follows from the observation that $\tau^{\leq r}C_{\mathrm{syn}}(r)$ is naturally isomorphic to the filtered colimit over $i \geq r$ of $\tau^{\leq r}R\Gamma_{\mathrm{syn}}(X, i)$ by (3.2.14, ii), the exact sequence (3.2.15.2) and the diagram (3.2.11.1) but assuming in the last diagram the exactness of the first two rows only at the middle terms. □

3.2.17 - Proposition (Colmez–Nizioł, Bosco). *The square* ($\mathrm{FD}_{i,r}$) *is bicartesian for all* $0 \leq i \leq r$ *in the following cases*:

(i) *X is a proper rigid space over C.*
(ii) *X is a smooth dagger affinoid rigid space over C.*
(iii) *X is a smooth Stein (dagger[26]) rigid space over C.*

[26] For partially proper dagger rigid space $X \in \mathrm{Rig}_C^\dagger$, there is a natural isomorphism $R\Gamma(X, \mathcal{F}^\dagger) \to R\Gamma(\widehat{X}, \mathcal{F})$ for éh-sheaves $\mathcal{F} \in \{R\Gamma_{\mathrm{HK}}(-), F^\bullet R\Gamma_{\inf}(-/B^+_{\mathrm{dR}})\}$ on $\mathrm{Rig}_C^\dagger$ (2.2.1).Therefore, there is no need here to distinguish between dagger and genuine rigid spaces in the partially proper case.

Proof. (i) For X proper, the semistable comparison (cf. [24, Theorem 6.2], [11, Theorem 7.4]) and the degeneration at the E_1-page of Hodge-to-de Rham spectral sequence imply that the diagram ($\mathrm{FD}_{i,r}$) is bicartesian for any $i,r \geq 0$. Indeed, by *loc. cit.*, for any $i \geq 0$, we have $H^i_{\text{proét}}(X,\mathbf{Q}_p) \simeq H^i_{\text{ét}}(X,\mathbf{Q}_p)$ which is a finite-dimensional $\mathbf{Q}_p$-vector space, and we have a natural isomorphism

$$H^i_{\text{ét}}(X,\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\tfrac{1}{t}] \simeq H^i_{\mathrm{HK}}(X) \otimes_{\breve{F}} B_{\log}[\tfrac{1}{t}]$$

compatible with (φ, N)-action, which induces a natural isomorphism

$$H^i_{\text{ét}}(X,\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq H^i_{\inf}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$$

compatible with filtrations. In fact, the first natural isomorphism is induced by $R\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r)) \simeq \tau^{\leq r} R\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r)) \simeq R\Gamma_{\mathrm{syn}}(X,r) \to R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]$ for $r \gg 0$.

(ii) For X smooth dagger affinoid, the diagram ($\mathrm{FD}_{i,r}$) is bicartesian for any $i,r \geq 0$. Indeed, we may freely use Tate twist to reduce to the case $i = r$. The bicartesiannes then follows from the bicartesianness of the ($\mathrm{FD}^+_{i,r}$) and [11, Formula (7.15)]. More precisely, we may assume by smoothness and [31, Theorem 7, Remark 2] that $X = Z_C$ for some smooth dagger affinoid rigid space over some finite extension L/K. We have

$$\begin{tikzcd}
0 \arrow[r] & H^r_{\mathrm{syn,FF}}(X,r) \arrow[r] \arrow[d, "\simeq"', "(3.2.16)"] & (H^r_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0} \arrow[d, hook] & \oplus & H^r(\mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/L) \otimes_L B^+_{\mathrm{dR}})) \arrow[r] \arrow[d, "\simeq"', "(3.2.14)"] & H^r_{\mathrm{dR}}(Z/L) \otimes_L B^+_{\mathrm{dR}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & H^r(C_{\mathrm{syn}}(r)) \arrow[r] & (H^r_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}])^{\varphi=p^r,N=0} \arrow[d, two heads] & \oplus & H^r(\mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/L) \otimes_L B_{\mathrm{dR}})) \arrow[r] \arrow[d] & H^r_{\mathrm{dR}}(Z/L) \otimes_L B_{\mathrm{dR}} \arrow[r] \arrow[d, two heads] & 0 \\
& & H^r_{\mathrm{dR}}(Z/L) \otimes_L B_{\mathrm{dR}}/B^+_{\mathrm{dR}} & \oplus & 0 \arrow[r, "(\mathrm{id},0)"] & H^r_{\mathrm{dR}}(Z/L) \otimes_L B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \arrow[r] & 0,
\end{tikzcd}$$

whose first row and all columns are exact. Here, the first row is exact by the bicartesianness of the ($\mathrm{FD}^+_{i,r}$) in the case of X smooth dagger affinoid, and the second column is exact by (3.2.15). We deduce from it the exactness of the second row.

(iii) For X smooth Stein, the proof is the same as in (ii). □

3.2.18 - Remark. The proof of (ii) actually shows that for partially proper $X \in \mathrm{Rig}_C$, by using suitable Tate twists, the bicartesianness of ($\mathrm{FD}^+_{i,r}$) implies formally that of ($\mathrm{FD}_{i,r}$) thanks to (3.2.14), (3.2.15) and (3.2.16). Therefore, any partially proper $X \in \mathrm{Rig}_C$ satisfying C_{st}-conjecture, such as small varieties[27] listed in [24, Theorem 8.1], proper or smooth Stein rigid spaces, smooth dagger affinoids and the varieties that are products of proper varieties and Stein varieties [24, Proposition 8.17][28], are immediate instances at hand.

We are now ready to construct morphisms of spectral sequences.

3.2.19. Assume that ($\mathrm{FD}_{i,r}$) is bicartesian for any $i,r \geq 0$. Then there exists by (3.2.3) a natural morphism of spectral sequences

$$\begin{tikzcd}
{}^{\lim}E^{i,j}_t \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d, equal] \\
{}^{\mathrm{hyp}}E^{i,j}_t \arrow[r, Rightarrow] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}})
\end{tikzcd}$$

starting from the E_2-page with

$$\begin{aligned}
{}^{\lim}E^{i,j}_1 &:= H^j(\mathrm{gr}^i\, C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \\
{}^{\mathrm{hyp}}E^{i,j}_2 &:= H^i(\underline{\mathscr{G}_K}, H^j(C_{\mathrm{syn}}(r))).
\end{aligned}$$

[27] Recall that *small* varieties include proper varieties, quasi-compact quasi-separated dagger affinoids, analytification of algebraic varieties, certain tubular neighbourhoods of subvarieties of proper varieties or complements of such tubular neighbourhoods. More concretely speaking, a (smooth) dagger variety is said to be *small* if its de Rham cohomology is finite-dimensional.

[28] In *loc. cit.*, the proper factor was assumed to be smooth. But in fact, the argument there goes through even if we drop out the smoothness of the proper factor, cf. [11, Proof of Theorem 7.4].

3.2.20. Before Postnikov limit spectral sequence, we have another natural morphism of spectral sequences mapping to it

$$\begin{tikzcd} {}^{\mathrm{syn}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(R\Gamma_{\mathrm{syn}}(Z,r)) \arrow[d] & \simeq & H^{i+j}_{\mathrm{syn}}(Z,r) \\ {}^{\mathrm{lim}}E_t^{i,j} \arrow[r, Rightarrow] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) & & \end{tikzcd}$$

starting from the E_1-page from the Postnikov limit type *syntomic descent spectral sequence* associated to the similar Postnikov system on $R\Gamma_{\mathrm{syn}}(Z,r)$

$${}^{\mathrm{syn}}E_1^{i,j} := H^j(\mathrm{gr}^i\, R\Gamma_{\mathrm{syn}}(Z,r)),$$

whose E_2-page yields

$${}^{\mathrm{syn}}E_2^{i,j} := H^i(\mathrm{Syn}_{(H^j_{\mathrm{HK}}(Z), H^j_{\mathrm{dR}}(Z/K), \iota^{\mathrm{arith}}_{\mathrm{HK}}, H^j(F^r R\Gamma_{\mathrm{dR}}(Z/K)))}).$$

3.2.21. The hypercohomology spectral sequence maps naturally to the Hochschild–Serre spectral sequence for the proétale cohomology

$$\begin{tikzcd} {}^{\mathrm{hyp}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d] & & \\ {}^{\mathrm{HS}}E_t^{i,j} \arrow[r, Rightarrow] & H^{i+j}(R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))^{\underline{\mathscr{G}_K}}) & \simeq & H^{i+j}_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r)) \end{tikzcd}$$

starting from the E_2-page by (3.1.3.1), with

$${}^{\mathrm{HS}}E_2^{i,j} := H^i(\underline{\mathscr{G}_K}, H^j_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))).$$

3.2.22. Morphisms of spectral sequences. Combining the constructions (3.2.20), (3.2.19) and (3.2.21) altogether, one obtains a sequence of natural morphisms of spectral sequences

$$\begin{tikzcd} {}^{\mathrm{syn}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}_{\mathrm{syn}}(Z,r) \arrow[d] \\ {}^{\mathrm{lim}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d, equal] \\ {}^{\mathrm{hyp}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d] \\ {}^{\mathrm{HS}}E_t^{i,j} \arrow[r, Rightarrow] & H^{i+j}_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r)) \end{tikzcd}$$

starting at the E_2-page whenever the fundamental diagram ($\mathrm{FD}_{i,r}$) is bicartesian for Z_C and for any $i,r \geq 0$; this is satisfied for example if Z is partially proper and if the C_{st}-conjecture (3.2.9) holds for Z_C, by (3.2.18) and (3.2.11).

3.3 Syntomic descent spectral sequence

We are going to interpret the terms ${}^{\mathrm{syn}}E_1^{i,j}$ of the first page of the syntomic spectral sequence, which are expressed in terms of the arithmetic rigid space Z, as Galois invariants of cohomology groups of the geometric rigid space, in such a way that when Z is proper over K, the terms ${}^{\mathrm{syn}}E_2^{i,j}$ of the next page can be identified with *geometric Selmer groups* of Bloch–Kato, which we shall recall now.

Let us first make a digression to certain Galois cohomology groups, following [36, chapitre I, §3].

3.3.1 - Definition (Fontaine–Perrin-Riou). Let V be a solid $\mathbf{Q}_p$-representation of $\underline{\mathscr{G}_K}$.

(i) We define $H^i_{\mathrm{st}}(\mathscr{G}_K, V)$ as the i-th cohomology group of the homotopy limit

$$\left[\begin{tikzcd} & H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \arrow[d] \\ H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \arrow[r, "(\varphi-1,\iota)"] \arrow[d, "N"] & H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \oplus V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}) \arrow[d, "(N,0)"] \\ H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \arrow[r, "p\varphi-1"] & H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \end{tikzcd}\right]$$

and call it *modified geometric continous Galois cohomology of V*.

(ii) An extension $0 \to V \to W \to W' \to 0$ of solid $\mathbf{Q}_p$-representations of $\underline{\mathscr{G}_K}$ is called an st*-exact extension* if the sequence

$$0 \to H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to H^0(\underline{\mathscr{G}_K}, W' \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to 0$$

is exact in $\mathrm{Mod}^{\blacksquare}_F$.

(iii) An extension $0 \to V \to W \to W' \to 0$ of solid $\mathbf{Q}_p$-representations of $\underline{\mathscr{G}_K}$ is called an g*-exact extension* if the sequence

$$0 \to H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}}) \to H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}}) \to H^0(\underline{\mathscr{G}_K}, W' \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}}) \to 0$$

is exact in $\mathrm{Mod}^{\blacksquare}_F$.

(iv) An extension $0 \to V \to W \to W' \to 0$ of solid $\mathbf{Q}_p$-representations of $\underline{\mathscr{G}_K}$ is called an B^+_{dR}*-exact extension* if the sequence

$$0 \to H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, W' \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to 0$$

is exact in $\mathrm{Mod}^{\blacksquare}_K$.

3.3.2 - Remark. If V is a continuous *finite-dimensional* $\mathbf{Q}_p$-representation of $\mathscr{G}_K$, then $H^0(\underline{\mathscr{G}_K}, V \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}])$ is always finite-dimensional with F-dimension at most equal to $\dim_{\mathbf{Q}_p} V$ by Fontaine's period ring formalism.

3.3.3 - Remark. Let S be any profinite set and $W' = \mathbf{Q}_p[S]^{\blacksquare}$ with trivial $\underline{\mathscr{G}_K}$-action. Then an extension $0 \to V \to W \to \mathbf{Q}_p \to 0$ is an st-exact extension (*resp.* a B^+_{dR}-exact extension) if and only if there is a Galois equivariant $\mathbf{Q}_p$-linear section of $X \to \mathbf{Q}_p[S]^{\blacksquare}$, where X is defined as the pushout

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \arrow[r] & X \end{tikzcd} \quad \left(\textit{resp.} \quad \begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}} \arrow[r] & X \end{tikzcd} \right)$$

in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}$. Indeed, the "exact" extension property amounts to a F-linear (*resp.* K-linear) section of $H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to F[S]^{\blacksquare}$ (*resp.* of $H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to K[S]^{\blacksquare}$) by (the proof of) [36, I, Proposition 3.3.7]. Using the fact that $H^0(\underline{\mathscr{G}_K}, \mathbf{Q}_p[S]^{\blacksquare} \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) = F[S]^{\blacksquare}$ (*resp.* $H^0(\underline{\mathscr{G}_K}, \mathbf{Q}_p[S]^{\blacksquare} \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) = K[S]^{\blacksquare}$) which is a compact projective object in $\mathrm{Mod}^{\blacksquare}_F$ (*resp.* in $\mathrm{Mod}^{\blacksquare}_K$), this is equivalent to the desired existence of section.

3.3.4 - Remark. Let V be a flat solid $\mathbf{Q}_p$-representation of $\mathscr{G}_K$. We compare the group $H^i_{\mathrm{st}}(\mathscr{G}_K, V)$ with the closely related $H^i_g(\mathscr{G}_K, V)$ defined by Fontaine–Perrin-Riou in [36, I, 3.3.3].

First, recall the definition of *loc. cit.*: consider the sequence in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p[\mathscr{G}_K]}$

$$(S_g) \qquad 0 \to \mathbf{Q}_p \to B_{\mathrm{st}} \to B_{\mathrm{st}} \oplus B_{\mathrm{st}} \oplus B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to B_{\log}[\frac{1}{t}] \to 0$$

with $a \mapsto a$, $b \mapsto ((\varphi-1)b, Nb, \iota_p(b))$, $(b_1, b_2, c) \mapsto (Nb_1 - (p\varphi-1)b_2)$, which is exact; tensoring this with V, we still get an exact sequence in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p[\underline{\mathscr{G}_K}]}$

$$(S_g(V)) \qquad 0 \to V \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}} \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\mathrm{st}} \oplus B_{\mathrm{st}} \oplus B_{\mathrm{dR}}/B^+_{\mathrm{dR}}) \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}} \to 0.$$

The termwise Galois invariant $H^0(\underline{\mathscr{G}_K}, -)$ of this resolution complex calculates the cohomology $H^i_g(\underline{\mathscr{G}_K}, V)$.

Mimicking this procedure, we have exact sequences in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p[\underline{\mathscr{G}_K}]}$

$$(S_{\mathrm{st}}) \qquad 0 \to \mathbf{Q}_p \to B_{\log}[\frac{1}{t}] \oplus B^+_{\mathrm{dR}} \to B_{\log}[\frac{1}{t}] \oplus B_{\log}[\frac{1}{t}] \oplus B_{\mathrm{dR}} \to B_{\log}[\frac{1}{t}] \to 0$$

with $a \mapsto (a, a)$, $(b, c) \mapsto ((\varphi - 1)b, Nb, \iota_p(b) - c)$, $(b_1, b_2, c) \mapsto (Nb_1 - (p\varphi - 1)b_2)$, and

$$(S_{\mathrm{st}}(V)) \quad 0 \to V \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\log}[\frac{1}{t}] \oplus B^+_{\mathrm{dR}}) \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\log}[\frac{1}{t}] \oplus B_{\log}[\frac{1}{t}] \oplus B_{\mathrm{dR}}) \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \to 0.$$

By our definition, the termwise Galois invariant $H^0(\underline{\mathscr{G}_K}, -)$ of this resolution complex calculates the cohomology $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, V)$.

3.3.5 - Proposition. *Let V be a flat solid $\mathbf{Q}_p$-representation of $\underline{\mathscr{G}_K}$. Let $\nu \in \{g, \mathrm{st}\}$.*

(i) *The natural map $H^0_\nu(\underline{\mathscr{G}_K}, V) \to H^0(\underline{\mathscr{G}_K}, V)$ is an isomorphism.*

(ii) *The natural map $H^1_\nu(\underline{\mathscr{G}_K}, V) \to H^1(\underline{\mathscr{G}_K}, V)$ is injective, and identifies $H^1_\nu(\underline{\mathscr{G}_K}, V)(S)$ with the sub-$\mathbf{Q}_p$-vector space of $H^1(\underline{\mathscr{G}_K}, V)(S)$ classifying the g-exact extensions if $\nu = g$* (resp. st*-exact and B^+_{dR}-exact extensions if* $\nu =$ st) *W of $\mathbf{Q}_p[S]^{\blacksquare}$ by V.*

In particular, $H^1_{\mathrm{st}}(\underline{\mathscr{G}_K}, V) \hookrightarrow H^1_g(\underline{\mathscr{G}_K}, V) \hookrightarrow H^1(\underline{\mathscr{G}_K}, V)$.

We ignore whether the map $H^1_{\mathrm{st}}(\underline{\mathscr{G}_K}, V) \hookrightarrow H^1_g(\underline{\mathscr{G}_K}, V)$ is in practice an isomorphism or not.

Proof. For $\nu = g$, it is essentially contained in [36, I, Proposition 3.3.7]. It remains to establish the identification statement in the condensed setting. Notice that $H^1(\underline{\mathscr{G}_K}, V)(S) = \mathrm{Ext}^1_{\mathbf{Q}_p[\underline{\mathscr{G}_K}]}(\mathbf{Q}_p[S]^{\blacksquare}, V)$ and $H^1(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}})(S) = \mathrm{Ext}^1_{\mathbf{Q}_p[\underline{\mathscr{G}_K}]}(\mathbf{Q}_p[S]^{\blacksquare}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{st}})$. Then we may argue exactly as in the proof of *loc. cit.*, simply by replacing $\mathbf{Q}_p$ by $\mathbf{Q}_p[S]^{\blacksquare}$, using the remark (3.3.3).

For $\nu =$ st, the proof is the similar, noticing that $[W] \in \ker(H^1(\underline{\mathscr{G}_K}, V) \to H^1(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\log}[\frac{1}{t}] \oplus B^+_{\mathrm{dR}}))$ if and only if $[W] \in \ker(H^1(\underline{\mathscr{G}_K}, V) \to H^1(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}])$ as well as $[W] \in \ker(H^1(\underline{\mathscr{G}_K}, V) \to H^1(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}})$, so if and only if $0 \to V \to W \to \mathbf{Q}_p \to 0$ is an st-exact and B^+_{dR}-exact extension by (3.3.3). Similarly for S-valued points. □

Now let us come back to our syntomic descent spectral sequence. We are going to interpret the E_2-page of the syntomic descent spectral sequence (3.2.20) as Galois cohomology groups.

3.3.6. First, we interpret each term of ${}^{\mathrm{syn}}E_1^{i,j}$, which are on the arithmetic level, as the Galois invariants of geometric objects.

(i) For de Rham cohomology, we have $H^j_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \simeq H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})$ for any $Z \in \mathrm{Rig}^{(\dagger)}_K$ by flatness of B^+_{dR} in $\mathrm{Mod}^{\blacksquare}_K$. Using the nuclearity of $H^j_{\mathrm{dR}}(Z/K)$ over K, we obtain by (1.2.5) that

$$H^j_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K H^0(\underline{\mathscr{G}_K}, B^{(+)}_{\mathrm{dR}}) \simeq H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^{(+)}_{\mathrm{dR}}),$$

whence

$$H^j_{\mathrm{dR}}(Z/K) \simeq H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})) \simeq H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}).$$

(ii) As for the cohomology of the filtration part, we have that

$$H^j(F^r R\Gamma_{\mathrm{dR}}(Z/K)) \simeq H^0(\underline{\mathscr{G}_K}, H^j(F^r R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}))) \simeq H^0(\underline{\mathscr{G}_K}, H^j(F^r(R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})))$$

for any $Z \in \mathrm{Rig}^{(\dagger)}_K$. Indeed, the filtration $F^\bullet R\Gamma_{\mathrm{dR}}(Z/K)$ is finite (and separated), because Z is éh-locally smooth of the same dimension d, so the naive truncation filtration on $F^\bullet\Omega^\bullet_{Z/K,\text{éh}}$ becomes zero at F^{d+1}. We can write $\mathrm{Fil}^r_{\mathrm{Hdg}} R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}) \simeq F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})$ as an iterated extension of $\mathrm{gr}^0 R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K t^r B^+_{\mathrm{dR}}$ by $\mathrm{gr}^1 R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K t^{r-1} B^+_{\mathrm{dR}}$, ..., $\mathrm{gr}^{r-1} R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K tB^+_{\mathrm{dR}}$ and then by

$F^r R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}$; then applying the Galois cohomology computation (1.2.12, i), we get the first isomorphism. For the second isomorphism, since the filtration $F^{\bullet}R\Gamma_{\mathrm{dR}}(Z/K)$ is bounded from below, say stablising from $\bullet \geq N+1$ on for some $N \geq r$; so $F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})$ is an iterated extension of $F^N R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}}/t^{r-N}B^+_{\mathrm{dR}}$ by $F^{N-1}R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C(r-N+1)$, ..., $F^{r+1}R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C(-1)$ then by $F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})$; then applying the Galois cohomology computation (1.2.12, i), we get the second isomorphism.

(iii) For the Hyodo–Kato part, we have

$$H^j_{\mathrm{HK}}(Z) \simeq H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\mathrm{st}}) \simeq H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\log}[\frac{1}{t}])$$

for $Z \in \mathrm{Rig}^{(\dagger)}_K$ which are qcqs (2.1.22) (2.2.6) or partially proper (2.2.7).

As a result, for any $Z \in \mathrm{Rig}^{(\dagger)}_K$ that is qcqs or partially proper, the E_2-term ${}^{\mathrm{syn}}E_2^{i,j}$ is the i-th cohomology group of the homotopy limit

(3.3.6.1)
$$\left[\begin{array}{ccc} & & H^0(\underline{\mathscr{G}_K}, H^j(\mathrm{Fil}^r(R\Gamma_{\inf}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}))) \\ & & \downarrow \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\mathrm{st}}) & \xrightarrow{(\varphi - p^r, \iota)} & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\mathrm{st}} \oplus H^j_{\inf}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\mathrm{st}}) & \xrightarrow{p\varphi - p^r} & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\mathrm{st}}) \end{array}\right]$$

or, equivalently by (iii), that of the homotopy limit

(3.3.6.2)
$$\left[\begin{array}{ccc} & & H^0(\underline{\mathscr{G}_K}, H^j(\mathrm{Fil}^r(R\Gamma_{\inf}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}))) \\ & & \downarrow \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\log}[\frac{1}{t}]) & \xrightarrow{(\varphi - p^r, \iota)} & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\log}[\frac{1}{t}] \oplus H^j_{\inf}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\log}[\frac{1}{t}]) & \xrightarrow{p\varphi - p^r} & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve F} B_{\log}[\frac{1}{t}]) \end{array}\right],$$

thus finishing our interpretation.

Finally, we focus on the proper case.

3.3.7. Proper case. Let $Z \in \mathrm{Rig}_K$ be proper. By the semistable comparison theorem (3.2.17, i), the diagram (3.3.6.2) is identified with

$$\left[\begin{array}{ccc} & & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} t^r B^+_{\mathrm{dR}}) \\ & & \downarrow \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) & \xrightarrow{(\varphi - p^r, \iota)} & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \oplus H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) & \xrightarrow{p\varphi - p^r} & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]). \end{array}\right]$$

Hence, we can identify

(3.3.7.1)
$${}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^j_{\mathrm{\acute et}}(Z_C, \mathbf{Q}_p(r))).$$

A sanity check through the constructions implies that the morphism of spectral sequences on the E_2-page

${}^{\mathrm{syn}}E_2^{i,j} \to {}^{\mathrm{HS}}E_2^{i,j}$ is identified with the natural map $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p(r))) \to H^i(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p(r)))$.

3.3.8 - Remark. As we can see by degeneration of Hodge-to-de Rham spectral sequence at the E_1-page for proper $Z \in \mathrm{Rig}_K$, the map $H^j(F^r R\Gamma_{\mathrm{dR}}(Z/K)) \to H^j_{\mathrm{dR}}(Z)$ is injective onto $F^r H^j_{\mathrm{dR}}(Z)$, with cokernel identified with $H^j_{\mathrm{dR}}(Z/K)/F^r \simeq H^0(\underline{\mathscr{G}_K}, (H^j_{\mathrm{dR}}(Z/K) \otimes_K B_{\mathrm{dR}})/F^r) \simeq H^0(\underline{\mathscr{G}_K}, (H^j_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})/F^r)$. Then similar to (3.3.6.1), the E_2-term ${}^{\mathrm{syn}}E_2^{i,j}$ can be identified with the i-th cohomology group of the homotopy limit

$$\text{(3.3.8.1)}\qquad \left[\begin{tikzcd} H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{st}}) \arrow[r, "(\varphi - p^r, \iota)"] \arrow[d, "N"] & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{st}} \oplus H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}/B^+_{\mathrm{dR}}) \arrow[d, "(N,0)"] \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{st}}) \arrow[r, "p\varphi - p^r"] & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{st}}) \end{tikzcd}\right]$$

The same argument as above using the semistable comparison theorem (3.2.17, i) shows that we can also identify

$${}^{\mathrm{syn}}E_2^{i,j} = H^i_g(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(X, \mathbf{Q}_p(r))).$$

4 Chern classes of vector bundles and regulators

We are going to study the étale Chern classes and to show that the étale regulator maps factors through their syntomic counterparts.

Firstly, Chern class maps are classically constructed from certain first Chern class map and the projective bundle formula. We shall follow this approach, define various first Chern class maps and show the corresponding projective bundle formulae. Then we shall prove that the syntomic and étale Chern classes are compatible with the period maps that we have defined in the previous section.

For simplicity, we will only be considering the most *naive* zeroth K-group of a rigid space, namely the one defined as the Grothendieck group of the set of isomorphism classes of the étale (or equivalently, analytic) vector bundles on it modulo relations generated by short exact sequences. A higher analogue of it, which makes use of the recent condensed approach to analytic and continuous K-theories, will be the subject of a subsequent article.

4.1 First Chern class maps

In this subsection, we shall recall some elementary constructions of log-crystalline, de Rham and étale first Chern class maps, the first two of which will be used to define the syntomic first Chern class maps in two different but equivalent ways for rigid spaces over K or C.

4.1.1. Crystalline construction. Let us first recall Tsuji's construction of the log-crystalline first Chern class (following Kato) [68, (2.2.3)]. Let $\mathscr{X}$ be an *integral* [45, 2.1] and *quasi-coherent* [45, 2.1] log-scheme over a quasi-coherent log pd-scheme $S^\sharp = (S, \mathcal{L}, I, \gamma)$ with $p \in \mathcal{O}_S$ nilpotent.

Assume first that there is a log pd-$S^\sharp$-smooth thickening $\mathscr{X} \hookrightarrow P$ with log pd-envelope $\mathscr{X} \hookrightarrow D$ (if $\mathscr{X}$ is moreoever *log affine*[29], then there is a universal *coordinate pd-$S^\sharp$-thickening*[30] P^{univ} [6, 1.4, Remark (iii)], which however is not necessarily of finite type). Our goal is to construct a map

$$M^{\mathrm{gp}}_{\mathscr{X}} \to \mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^\bullet_{P/S^\sharp}[1]$$

[29] Recall [6, 1.1] that being *log affine* means that the underlying scheme is affine and that its log-structure $M_{\mathscr{X}}$ is generated by its global sections $\Gamma(\mathscr{X}, M_{\mathscr{X}})$.

[30] Recall [6, 1.4, Remark (i)] that a *coordinate pd-$S^\sharp$-thickening* is the pd-$S^\sharp$-envelope of a closed embedding into $\mathbf{A}^I_{(S,\mathcal{L})} \times_{(S,\mathcal{L})} \mathbf{A}^{(J)}_{(S,\mathcal{L})}$, the product of the affine space $\mathbf{A}^I_{(S,\mathcal{L})}$ and the logarithmic affine space $\mathbf{A}^{(J)}_{(S,\mathcal{L})}$ over $(S,\mathcal{L})$, indexed respectively by two (not necessarily finite) sets I and J.

in $\mathscr{D}(\mathscr{X}_{\text{ét}}, \mathbf{Z})$. This map will be constructed only in the derived category, as the composition

(4.1.1.1) $$M^{\text{gp}}_{\mathscr{X}} \xleftarrow{\simeq} (1 + J_D \to M^{\text{gp}}_D) \xrightarrow{(\log, \text{dlog})} (\mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^1_{P/S^\sharp} \to \cdots) = \mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^\bullet_{P/S^\sharp}[1]$$

in $\mathscr{D}(\mathscr{X}_{\text{ét}}, \mathbf{Z})$, where $1 + J_D$ and M^{gp} sit respectively at cohomological degrees -1 and 0.

In general, we follow the procedure of [6, 1.6, Remark] using an embedding system $E = \{\mathscr{X}_\bullet \hookrightarrow P_\bullet\}$; such liftings form a cofiltered system [44, (2.21)]. Let $\mathscr{X}_\bullet \to D_\bullet$ be the log pd-envelopes of $\mathscr{X}_\bullet \to P_\bullet$, defined by the pd-ideals $J_{D_\bullet} := \ker(\mathcal{O}_{D_\bullet} \to \mathcal{O}_{\mathscr{X}_\bullet})$. We have an adjoint pair of topoi

$$\theta^* \colon \mathscr{X}^\sim_{\text{ét}} \rightleftarrows \mathscr{X}^\sim_{\bullet,\text{ét}} \colon \theta_*$$

and their derived functors $\theta^* \dashv R\theta_*$. We have a map by (4.1.1.1)

(4.1.1.2) $$\theta^* M^{\text{gp}}_{\mathscr{X}} = M^{\text{gp}}_{\mathscr{X}_\bullet} \to \mathcal{O}_{D_\bullet} \otimes_{\mathcal{O}_{P_\bullet}} \omega^\bullet_{P_\bullet/S^\sharp}[1]$$

in $\mathscr{D}(\mathscr{X}_{\text{ét}}, \mathbf{Z})$, which induces by adjunction the *log-crystalline first Chern class map*

(4.1.1.3) $$c^{\text{cris}}_{1,\mathscr{X}/S^\sharp} \colon M^{\text{gp}}_{\mathscr{X}} \to R\theta_*(\mathcal{O}_{D_\bullet} \otimes_{\mathcal{O}_{P_\bullet}} \omega^\bullet_{P_\bullet/S^\sharp})[1] \simeq Ru^{\log}_{\mathscr{X}/S^\sharp *}\mathcal{O}_{\mathscr{X}/S^\sharp}[1]$$

where the last canonical isomorphism is given by [44, Proposition 2.20].

4.1.2 - Lemma (Compatibility of log-crystalline first Chern class maps). *For any commutative diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "g"] \arrow[d] & \mathscr{X}' \arrow[d] \\ S^\sharp \arrow[r, "f"] & S'^\sharp \end{tikzcd}$$

with $\mathscr{X}'$ an integral and quasi-coherent log-scheme over another quasi-coherent log pd-scheme $S'^\sharp$ with $p \in \mathcal{O}_{S'}$ nilpotent such that the underlying morphism of schemes is locally of finite type, the following diagram

$$\begin{tikzcd} M^{\text{gp}}_{\mathscr{X}} \arrow[d, "c^{\text{cris}}_{1,\mathscr{X}/S^\sharp}"] & & g^* M^{\text{gp}}_{\mathscr{X}'} \arrow[ll] \arrow[d, "c^{\text{cris}}_{1,\mathscr{X}'/S'^\sharp}"] \\ Ru^{\log}_{\mathscr{X}/S^\sharp *}\mathcal{O}_{\mathscr{X}/S^\sharp}[1] & & g^* Ru^{\log}_{\mathscr{X}'/S'^\sharp *}\mathcal{O}_{\mathscr{X}'/S'^\sharp}[1] \arrow[ll] \end{tikzcd}$$

in $\mathscr{D}(\mathscr{X}_{\text{ét}}, \mathbf{Z})$ commutes.

Proof. It is easily checked by choosing compatible embedding systems, which is possible by existence of universal coordinate thickenings. □

4.1.3. Characteristic p setting. By this we mean that $S^\sharp$ is a fine log pd-scheme endowed with a Frobenius action F lifting that of $S^\sharp/p$ and $\mathscr{X}$ is a fine log-scheme in characteristic p over $S^\sharp$.

4.1.4 - Lemma. *Let $\mathscr{X} \to S^\sharp$ be a characteristic p setting. Consider the natural Frobenius action φ on $Ru_*\mathcal{O}_{\mathscr{X}/S^\sharp}$. Then $c^{\text{cris}}_{1,\mathscr{X}/S^\sharp}$ factors through*

$$M^{\text{gp}}_{\mathscr{X}} \to (Ru_*\mathcal{O}_{\mathscr{X}/S^\sharp})^{\varphi=p}.$$

Proof. The embedding systems considered in (4.1.1) can be taken to be equipped with compatible Frobenius liftings $E = \{\mathscr{X}_\bullet \hookrightarrow P_\bullet\}$, whose log pd-envelopes $\mathscr{X}_\bullet \hookrightarrow D_\bullet$ are equipped with induced compatible Frobenius liftings $F_{D_\bullet}$. The Frobnius action on $Ru_*\mathcal{O}_{\mathscr{X}/S^\sharp} \simeq R\theta_*(\mathcal{O}_{D_\bullet} \otimes_{\mathcal{O}_{P_\bullet}} \omega^\bullet_{P_\bullet/S^\sharp})$ is induced by $F_{D_\bullet}$ and $F_{P_\bullet}$. The map (4.1.1.2) whence (4.1.1.3) is Frobenius equivariant. Then we are done since $\varphi = p$ on $M^{\text{gp}}_{\mathscr{X}}$ for $\mathscr{X}$ in characteristic p. □

4.1.5 - Example. Let L be a finite extension of $\mathbf{Q}_p$ with residue field k_L. Let $\mathfrak{Z}$ be a semistable formal scheme over $\mathcal{O}_L$ with log-structure $M_{\mathfrak{Z}} \to \mathcal{O}_{\mathfrak{Z}}$ induced by its special fibre. Let $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L^\times} \mathcal{O}_C^\times$.

Consider the following p-adic formal log pd-schemes:

- $\operatorname{Spf}\mathbf{Z}_p$ with trivial log-structure,
- $\operatorname{Spf}\mathcal{O}_{F_L}^{\mathrm{triv}}$,
- $\operatorname{Spf} r_L^{\mathrm{PD},0}$ with log-structure induced by the t_a's for $a \in (\mathfrak{m}_L/\mathfrak{m}_L^2)\backslash\{0\}$, together with log pd-thickening $p_0\colon \operatorname{Spec}\mathcal{O}_{F_L,1}^0 \hookrightarrow \operatorname{Spf} r_L^{\mathrm{PD},0}$,
- $\operatorname{Spf} r_L^{\mathrm{PD}}$ with log-structure induced by the t_a's for $a \in (\mathfrak{m}_L/\mathfrak{m}_L^2)\backslash\{0\}$, where l is a $\mathcal{O}_{F_L}$-class in $(\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$ determining a log pd-thickening $p_l\colon \operatorname{Spec}\mathcal{O}_{L,1}^{\times} \hookrightarrow \operatorname{Spf} r_L^{\mathrm{PD}}$ lifting p_0,
- $\operatorname{Spf}\mathcal{O}_{F_L}^0 := W_n(k_L)^0$,
- $\operatorname{Spf}\mathcal{O}_L^{\times}$,
- $\operatorname{Spf} A_{\mathrm{cris}}^{\times}$,
- $\operatorname{Spf}\widehat{A}_{l,\mathrm{st}}$, where l is a $\mathcal{O}_{\breve{F}}$-class in $(\mathfrak{m}_C/p\mathfrak{m}_C)\backslash\{0\}$.

They have respectively $\operatorname{Spec}\mathbf{F}_p$, $\operatorname{Spec} k_L^{\mathrm{triv}}$, $\operatorname{Spec} k_L\{t_a\}$ and $\operatorname{Spec} k_L^0$ as reduction modulo p which all receive maps from $\mathfrak{Z}_1^0$, and are related by morphisms of p-adic formal log pd-rings

$$\mathbf{Z}_p \to \mathcal{O}_{F_L}^{\mathrm{triv}} \to r_L^{\mathrm{PD},0} \xrightarrow{p_0} \mathcal{O}_{F_L}^0.$$

Applying the construction (4.1.1) to these settings, one obtains respective first Chern class maps

- $c_{1,\mathfrak{Z}_n}^{\mathrm{cris}}$ for $\mathfrak{Z}_n$ over $S^\sharp = \mathbf{Z}_p$ or $\mathcal{O}_{F_L}^{\mathrm{triv}}$,
- $c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}$ for $S^\sharp = r_L^{\mathrm{PD},0}$, and $c_{1,\mathfrak{Z}_1,l}^{\mathrm{PD}}$ for $S^\sharp = r_L^{\mathrm{PD}}$,
- $c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}$ for $S^\sharp = \mathcal{O}_{F_L}^0$,
- $c_{1,\mathfrak{Z}_n}^{\mathrm{crdR}}$ for $\mathfrak{Z}_n$ over $S^\sharp = \mathcal{O}_L^{\times}$,
- $c_{1,\mathfrak{X}_n}^{A_{\mathrm{cris}}}$ for $\mathfrak{X}_n$ over $S^\sharp = A_{\mathrm{cris}}^{\times}$,
- $c_{1,\mathfrak{X}_1,l}^{\mathrm{st}}$ for $S^\sharp = \widehat{A}_{l,\mathrm{st}}$.

4.1.6 - Lemma. *The first Chern class maps $c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}$ and $c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}$ factor respectively through*

$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}\colon M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}})^{\varphi=p,N=0}[1],$$
$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}\colon M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{HK},0}})^{\varphi=p,N=0}[1].$$

Proof. The compatibility (4.1.2) affirms that the diagram

$$\begin{tikzcd}
M_{\mathfrak{Z}_1}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1}^{\mathrm{cris}}"] \arrow[d, equal] & Ru_*\mathcal{O}_{\mathfrak{Z}_1/\mathbf{Z}_p}[1] \arrow[d, "\simeq"] \\
M_{\mathfrak{Z}_1}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1}^{\mathrm{cris}}"] \arrow[d] & Ru_*\mathcal{O}_{\mathfrak{Z}_1/\mathcal{O}_{F_L}^{\mathrm{triv}}}[1] \arrow[d] \\
M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1^0}^{\mathrm{cris}}"] \arrow[d, equal] & Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^{\mathrm{triv}}}[1] \arrow[d] \\
M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}"] \arrow[d, equal] & Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}}[1] \arrow[d] \\
M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}"] & Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0}[1]
\end{tikzcd}$$

commutes. The fourth right vertical map is N-equivariant. Moreover, by [46, Lemma 4.2], the third right vertical map factors through $Ru_*\mathcal{O}_{\mathfrak{Z}_1/\mathcal{O}_{F_L}^{\mathrm{triv}}} \xrightarrow{\simeq} (Ru_*\mathcal{O}_{\mathfrak{Z}_1/r_L^{\mathrm{PD},0}})^{N=0}$. Hence $c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}$ and $c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}$ factor respectively through

$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}\colon M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}})^{N=0}[1]$$

$$c^{\mathrm{HK}}_{1,\mathfrak{Z}^0_1}\colon M^{\mathrm{gp}}_{\mathfrak{Z}^0_1} \to (Ru_*\mathcal{O}_{\mathfrak{Z}^0_1/r^{\mathrm{HK},0}_L})^{N=0}[1].$$

Then (4.1.4) helps to conclude. □

4.1.7. De Rham construction. Next, let us recall the construction of the *rigid-analytic de Rham first Chern class map*

$$c^{\mathrm{dR}}_1\colon \mathcal{O}^\times_Z \to \mathrm{Fil}^1\Omega^\bullet_{Z/K}[1] \tag{4.1.7.1}$$

of éh-sheaves over Z for an rigid-analytic variety Z over K, and that of the *infinitesimal B^+_{dR} first Chern class map*

$$c^{\mathrm{inf}}_1\colon \mathcal{O}^\times_X \to Ru_{X/B^+_{\mathrm{dR}}*}\mathcal{O}_{\mathrm{inf},X/B^+_{\mathrm{dR}}}[1] \tag{4.1.7.2}$$

of éh-sheaves over X for an rigid-analytic variety X over C. Here, $u_{X/B^+_{\mathrm{dR}}*}$ denotes the canonical morphism of sites from $(X/B^+_{\mathrm{dR}})_{\mathrm{inf}}$ to $X_{\text{éh}}$.

Assume first that everything is smooth. In the geometric case, one can proceeds as in the crystalline case using embedding systems, so as to reduce to the case where there exists $X \hookrightarrow P$ with latter smooth over B^+_{dR}. In this case, letting D be the envelope of X in P [41, Definition 2.2.1, and §4.1], our map is then constructed as

$$\mathcal{O}^\times_X \xleftarrow{\simeq} (1+\mathcal{J}_{\mathrm{inf},D} \to \mathcal{O}^\times_{\mathrm{inf},D}) \xrightarrow{(\log,\mathrm{dlog})} (\mathcal{O}_{\mathrm{inf},D} \to \mathcal{O}_{\mathrm{inf},D}\otimes_{\mathcal{O}_{\mathrm{inf},P}}\Omega^1_{P/B^+_{\mathrm{dR}}} \to \cdots) = \mathcal{O}_{\mathrm{inf},D}\otimes_{\mathcal{O}_{\mathrm{inf},P}}\Omega^\bullet_{P/B^+_{\mathrm{dR}}}[1]. \tag{4.1.7.3}$$

in $\mathscr{D}(X_{\text{ét}},\mathbf{Z})$, where the last complex is identified with $R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}})$ [41, Theorem 4.1.1]. The logarithm is well-defined. Indeed, by definition, if we let I be defining ideal of X in P, which is a coherent ideal, then $D := \varinjlim_{n\in\mathbf{N}} h_{P_n}$ where P_n is the n-th infinitesimal neighbourhood of X in P, defined by the ideal I^{n+1}; and $\mathcal{O}_D := \varprojlim_{n\in\mathbf{N}}\mathcal{O}_{P_n}$, $\mathcal{J}_{\mathrm{inf},D} := \ker(\mathcal{O}_D \to \mathcal{O}_X) = \varprojlim_{n\in\mathbf{N}}\ker(P_n \to \mathcal{O}_X)$. The logarithm is well-defined on $1+I^{n+1} \to \mathcal{O}_{P_n}$ by I-nilpotency and since we are over $\mathbf{Q}_p$; then it suffices to pass to the limit.

For the arithmetic case, one could do the same using the infinitesimal site $(Z/K)_{\mathrm{inf}}$ which calculates de Rham cohomology [41, Theorem 1.2.1 (iii)], but it would be far more elementary if we define c^{dR}_1 directly as the map of genuine complexes

$$\mathcal{O}^\times_Z \xrightarrow{\mathrm{dlog}} (\Omega^1_{Z/K} \to \Omega^2_{Z/L} \to \cdots) = \mathrm{Fil}^1\Omega^\bullet_{Z/K}[1].$$

In general for singular X, éh-descent suffices to conclude.

Using infinitesimal interpretation, we see that c^{dR}_1 and c^{inf}_1 are compatible.

4.1.8 - Lemma. *The map c^{dR}_1 factors through*

$$c^{\mathrm{inf}}_1\colon \mathcal{O}^\times_X \to \mathrm{Fil}^1_{\mathrm{Hdg}}\Omega^\bullet_{X/B^+_{\mathrm{dR}}}.$$

Proof. In the case where there exists $X \hookrightarrow P$ with latter smooth over B^+_{dR} and D is the envelope of X in P, it is easily from the expression that the second map in (4.1.7.3) factors through $(\mathcal{J}_{\mathrm{inf},D} \to \mathcal{O}_{\mathrm{inf},D}\otimes_{\mathcal{O}_{\mathrm{inf},P}}\Omega^1_{P/B^+_{\mathrm{dR}}} \to \cdots) = \mathrm{Fil}^1(\mathcal{O}_{\mathrm{inf},D}\otimes_{\mathcal{O}_{\mathrm{inf},P}}\Omega^\bullet_{P/B^+_{\mathrm{dR}}})[1]$. In general, it follows from simplicial construction then éh-descent. □

4.1.9. Syntomic construction. For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$, we want to define its *syntomic first Chern class map*

$$c^{\mathrm{syn}}_{1,\mathfrak{Z}}\colon M^{\mathrm{gp}}_{\mathfrak{Z}} \to R\Gamma_{\mathrm{syn}}(\mathfrak{Z}_\eta,1)[1] \tag{4.1.9.1}$$

in $\mathscr{D}((\mathfrak{Z}_\eta)_{\text{ét}},\mathbf{Z})$.

By (2.3.3), we may use the natural identification $R\Gamma_{\mathrm{syn}}(\mathfrak{Z}_\eta,1) \simeq [R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p}_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)/\mathrm{Fil}^1]$. Consider the following diagram

$$\begin{tikzcd} \Gamma(\mathfrak{Z},M^{\mathrm{gp}}_{\mathfrak{Z}}) \arrow[r, "c^{\mathrm{cris}}_1"] \arrow[d, "\simeq"] \arrow[rr, bend right, "c^{\mathrm{crdR}}_1"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p}_{\mathbf{Q}_p}[1] \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}^\times_L)_{\mathbf{Q}_p}[1] \arrow[d, "\simeq"] \\ \Gamma(\mathfrak{Z}_\eta,\mathcal{O}^\times_{\mathfrak{Z}_\eta}) \arrow[r, "c^{\mathrm{dR}}_1"] & \mathrm{Fil}^1R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1] \arrow[r] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1] \end{tikzcd} \tag{4.1.9.2}$$

where the left vertical isomorphism results from $M_{\mathfrak{Z}}^{\mathrm{gp}} \simeq j_* \mathcal{O}_{\mathfrak{Z}_\eta}^{\times}$ (where $j\colon (\mathfrak{Z}_\eta)_{\text{ét}} \to \mathfrak{Z}_{\text{ét}}$ is the natural morphism of sites), and the right vertical isomorphism will be recalled below. Now, for commutativity, the maps c_1^{cris} and c_1^{crdR} are naturally compatible by (4.1.2), and we will now show that c_1^{crdR} and c_1^{dR} are compatible by similarity between constructions of crystalline and infinitesimal cohomologies. Let $\alpha\colon (\mathfrak{Z}_\eta/L)_{\mathrm{inf}} \to (\mathfrak{Z}/\mathcal{O}_L^{\times})_{\mathrm{cris}}$ be the morphism of sites defined via the generic fibre functor sending $(\mathfrak{U} \hookrightarrow \mathfrak{T})$ over $\mathcal{O}_L^{\times}$ to $\mathfrak{U}_\eta \hookrightarrow \mathfrak{T}_\eta$ over L. Then α induces a morphism $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^{\times}) \xrightarrow{\simeq} R\alpha_* R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L)$. Since $\mathfrak{Z}$ is semistable, $\mathfrak{Z}$ is log-smooth over $\mathcal{O}_L^{\times}$ and $\mathfrak{Z}_\eta$ is smooth over L, hence in our definition of first Chern class maps, one can choose $\mathfrak{Z} \to \mathfrak{P}$ and $\mathfrak{Z}_\eta \hookrightarrow P$ to be identities. This gives the following commutative diagram

$$\begin{tikzcd}
M_{\mathfrak{Z}}^{\mathrm{gp}} \arrow[d, "\simeq"] & (1 \to M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[l, "\simeq"'] \arrow[r, "(\log,\mathrm{dlog})"] \arrow[d, "\simeq"] & (\mathcal{O}_{\mathfrak{Z}} \to \omega^1_{\mathfrak{Z}/\mathcal{O}_L^{\times}} \to \cdots) \arrow[r, equal] \arrow[d, "\simeq"] & \omega^{\bullet}_{\mathfrak{Z}/\mathcal{O}_L^{\times}}[1] \arrow[d, "\simeq \text{ after } (-)_{\mathbf{Q}_p}"] \\
\alpha_* \mathcal{O}_{\mathfrak{Z}_\eta}^{\times} & (1 \to \alpha_* \mathcal{O}^{\times}_{\mathrm{inf},\mathfrak{Z}_\eta}) \arrow[l, "\simeq"'] \arrow[r, "(\log,\mathrm{dlog})"] & (\alpha_* \mathcal{O}_{\mathrm{inf},\mathfrak{Z}_\eta} \to \alpha_* \Omega^1_{\mathfrak{Z}_\eta/L} \to \cdots) \arrow[r, "\simeq"] & R\alpha_* \Omega^{\bullet}_{\mathfrak{Z}_\eta/L}[1]
\end{tikzcd}$$

which calculates the diagram at the beginning of the proof involving c_1^{crdR} and c_1^{dR}; for the right vertical isomorphism, we notice that $R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathfrak{Z}/\mathcal{O}_L^{\times}})_{\mathbf{Q}_p} \xrightarrow{\simeq} R\Gamma(\mathfrak{Z}_\eta, \Omega^{\bullet}_{\mathfrak{Z}_\eta/K})$ for affine $\mathfrak{Z}$ semistable over $\mathcal{O}_L$).

By commutativity of (4.1.9.2), the morphism crystalline Chern class map c_1^{cris} factors through the fibre $[R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}^{\varphi=p} \to R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)/\mathrm{Fil}^1] = R\Gamma_{\mathrm{syn}}(\mathfrak{Z}_\eta, 1)$, thus defining (4.1.9.1).

Finally, for $Z \in \mathrm{Rig}_K$, by éh-descent, one defines its *syntomic first Chern class map*

$$c_1^{\mathrm{syn}}\colon \mathcal{O}_Z^{\times} \to R\Gamma_{\mathrm{syn}}(Z,1)[1] \tag{4.1.9.3}$$

in $\mathscr{D}(Z_{\text{éh}}, \mathbf{Z})$. Taking first cohomology, we obtain (by abuse of notation) a morphism

$$c_1^{\mathrm{syn}}\colon \mathrm{Pic}(Z) \simeq H^1_{\text{ét}}(Z, \mathcal{O}_Z^{\times}) \to H^2_{\mathrm{syn}}(Z,1). \tag{4.1.9.4}$$

In other words, we may associate with any line bundle $\mathcal{L}$ on $Z_{\text{ét}}$ a class $c_1^{\mathrm{syn}}(\mathcal{L}) \in H^2_{\mathrm{syn}}(Z,1)$.

4.1.10 - Remark. We could also have defined syntomic first Chern class using c_1^{HK} and c_1^{dR} if we have proven the compatibility between them, whose construction is shown as follows.

4.1.11 - Proposition. *For $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L, there is a natural commutative diagram*

$$\begin{tikzcd}
R\Gamma(\mathfrak{Z}, M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[r] \arrow[d] & R\Gamma(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \arrow[r, "c_1^{\mathrm{HK}}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}[1] \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{arith}}"] \\
R\Gamma(\mathfrak{Z}_\eta, \mathcal{O}_{\mathfrak{Z}_\eta}^{\times}) \arrow[r, "c_1^{\mathrm{dR}}"] & \mathrm{Fil}^1 R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1] \arrow[r] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1].
\end{tikzcd}$$

Moreover, the syntomic first Chern class map defined by this is equivalent to that of (4.1.9.1).

Proof. Once the natural commutativity is established, the last statement on natural equivalence will follow thanks to (the proof of) the equivalence (2.3.3), the compatibility between c_1^{HK} and c_1^{cris} (4.1.2), and that between c_1^{cris} and c_1^{dR} (4.1.9.2). So we are only left to treat the natural commutativity of the diagram.

We may assume $\mathfrak{Z}$ to be qcqs by Zariski descent; then $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)$ is represented by a bounded complex of K-Banach spaces, so that (1.2.12, ii) applies. Let $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C \in \mathcal{M}_C^{\mathrm{ss},b}$. By definition of Hyodo–Kato morphisms (2.1.4) and compatibility between c_1^{dR} and c_1^{inf} (4.1.7), it suffices to prove that the following diagram

$$\begin{tikzcd}
R\Gamma(\mathfrak{Z}, M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[r] \arrow[d] & R\Gamma(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}})_{\mathbf{Q}_p} \arrow[r, "c_1^{\mathrm{HK}}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}[1] \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \\
R\Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta}^{\times}) \arrow[rr, "c_1^{\mathrm{inf}}"] & & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+)[1]
\end{tikzcd}$$

commutes naturally and $\underline{\mathscr{G}_L}$-equivariantly.

Let us consider the $\underline{\mathscr{G}_L}$-equivariant diagram (2.1.3.3) which defines the geometric Hyodo–Kato isomorphism $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$ (see the proof of theorem (2.1.3) for further explanation of notation), and record it here below for clarity of the proof:

(4.1.11.1)

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[d, "\simeq"', "(\mathrm{Frob}^n)^*"] \arrow[r, bend right, "\iota_0"'] & R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/r^{\mathrm{PD},0}_L)^{N\text{-nilp}}_{\mathbf{Q}_p} \arrow[l, "p_0^*"'] \arrow[drr, "\pi^*"] & & \\
R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} & & & R\Gamma_{\mathrm{cris}}(\mathscr{X}/\widehat{A}_{l_\pi,\mathrm{st}})^{N\text{-nilp}}_{\mathbf{Q}_p} & R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^{\times}_{\mathrm{cris}})\otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}}. \arrow[l, "\kappa^*_{l_\pi}"', "\simeq"]
\end{tikzcd}$$

Recall that here, ι_0 is the Hyodo–Kato section, which is (φ, N)-equivariant [23, Theorem 2.12, Proposition 2.14].

By naturality (4.1.2), c_1^{PD} is compatible with $c^{\mathrm{st}}_{1,l_\pi}$, and $c^{\mathrm{st}}_{1,l_\varpi}$ is compatible with $c_1^{A_{\mathrm{cris}}}$, which is in turn compatible with c_1^{inf} by similarity between crystalline and infinitesimal cohomologies via the morphism of sites $(\mathfrak{X}_\eta/B^+_{\mathrm{dR},m})_{\mathrm{inf}} \to (\mathfrak{X}/A^{\times}_{\mathrm{cris}})_{\mathrm{cris}}$ which is defined by the functor sending a pair $(\mathfrak{U} \hookrightarrow \mathfrak{T} = \mathrm{Spf}\,P)$ over $A^{\times}_{\mathrm{cris}}$ to $(\mathfrak{U}_\eta \hookrightarrow \mathfrak{T}_{B^+_{\mathrm{dR},m}} := \mathfrak{T}_\eta \times_{\mathrm{Spa}\,B^+_{\mathrm{cris}}} \mathrm{Spa}\,B^+_{\mathrm{dR},m} = \mathrm{Spa}(B\hat{\otimes}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR},m}))$ over $B^+_{\mathrm{dR},m}$. On the other hand, similarly by naturality, we have $(\mathrm{Frob}^n)^*(c_1^{\mathrm{HK}})_{\mathbf{Q}_p} \simeq p(c_1^{\mathrm{HK}})_{\mathbf{Q}_p}$.

It remains to show that there is a natural equivalence $\iota_0 \circ (c_1^{\mathrm{HK}})_{\mathbf{Q}_p} \simeq (c_1^{\mathrm{PD}})_{\mathbf{Q}_p}$ commuting with the N-action. It suffices to show that their difference $\delta := \iota_0 \circ (c_1^{\mathrm{HK}})_{\mathbf{Q}_p} - (c_1^{\mathrm{PD}})_{\mathbf{Q}_p}$ is homotopic to zero. By compatibility of c_1^{PD} and c_1^{HK} via the natural map $p_0 \colon r^{\mathrm{PD},0}_L \to \mathcal{O}^0_{F_L}$, and by the (φ, N)-equivariance of the isomorphism $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} r^{\mathrm{PD},0}_L[\frac{1}{p}] \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r^{\mathrm{PD},0}_L)_{\mathbf{Q}_p}$, the difference δ factors φ-equivariantly through

$$\delta' \colon R\Gamma(\mathfrak{Z}^0_1, M^{\mathrm{gp}}_{\mathfrak{Z}^0_1}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_1[1]$$

where $I_1 = \ker(r^{\mathrm{PD},0}_L \to \mathcal{O}_{F_L})[\frac{1}{p}]$, which is a F_L-Banach space.

Now, we are reduced to showing that the map (by abuse of notation) on cohomology groups

$$\delta' \colon H^i_{\text{ét}}(\mathfrak{Z}^0_1, M^{\mathrm{gp}}_{\mathfrak{Z}^0_1}) \to H^{i+1}_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_1 \tag{4.1.11.2}$$

is equal to zero.

For this, we need more notation. For $n \in \mathbf{N}$, denote

$$I_n := \left\{ \sum_{i=n}^{+\infty} a_i \frac{t_a^i}{i!} \mid a_i \in F_L, \lim_{i \to +\infty} a_i = 0 \right\} \subset r^{\mathrm{PD},0}_L[\frac{1}{p}]. \tag{4.1.11.3}$$

This is a F-linear closed (whence Banach) subspace (even an ideal) with finite-dimensional complement $\bigoplus_{i=0}^{n-1} F_L \frac{t_a^i}{i!}$, and does not depend on the choice of $a \in (\mathfrak{m}_L/\mathfrak{m}_L^2)\backslash\{0\}$. The n-th power of frobenius φ^n on $r^{\mathrm{PD},0}_L$ when restricted to I_1 factors through I_n, i.e. $\varphi^n(I_1) \subset I_n$.

Now we treat the vanishing of (4.1.11.2). Knowing that $\varphi(c_1^{\mathrm{HK}}) = pc_1^{\mathrm{HK}}$ and $\varphi(c_1^{\mathrm{PD}}) = pc_1^{\mathrm{PD}}$, we have $\varphi(\delta') = p\delta'$. By φ-equivariance of δ' and invertibility of $\varphi = p$ on $R\Gamma_{\text{ét}}(\mathfrak{Z}^0_1, M^{\mathrm{gp}}_{\mathfrak{Z}^0_1})_{\mathbf{Q}_p}$, we have $\delta' = \frac{1}{p^n}\varphi^n\delta'$ which factors as

$$\delta' \colon H^i_{\text{ét}}(\mathfrak{Z}^0_1, M^{\mathrm{gp}}_{\mathfrak{Z}^0_1}) \to H^{i+1}_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_n \to H^{i+1}_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_1.$$

Hence the map δ' factors through

$$\delta'' \colon H^i_{\text{ét}}(\mathfrak{Z}^0_1, M^{\mathrm{gp}}_{\mathfrak{Z}^0_1}) \to \varprojlim_n \left(H^{i+1}_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} t_a^{p^n-1} r^{\mathrm{PD},0}_L[\frac{1}{p}] \right),$$

where the transition maps are induced by inclusions of ideals $I_n = t_a^{p^n-1} r^{\mathrm{PD},0}_L \subset r^{\mathrm{PD},0}_L$. Since $\mathfrak{Z}$ is affine (hence qcqs), the $H^{i+1}_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$ is a direct summand of the cokernel of a morphism between F_L-Banach spaces (2.1.8), so that Lemma (1.1.8.1) allows us to conclude. $\square$

4.1.12 - Remark. We could also have resorted to the derived limit approach, knowing that $R\Gamma_{\mathrm{HK}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$ is a bounded complex of F_L-Banach spaces, for which tensor product commutes with countable products on each factor. Apparently however, this will not work, because $R\varprojlim_n t_a^{p^n-1} r_L^{\mathrm{PD},0}[\frac{1}{p}] \simeq R^1\varprojlim_n t_a^{p^n-1} r_L^{\mathrm{PD},0}[\frac{1}{p}][-1]$ which does not vanish by an easy computation, e.g. $F\langle T\rangle \to \prod_{\mathbf{N}} F$ is not surjective. Nevertheless, the last map is injective, which was our principal motivation of using the vanishing results in (1.1.8.2).

4.1.13. Hyodo–Kato construction. Globalising the composed map

$$\Gamma(\mathfrak{Z}, M_{\mathfrak{Z}}^{\mathrm{gp}}) \to R\Gamma(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}})_{\mathbf{Q}_p} \xrightarrow{c_1^{\mathrm{HK}}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$$

for $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$, knowing that $M_{\mathfrak{Z}}^{\mathrm{gp}} \simeq j_*\mathcal{O}_{\mathfrak{Z}_\eta}^\times$ (where $j\colon (\mathfrak{Z}_\eta)_{\text{ét}} \to \mathfrak{Z}_{\text{ét}}$ is the natural morphism of sites), we obtain the Hyodo–Kato first Chern class map

$$c_1^{\mathrm{HK}}\colon \Gamma(Z, \mathcal{O}_Z^\times) \to R\Gamma_{\mathrm{HK}}(Z)[1].$$

As in the semistable case, it factors through $\Gamma(Z,\mathcal{O}_Z^\times) \to R\Gamma_{\mathrm{HK}}(Z)^{\varphi=p, N=0}[1]$.

4.1.14 - Corollary. *For $Z \in \mathrm{Rig}_K$, there is a natural commutative diagram*

$$\begin{tikzcd}
\Gamma(Z,\mathcal{O}_Z^\times) \arrow[r, "c_1^{\mathrm{HK}}"] \arrow[dr, "c_1^{\mathrm{dR}}"'] & R\Gamma_{\mathrm{HK}}(Z)[1] \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{arith}}"] \\
& R\Gamma_{\mathrm{dR}}(Z/K)[1].
\end{tikzcd}$$

Moreover, the syntomic first Chern class map defined by this is equivalent to that of (4.1.9.3).

Proof. This follows from the above proposition by éh-descent. □

4.1.15. Étale construction. For Z either schemes or analytic adic spaces (so that $\mathcal{O}_Z^\times$ is an étale sheaf [43, 2.2.6]) over K (which is a nonarchimedean field of characteristic 0), there is a short exact sequence of étale sheaves named *Kummer sequence* on $Z_{\text{ét}}$

$$0 \to \mu_{p^n} \to \mathcal{O}_Z^\times \to \mathcal{O}_Z^\times \to 0$$

for any $n \in \mathbf{N}$, from which we obtain a mod p^n *étale first Chern class map* $\mathcal{O}_Z^\times \to \mu_{p^n}[1]$ in $\mathscr{D}(Z_{\text{ét}}, \mathbf{Z}/p^n) \hookrightarrow \mathscr{D}(Z_{\text{ét}}, \mathbf{Z}_p)$, whence an *étale first Chern class map*

$$c_1^{\text{ét}}\colon \mathcal{O}_Z^\times \to \mathbf{Z}_p(1)[1].$$

The algebraic and analytic étale first Chern class maps are compatible through the analytification functor.

4.2 Projective bundle formula and $\mathbf{A}^1$-homotopy invariance

We are going to prove the projective bundle formula and $\mathbf{A}^1$-homotopy invariance for Hyodo–Kato, de Rham and étale cohomology theories. By contrast, for syntomic cohomology and pro-étale cohomology, the projective bundle formula holds but the $\mathbf{A}^1$-homotopy invariance fails.

First, we need to recall the natural product structures on them.

4.2.1. The (relative) cup product is defined in general for morphisms of ringed spaces [66, Remark 0B68]. In many cases, it can be quite explicit and such formula could be useful for proving compatibilities.

The relative cup product is compatible with respect to commutative squares, namely, for any commutative diagram of ringed spaces

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

and $F, G \in \mathrm{Mod}_{\mathcal{O}_X}$, the following diagram commutes naturally

$$\begin{tikzcd}
g^*(Rf_*F \otimes^L Rf_*G) \arrow[r, "g^*(-\cup-)"] \arrow[d] & g^*Rf_*(F \otimes G) \arrow[d] \\
(Rf'_*g'^*F) \otimes^L (Rf'_*g'^*G) \arrow[r, "-\cup-"] & Rf'_*(g'^*F \otimes^L g'^*G).
\end{tikzcd}$$

This follows formally from *loc. cit.*. Indeed, the vertical maps are induced by $g^*Rf_* \to Rf'_*g'^*$, adjoint to $f'^*g^*Rf_* \simeq g'^*f^*Rf_* \to g'^*$ the $g'^* \circ$ counit. The adjoint diagram (associated to $f'^* \dashv Rf'_*$) of the above is then

$$\begin{tikzcd}
g'^*f^*(Rf_*F \otimes^L Rf_*G) \arrow[r, "g'^*f^*(-\cup-)"] \arrow[d, "\mathrm{unit}\circ g'^*\circ\mathrm{counit}"] & g'^*f^*Rf_*(F \otimes G) \arrow[d, "g'^*\circ\mathrm{counit}"] \\
f'^*(Rf'_*g'^*F) \otimes^L f'^*(Rf'_*g'^*G) \arrow[r, "\mathrm{counit}\circ g'^*"] & g'^*F \otimes^L g'^*G.
\end{tikzcd}$$

The lower left two morphisms composed to a diagonal morphism $g'^* \circ$ counit, hence its natural commutativity is reduced to the commutativity of the upper triangle, and ***a fortiori*** that of the triangle

$$\begin{tikzcd}
f^*(Rf_*F \otimes^L Rf_*G) \arrow[r, "f^*(-\cup-)"] \arrow[dr, "\mathrm{counit}"'] & f^*Rf_*(F \otimes^L G) \arrow[d, "\mathrm{counit}"] \\
& F \otimes^L G.
\end{tikzcd}$$

But the diagonal morphism here is by definition adjoint to the cup product $-\cup-$, whence the commutativity.

4.2.2. Product structure on log-crystalline cohomology. Let $\mathscr{X}$ be an integral and quasi-coherent log-scheme over a quasi-coherent log pd-scheme $S^\sharp$, as in the setting of (4.1.1. The product structure on log-crystalline cohomology $R\Gamma_{\mathrm{cris}}(\mathscr{X}/S^\sharp)$ is described as induced from the product structure of $\mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^\bullet_{P/S^\sharp}$ when $\mathscr{X} \hookrightarrow P$ is a log pd-$S^\sharp$-smooth thickening with log pd-envelope $\mathscr{X} \hookrightarrow D$, then by étale hyperdescent in general cases. It satisfies higher associativity relations. The product structure thus defined is natural with respect to morphism of log-crystalline sites, and is compatible with Frobenius action in the characteristic p setting (4.1.3).

4.2.3 - Example. Let us continue the example (4.1.5): let $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L; let l be a $\mathcal{O}_{F_L}$-class in $(\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$. The natural morphisms

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_l \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)$$

are compatible with φ-equivariant product structures.

4.2.4 - Lemma. *Consider a $\mathcal{O}_{F_L}$-class $l \in (\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$ associated with $i_l^*\colon r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}_{L,1}^\times$ lifting $r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}_{F_L,1}^0$ along $\mathcal{O}_{L,1}^\times \twoheadrightarrow \mathcal{O}_{F_L,1}^0$. Then the Hyodo–Kato section $\iota_{0,l}\colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}$ is compatible with product structures.*

Proof. The same strategy of the last part of the proof of (4.1.11) applies: one looks at the difference of two maps δ, which factors through $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_1[1]$. By φ-equivariance of δ' and invertibility of φ on $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$, we have $\delta' = \varphi^n\delta'\varphi^{-n}$ factoring as

$$\delta'\colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes^{L\blacksquare}_{F_L} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_n[1] \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L} I_1[1].$$

Then we apply (1.1.8.1) to conclude. □

4.2.5 - Example. Let us continue the example (4.2.3), focusing now on the monodromy operators. We claim that for $M \in \{R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}, R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}, R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}\}$, the product structure is compatible with the monodromy operator in the sense that the following diagram commutes

$$\begin{tikzcd} M \otimes^{L\blacksquare}_{F_L} M \arrow[r] \arrow[d, "N\otimes \mathrm{id}_M + \mathrm{id}_M \otimes N"] & M \arrow[d, "N"] \\ M \otimes^{L\blacksquare}_{F_L} M \arrow[r] & M \end{tikzcd}$$

and satisfies higher associativity relations. Indeed, in these cases, the monodromy operators are induced (in a canonical way[31]) from the Lie algebra action of $\mathbf{G}_m^\natural$-action on such M (cf. [23, Paragraphs around (2.15)]), and the concerned product structures are all $\mathbf{G}_m^\natural$-equivariant. The equivariance of the Lie algebra action is then illustrated exactly by the diagram above.

4.2.6 - Lemma. *Let (A,ϕ), (B,ψ) and (C,χ) be three objects with endormorphism in a $\otimes$-stable ∞-category (hence have fibre sequences and additive structure on* Hom*'s), together with a pairing $\mu\colon A\otimes B \to C$ such that $\psi_A \otimes \chi_B + \chi_A \otimes \psi_B$ is compatible with ψ_C, where $\chi_A \in \mathrm{End}(A)$ and $\chi_B \in \mathrm{End}(B)$. Then μ induces a unique pairing* $\mathrm{fib}(\psi_A) \otimes \mathrm{fib}(\psi_B) \to \mathrm{fib}(\chi)$.

Proof. This is formal, and can be read off from the following commutative diagram

$$\begin{tikzcd} \mathrm{fib}(\psi_A) \otimes \mathrm{fib}(\psi_B) \arrow[r, dashed] \arrow[d] & \mathrm{fib}(\psi_C) \arrow[d] \\ A \otimes B \arrow[r, "\mu"] \arrow[d, "\psi_A\otimes\chi_B + \chi_A\otimes\psi_B"] & C \arrow[d, "\psi_C"] \\ A \otimes B \arrow[r, "\mu"] & C \end{tikzcd}$$

whose left column composes naturally to zero. □

4.2.7 - Example. The natural morphisms

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})^{N=0}_{l,\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})^{N=0}_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}^0_{F_L})^{N=0}_{\mathbf{Q}_p}$$

are compatible with φ-equivariant product structures, where the last three objects are endowed with the canonical product structure by (4.2.6).

4.2.8 - Example. We want to study product structures on Frobenius fixed points $(-)^{\varphi=p^r}$ for $r\in\mathbf{Z}$. Intuitively speaking, the product of a $\varphi=p^r$ eigenvector and a $\varphi=p^s$ eigenvector should be a $\varphi=p^{r+s}$ eigenvector, so we should expect a product structure on Frobenius fixed points of the form

$$A^{\varphi=p^r}\otimes B^{\varphi=p^s} \to C^{\varphi=p^{r+s}}.$$

We claim that it is indeed the case. For this, in order to apply (4.2.6), consider $\psi_A = \varphi_A - p^r$, $\chi_B = p^s$, $\chi_A = p^r\varphi_A$ and $\psi_B = \varphi_B - p^s$. Then $\psi_A\otimes\chi_B + \chi_A\otimes\psi_B = \varphi_A\otimes\varphi_B - p^{r+s}$ is compatible with $\varphi_C - p^{r+s}$. It is direct but probably tedious to check that the product structure thus defined satisfies higher associativity relations.

As a result, the natural morphisms

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})^{\varphi=p^r,N=0}_{l,\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}^0_{F_L})^{\varphi=p^r,N=0}_{\mathbf{Q}_p}$$

for $r\in\mathbf{Z}$ are compatible with product structures.

4.2.9. Product structure on de Rham cohomology. For $Z\in\mathrm{Rig}_K$, we have a product structure

$$\mathrm{Fil}^r R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{L\blacksquare}_K \mathrm{Fil}^s R\Gamma_{\mathrm{dR}}(Z/K) \to \mathrm{Fil}^{r+s}R\Gamma_{\mathrm{dR}}(Z/K)$$

[31] We took the rational coefficients in order to make the monodromy action canonical, hence $N = \frac{1}{e_L}t\partial_t$ [23, The paragraph after (2.15)].

for $r,s\in\mathbf{Z}$ induced from the one on the level of complexes of sheaves. For $X\in\mathrm{Rig}_C$, similarly (combining the the log-crystalline setup (4.2.2) and the de Rham setup above), we obtain a product structure

$$\mathrm{Fil}^r R\Gamma_{\inf}(X/B_{\mathrm{dR}}^+)\otimes_K^{L\blacksquare}\mathrm{Fil}^s R\Gamma_{\inf}(X/B_{\mathrm{dR}}^+)\to\mathrm{Fil}^{r+s}R\Gamma_{\inf}(X/B_{\mathrm{dR}}^+)$$

for $r,s\in\mathbf{Z}$. They all satisfy higher associativity relations.

For $\mathfrak{Z}\in\mathcal{M}_K^{\mathrm{ss}}$ with splitting field L and $\mathfrak{X}\in\mathcal{M}_C^{\mathrm{ss}}$, the natural "base change" morphisms $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^\times)\to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_\eta/K)$ and $R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A_{\mathrm{cris}}^\times)\to R\Gamma_{\inf}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+)$ are compatible with product structures, again by similarity between crystalline and infinitesimal constructions.

To induce the product structure on syntomic cohomologies from the above, one needs the following lemma.

4.2.10 - Lemma. *The Hyodo–Kato section $\iota_0\colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}\to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}$ is compatible with product structures.*

Proof. Adapt the proof of (4.2.4) (or apply *loc. cit.* by writing ι_0 as the composition of $\iota_{0,l}$ with the natural base change map $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}\to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})\mathbf{Q}_p$). □

4.2.11. Product structure on syntomic cohomology. Let $Z\in\mathrm{Rig}_K$ and $r,r'\in\mathbf{N}$, there is a natural product structure [68, §2.2]

$$R\Gamma_{\mathrm{syn}}(Z,r)\otimes_{\mathbf{Q}_p}^{L\blacksquare}R\Gamma_{\mathrm{syn}}(Z,r')\to R\Gamma_{\mathrm{syn}}(Z,r+r') \tag{4.2.11.1}$$

given by the formula $(x,y)\otimes(x',y')\mapsto(xx',(-1)^q xy'+y\varphi_{r'}(x'))$, but we explain it now using previous constructions. For this, taking the Bloch–Kato point of view, we need to show that $\iota_{\mathrm{HK}}^{\mathrm{arith}}\colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}^{\varphi=p^r,N=0}\to R\Gamma_{\mathrm{dR}}(Z/K)/\mathrm{Fil}^r$ is compatible with product structures hence induces a canonical product structure on its fibre $R\Gamma_{\mathrm{syn}}^{\mathrm{BK}}(Z,r)$. Again, we may work locally and assume that $Z=\mathfrak{Z}_\eta$ with $\mathfrak{Z}\in\mathcal{M}_K^{\mathrm{ss}}$ with splitting field L. Going through the construction of (2.1.4), using (4.2.3), (4.2.7), (4.2.8) and (4.2.9), we only need to prove that the Hyodo–Kato section $\iota_0\colon R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}\to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}$ is compatible with product structures; but this has just been done in (4.2.10).

Moreoever, the maps of the diagrams in the proof of (2.3.3) are compatible with product structures again by (4.2.3), (4.2.7), (4.2.8), (4.2.9) and (4.2.10), whence it is the same product structure as defined by $R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(\mathfrak{Z},r)=[R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}^{\varphi=p^r}\to R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}/\mathrm{Fil}^r]$.

4.2.12 - Lemma. *For $Z\in\mathrm{Rig}_K$, the natural syntomic-proétale period map $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ is compatible with the product structure on syntomic cohomology.*

Proof. In light of the construction of the syntomic-proétale period map (3.1.3) by taking Galois invariants of the diagram (3.1.3.1), it suffices to show that each morphism in this last diagram is compatible with product structures, and that the maps $R\Gamma_{\mathrm{syn}}(Z,r)\to R\Gamma_{\mathrm{syn}}(Z_C,r)$ is compatible with product structures. The second compatibility is clear by construction. The first compatibility follows from the (φ,N)-equivariance of the comparison map $R\Gamma_{\mathrm{HK}}(X)\otimes_{\breve F}^{\blacksquare}B_{\log}[\frac1t]\to R\Gamma_{\text{proét}}(X,\mathbf{B}_{\log}[\frac1t])$ as can be deduced from (3.1.2.4). □

4.2.13. Products with first Chern classes. Let $\mathcal{E}$ be an (analytic/étale) vector bundle of rank $d+1$, $d\geq 0$, on Z. Let $\pi\colon\mathbf{P}_Z(\mathcal{E})\to Z$ be its associated projective bundle and $\mathcal{O}(1)$ be its canonical bundle. The syntomic first Chern class (4.1.9.4) defines maps

$$c_1^{\mathrm{syn}}(\mathcal{O}(1))^i\cup\pi^*\colon R\Gamma_{\mathrm{syn}}(Z,r-i)\xrightarrow{\pi^*}R\Gamma_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}),r-i)\xrightarrow{c_1^{\mathrm{syn}}(\mathcal{O}(1))^i\cup-}R\Gamma_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}),r)[2i]$$

for $0\leq i\leq r$. Indeed, (4.2.11.1) induces a map

$$R\Gamma_{\mathrm{syn}}(Z,1)\to R\operatorname{Hom}(R\Gamma_{\mathrm{syn}}(Z,r-1),R\Gamma_{\mathrm{syn}}(Z,r))$$

hence taking H^2, one gets

$$H^2_{\mathrm{syn}}(Z,1)\to\operatorname{Hom}(R\Gamma_{\mathrm{syn}}(Z,r-1),R\Gamma_{\mathrm{syn}}(Z,r)[2]).$$

The image of a class c is the cup product denoted by $c \cup -$. By itertation on i, one obtains

$$\prod_i H^2_{\mathrm{syn}}(Z,1) \to \mathrm{Hom}(R\Gamma_{\mathrm{syn}}(Z,r-i), R\Gamma_{\mathrm{syn}}(Z,r)[2i])$$

for $0 \leq i \leq r$. The image of the constant tuple $c_1^{\mathrm{syn}}(\mathcal{O}(1))^{[[1,i]]}$ is denoted by $c_1^{\mathrm{syn}}(\mathcal{O}(1))^i \cup$.

Similarly, for Z a scheme or analytic adic space over K, the étale first Chern class (4.1.15) defines maps

$$c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*\colon R\Gamma_{\text{ét}}(Z, \mu_{p^n}^{\otimes(r-i)}) \to R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mu_{p^n}^{\otimes r})[2i]$$

whence maps

$$c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*\colon R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r-i)) \to R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Z}_p(r))[2i].$$

These commute with the analytification functor since so do $c_1^{\text{ét}}$ and $- \cup -$ by (4.2.1).

4.2.14 - Lemma. *Let $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L, and choose a uniformizer $\varpi \in L$. Then the product structures on $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)$ and $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)$ are compatible across the arithmetic Hyodo–Kato morphism $\iota_{\mathrm{HK}}^{\mathrm{arith}}$, whose homotopy depends on the class $l_\varpi = [\varpi]$ in $\mathfrak{m}_L/p\mathfrak{m}_L$.*

Proof. The same strategy as in the proof of (4.1.11), whose most essential part is dealt with in (4.2.4). ◻

4.2.15 - Proposition (Projective bundle formula). *For $r \geq d$, there is a natural isomorphism in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$:*

$$\bigoplus_{i=0}^d (c_1^{\mathrm{syn}}(\mathcal{O}(1))^i \cup \pi^*)\colon \bigoplus_{i=0}^d R\Gamma_{\mathrm{syn}}(Z, r-i)[-2i] \xrightarrow{\simeq} R\Gamma_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}), r).$$

Proof. By admissible descent, we may assume that $\mathcal{E} = \mathcal{O}_Z^{\oplus(d+1)}$, then $\mathbf{P}_Z(\mathcal{E}) = \mathbf{P}_Z^d$. By éh-descent, one may assume that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ is affine with splitting field L, which we may furthermore assume to be algebraizable to an affine scheme $\mathscr{Z}$ flat and η-smooth over $\mathcal{O}_L$ whose p-adic formal completion is $\mathfrak{Z}$ (by [31, Theorem 7], cf. [67, Corollary 3.3.2]). By compatiblity of first Chern class maps (4.1.11) and product structures (4.2.14), we are reduced to showing that

$$(\mathrm{Proj}_{\mathrm{HK}}) \qquad \bigoplus_{i=0}^d (c_1^{\mathrm{HK}}(\mathcal{O}(1))^i \cup \pi^*)\colon \bigoplus_{i=0}^d R\Gamma_{\mathrm{HK}}(Z)[-2i] \xrightarrow{\simeq} R\Gamma_{\mathrm{HK}}(\mathbf{P}_Z^d)$$

$$(\mathrm{Proj}_{\mathrm{dR},r}) \qquad \bigoplus_{i=0}^d (c_1^{\mathrm{dR}}(\mathcal{O}(1))^i \cup \pi^*)\colon \bigoplus_{i=0}^d \mathrm{Fil}^{r-i} R\Gamma_{\mathrm{dR}}(Z/K)[-2i] \xrightarrow{\simeq} \mathrm{Fil}^r R\Gamma_{\mathrm{dR}}(\mathbf{P}_Z^d/K)$$

are isomorphisms. Since Z has a semistable model $\mathfrak{Z}$, ($\mathrm{Proj}_{\mathrm{HK}}$) reduces to ($\mathrm{Proj}_{\mathrm{dR},r}$) with $r = 0$ by the Hyodo–Kato isomorphism (2.1.9). So it suffices to prove ($\mathrm{Proj}_{\mathrm{dR},r}$). Recall that the algebraic analogue of it holds for $\mathscr{Z}_\eta$, cf. [56, Proof of Proposition 5.2]; indeed, the statement, being analytically local, is refined to a sheaf theoretic isomorphism

$$(\mathrm{Proj}^{\mathrm{alg}}_{\mathrm{dR},r}) \qquad \bigoplus_{i=0}^d (c_1^{\mathrm{dR}}(\mathcal{O}(1))^i \cup \pi^{\mathrm{alg}*})\colon \bigoplus_{i=0}^d \Omega^{\bullet \geq r-i}_{\mathscr{Z}_\eta/K}[-2i] \xrightarrow{\simeq} R\pi_*^{\mathrm{alg}} \Omega^{\bullet \geq r}_{\mathbf{P}^d_{\mathscr{Z}_\eta}/K},$$

which can be easily checked on stalks [66, Proposition 0FMR, Proposition 0FMT]. By relative GAGA [25, Example 3.2.6, Appendix (A.1.1)] [49, §p. 43-53] and compatibility between algebraic and rigid-analytic de Rham first Chern class maps, one obtains

$$(\mathrm{Proj}'_{\mathrm{dR},r}) \qquad \bigoplus_{i=0}^d (c_1^{\mathrm{dR}}(\mathcal{O}(1))^i \cup \pi^{\mathrm{alg}*})\colon \bigoplus_{i=0}^d \Omega^{\bullet \geq r-i}_{(\mathscr{Z}_\eta)^{\mathrm{an}}/K}[-2i] \xrightarrow{\simeq} R\pi_* \Omega^{\bullet \geq r}_{\mathbf{P}^d_{(\mathscr{Z}_\eta)^{\mathrm{an}}}/K}.$$

Since $Z = \mathfrak{Z}_\eta = \mathscr{Z}_\eta^{\mathrm{rig}}$ is an affinoid open subspace of $(\mathscr{Z}_\eta)^{\mathrm{an}}$, one obtains ($\mathrm{Proj}_{\mathrm{dR},r}$) by taking $R\Gamma(Z, -)$ of this isomorphism. ◻

4.2.16 - Proposition (Projective bundle formula for (pro)étale cohomology). *For $r \geq d$, there is a natural* *isomorphism in* $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p})$:

$$\bigoplus_{i=0}^{d}(c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) \colon \bigoplus_{i=0}^{d} R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r-i))[-2i] \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Z}_p(r)).$$

In particular, there are natural isomorphisms in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$:

$$\bigoplus_{i=0}^{d}(c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) \colon \bigoplus_{i=0}^{d} R\Gamma_{\text{ét}}(Z, \mathbf{Q}_p(r-i))[-2i] \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Q}_p(r))$$

$$\bigoplus_{i=0}^{d}(c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) \colon \bigoplus_{i=0}^{d} R\Gamma_{\text{proét}}(Z, \mathbf{Q}_p(r-i))[-2i] \xrightarrow{\simeq} R\Gamma_{\text{proét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Q}_p(r)).$$

Proof. The second isomorphism is the first with p inverted. The third one reduces to the case where Z is affine by analytic descent, which agrees with the second by quasi-compactness of Z.

For the first isomorphism, it suffices to prove that

$$\bigoplus_{i=0}^{d}(c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) \colon \bigoplus_{i=0}^{d} \mu_{p^n}^{\otimes(r-i)}[-2i] \xrightarrow{\simeq} R\pi_*\mu_{p^n}^{\otimes r}. \tag{4.2.16.1}$$

As these are discrete sheaves, once proven, it can be upgraded to an isomorphism in $\mathscr{D}(Z_{\text{ét}}, \mathrm{CondAb})$.

The question being éh-local, we may assume that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z}$ is a p-adic formal scheme algebraizable to an affine scheme $\mathscr{Z}$ flat, of finite type and η-smooth over $\mathcal{O}_L$ for some finite extension L/K (cf. beginning of the proof of (4.2.15)). There is a projective bundle formula for mod p^n étale cohomology of schemes over K (cf. [69, Theorem 6.1.7]), so there is a natural isomorphism

$$\bigoplus_{i=0}^{d}(c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^{\text{alg}*}) \colon \bigoplus_{i=0}^{d} \mu_{p^n}^{\otimes(r-i)}[-2i] \xrightarrow{\simeq} R\pi_*\mu_{p^n}^{\otimes r}$$

in $\mathscr{D}((\mathscr{Z}_\eta)_{\text{ét}}, Z/p^n)$. By Huber's comparison [43, Theorem 3.8.1] and its compatibility between products with first Chern classes (4.2.13), this implies the desired (4.2.16.1). □

4.2.17 - Proposition ($\mathbf{A}^1$-homotopy invariance of integral étale cohomology). *Let* $\pi \colon \mathbf{A}^1_Z \to Z$ *be the relative analytic affine line over* Z. *Then for any* $r \in \mathbf{N}$, *the pullback induces a natural isomorphism*

$$\pi^* \colon R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r)) \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{A}^1_Z, \mathbf{Z}_p(r)).$$

In particular, the pullback induces a natural isomorphism

$$\pi^* \colon R\Gamma_{\text{ét}}(Z, \mathbf{Q}_p(r)) \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{A}^1_Z, \mathbf{Q}_p(r)).$$

Proof. The second isomorphism reduces to the first, since by definition $R\Gamma_{\text{ét}}(Z, \mathbf{Q}_p(r)) := R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r))_{\mathbf{Q}_p}$.

To prove the first, we may assume by éh-descent that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z}$ is a p-adic formal scheme algebraizable to an affine scheme $\mathscr{Z}$ flat, of finite type and η-smooth over $\mathcal{O}_L$ for some finite extension L/K (cf. beginning of the proof of (4.2.15)). By Huber's comparison theorem [43, Theorem 3.8.1], we have the natural base change isomorphism of étale sheaves

$$\varphi^*_{\mathscr{Z}_\eta} R\pi^{\text{alg}}_* \mu_{p^n}^{\otimes r} \xrightarrow{\simeq} R\pi_*\mu_{p^n}^{\otimes r},$$

where $\varphi_{\mathscr{Z}_\eta} \colon (\mathscr{Z}_\eta)^{\text{an}}_{\text{ét}} \to (\mathscr{Z}_\eta)_{\text{ét}}$ is the natural morphism of sites defined by the analytification functor. But we have $F \xrightarrow{\simeq} R\pi^{\text{alg}}_*\pi^{\text{alg}*}F$ by the $\mathbf{A}^1$-homotopy invariance of torsion sheaves F on $(\mathscr{Z}_\eta)_{\text{ét}}$ (since its base field L is of characteristic 0)[32]. Therefore, we obtain isomorphisms of étale sheaves

$$\mu_{p^n}^{\otimes r} = \varphi^*_{\mathscr{Z}_\eta}\mu_{p^n}^{\otimes r} \xrightarrow{\simeq} \varphi^*_{\mathscr{Z}_\eta} R\pi^{\text{alg}}_* \mu_{p^n}^{\otimes r} \xrightarrow{\simeq} R\pi_*\mu_{p^n}^{\otimes r}.$$

Since they are discrete objects on quasi-compact schemes, these isomorphisms upgrade to isomorphisms in

[32] Or by a seemingly stronger result: locally acyclicity of smooth morphisms of schemes.

$\mathscr{D}(Z_{\text{ét}},\text{CondAb})$. Evaluating them on $Z = \mathfrak{Z}_\eta = \mathscr{Z}_\eta^{\text{rig}}$, which is an affinoid open subspace of $(\mathscr{Z}_\eta)^{\text{an}}$, one gets the $\mathbf{A}^1$-homotopy invariance for the coefficients $\mu_{p^n}^{\otimes r}$, from which follows the desired isomorphism by taking limits over $n \in \mathbf{N}$. □

4.2.18 - Remark. Unfortunately, the natural morphism

$$\pi^*\colon R\Gamma_{\text{proét}}(Z,\mathbf{Q}_p(r)) \xrightarrow{\simeq} R\Gamma_{\text{proét}}(\mathbf{A}^1_Z,\mathbf{Q}_p(r))$$

is not an isomorphism since $R\Gamma_{\text{proét}}(\mathbf{A}^1_Z,\mathbf{Q}_p(r)) \not\simeq R\Gamma_{\text{ét}}(\mathbf{A}^1_Z,\mathbf{Z}_p(r))_{\mathbf{Q}_p}$. It fails even for $Z = \text{Spa}(K,\mathcal{O}_K)$; indeed, $H^1_{\text{proét}}(\mathbf{A}^1_K,\mathbf{Q}_p(1)) = 0$ while $H^1_{\text{proét}}(\mathbf{A}^1_K,\mathbf{Q}_p(1)) \xrightarrow{\simeq} \Omega^1(\mathbf{A}^1_K)$ by using Colmez–Nizioł's computation [22] and the Hochschild–Serre spectral sequence, and the K-vector $\Omega^1(\mathbf{A}^1_K)$ is large.

Nevertheless, we have a so-called *fundamental motivic spectrum* $(R\Gamma_{\text{proét}}(-,\mathbf{Q}_p(r)))_r$ in the sense of [4, Definition 2.3.2]. Recall that the ∞-category of "motivic spectra" is defined as

$$\text{Sp}_{\mathbf{P}^1} := \text{Sp}_{\mathbf{P}^1}(\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc}_*)) := \text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc}_*)[(\mathbf{P}^1)^{-1}]$$

for the $\mathbf{P}^1$-action by the pointed projective line $(\mathbf{P}^1,\infty)$ on $\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc})$[33]. A motivic spectra being "fundamental" means roughly that it satifies Bass fundamental exact sequence. Of course, for any $\mathscr{V} \in \text{Mod}_{\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc})}(\text{Pr}^{L,\otimes})$, i.e. any ∞-category presentably tensored over $\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc})$ (i.e. presentable ∞-category tensored together with a tensor action of $\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc})$ that preserves colimits in each variable), we can define $\text{Sp}_{\mathbf{P}^1}(\mathscr{V}_*)$ by formally inverting the $\mathbf{P}^1$-action on $\mathscr{V}_*$. For example, consider the presentably symmetric monoidal ∞-category $\mathscr{V} := \text{Shv}_{\text{Nis}}(\text{RigSm}_K,\text{Sp})$ which admits a symmetric monoidal functor from $\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc})$ via $\text{Spc} \to \text{Sp}$ and base change (left Kan extension) along $\text{Sm}_{\mathbf{Z}} \to \text{Sm}_{\mathcal{O}_K} \to \text{RigSm}_K$ where the last arrow is given by taking the rigid generic fibre $(-)^{\text{rig}}_\eta$. For $E = (E_r)_{r\in\mathbf{N}} := (R\Gamma_{\text{proét}}(-,\mathbf{Q}_p(r)))_{r\in\mathbf{N}}$, we have the following:

(i) The proétale cohomology satisfies Nisnevich descent, even éh-descent, on RigSm_K.

(ii) The system E forms naturally a $\mathbf{P}^1$-spectrum in $\mathscr{V}$, i.e. one can naturally make $E \in \text{Sp}_{\mathbf{P}^1}(\mathscr{V})$. Indeed, there is an equivalence

$$\pi^* \oplus c_1^{\text{ét}}(\mathcal{O}(1)) \cup \pi^*\colon R\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r)) \oplus R\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r-1))[-2] \xrightarrow{\simeq} R\Gamma_{\text{proét}}(\mathbf{P}^1_X,\mathbf{Q}_p(r))$$

provided by the projective bundle formula (4.2.16). So we have $c_1^{\text{ét}}(\mathcal{O}(1))\colon E_{r-1} \xrightarrow{\simeq} \underline{\text{Hom}}((\mathbf{P}^1_K,\infty),E)$.

(iii) It is a fundamental $\mathbf{P}^1$-spectrum, i.e. the Bass boundary map

$$\partial^*\colon E^{\mathbf{S}^1\otimes\mathbf{G}_m} \to E^{\mathbf{P}^1},$$

identified as

$$\partial^*_X\colon R\Gamma_{\text{proét}}(\mathbf{G}_{m,X},\mathbf{Q}_p(r))[-1] \to R\Gamma_{\text{proét}}(\mathbf{P}^1_X,\mathbf{Q}_p(r)),$$

admits a natural right inverse, namely there exists a natural section $s_X\colon R\Gamma_{\text{proét}}(\mathbf{P}^1_X,\mathbf{Q}_p(r)) \to R\Gamma_{\text{proét}}(\mathbf{G}_{m,X},\mathbf{Q}_p(r))[-1]$ such that naturally $\partial^*_X s_X \simeq \text{id}$. Indeed, there is a chain of morphisms

$$\begin{aligned} s_X\colon R\Gamma_{\text{proét}}(\mathbf{P}^1_X,\mathbf{Q}_p(r)) &\simeq \lim_{U\in\text{Affd}} R\Gamma_{\text{proét}}(\mathbf{P}^1_U,\mathbf{Q}_p(r)) \\ &\xleftarrow{\simeq} \lim_{U\in\text{Affd}} R\Gamma_{\text{ét}}(\mathbf{P}^1_U,\mathbf{Q}_p(r)) \\ &\xrightarrow{(s_U^{\text{ét}})_U} \lim_{U\in\text{Affd}} R\Gamma_{\text{ét}}(\mathbf{G}_{m,U},\mathbf{Q}_p(r))[-1] \\ &\to \lim_{U\in\text{Affd}} R\Gamma_{\text{proét}}(\mathbf{G}_{m,U},\mathbf{Q}_p(r))[-1] \\ &\simeq R\Gamma_{\text{proét}}(\mathbf{G}_{m,U},\mathbf{Q}_p(r))[-1] \end{aligned}$$

[33] Recall that the usual spectra construction inverts the usual action of the unit sphere $\mathbf{S}^1$ on Spc_*, which is the same as telescoping the action $\mathbf{S}^1 \otimes -$ since the cyclic action $\tau = (123)\colon (\mathbf{S}^1)^{\otimes 3} \to (\mathbf{S}^1)^{\otimes 3}$ is homotopic to id, as a result $(-)[\mathbf{S}^{-1}] \simeq \text{Tel}_{\mathbf{S}^1}(-) := \text{colim}_{\mathbf{S}^1\otimes -}(-)$ is equivalent to the telescope construction. However, by contrast, $\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc}_*)[(\mathbf{P}^1)^{-1}]$ is rather the *symmetric* telescoping procedure (or the so-called "*symmetric* spectra" $\text{Sp}^{\Sigma}(-)$ construction) and admits a forgetful map $\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc}_*)[(\mathbf{P}^1)^{-1}] \to \text{Tel}_{\mathbf{P}^1}(\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc}_*)) := \text{colim}_{\mathbf{P}^1\otimes -}(\text{Shv}_{\text{Zar}}(\text{Sm}_{\mathbf{Z}},\text{Spc}_*))$ forgetting the symmetric group action, which is conservative but not an equivalence.

where $s_U^{\text{ét}}$ is obtained by comparison with algebraic p-adic étale cohomology over the characteristic 0 field K, and satisfies $\partial_U^{*,\text{ét}} s_U^{\text{ét}} = \mathrm{id}$, whence $\partial_U^* s_U = \mathrm{id}$ thus $\partial_X^* s_X = \mathrm{id}$. Therefore, E is a fundamental motivic spectrum.

4.3 Chern classes for vector bundles

We consider Chern classes for vector bundles, which are actually defined on the naive zeroth K-group of rigid spaces, defined as the Grothendieck group of étale (or analytic) vector bundles on them modulo relations generated by short exact sequences. The main result here is the compatibility between syntomic and étale Chern classes through syntomic-étale period maps.

4.3.1. Syntomic Chern classes. Let $Z \in \mathrm{Rig}_K$. Using the projective bundle formula (4.2.15) and Chern class maps

(4.3.1.1) $$c_0^{\text{syn}} \colon \mathbf{Q}_p \xrightarrow{\text{can}} R\Gamma_{\text{syn}}(Z,0), \quad c_1^{\text{syn}} \colon \mathcal{O}_Z^{\times} \to R\Gamma_{\text{syn}}(Z,1)[1],$$

we obtain syntomic Chern classes $c_i^{\text{syn}}(\mathcal{E})$ for any locally free sheaf $\mathcal{E}$ on Z. More precisely, there are unique classes $c_i^{\text{syn}}(\mathcal{E}) \in H^{2i}_{\text{syn}}(Z,i)$ for $i = 1,\ldots,d+1$ such that

$$\sum_{i=0}^{d} c_1^{\text{syn}}(\mathcal{O}(1))^i \cup \pi^* c_{d+1-i}^{\text{syn}}(\mathcal{E}) = c_1^{\text{syn}}(\mathcal{O}(1))^{d+1} c_0^{\text{syn}}(1) \in H^{2(d+1)}_{\text{syn}}(\mathbf{P}_Z(\mathcal{E}), d+1).$$

And we put $c_i^{\text{syn}}(\mathcal{E}) = 0$ for $i > d+1$. In other words, $c_i^{\text{syn}}(\mathcal{E})$ can be read off from the minimal polynomial of $c_1^{\text{syn}}(\mathcal{O}(1))$ in the graded algebra $H^{2\bullet}_{\text{syn}}(\mathbf{P}_Z(\mathcal{E}),\bullet)$.

It is easily verified that $c_i^{\text{syn}}(\mathcal{E})$ depends only on the class of $\mathcal{E}$ in the naive[34] zeroth K-group of Z

$$K_0^{\text{naive}}(Z) := \frac{\{\text{Étale/Analytic vector bundles over } Z\}^{\text{gp}}}{\{[\mathcal{E}_1]+[\mathcal{E}_3]-[\mathcal{E}_2] \mid \mathcal{E}_2 \text{ is an } \mathcal{O}_Z\text{-extension of } \mathcal{E}_3 \text{ by } \mathcal{E}_1\}}.$$

Hence we obtain the i-th syntomic Chern class maps

$$c_i^{\text{syn}} \colon K_0^{\text{naive}}(Z) \to H^{2i}_{\text{syn}}(Z,i), \quad i \in \mathbf{N}$$

extending those in (4.3.1.1).

4.3.2. Étale Chern classes. Similarly as above, using the projective bundle formula (4.2.16) and Chern class maps

(4.3.2.1) $$c_0^{\text{ét}} \colon \mathbf{Z}_p \xrightarrow{\text{can}} R\Gamma_{\text{ét}}(Z,\mathbf{Z}_p), \quad c_1^{\text{ét}} \colon \mathcal{O}_Z^{\times} \to R\Gamma_{\text{syn}}(Z,\mathbf{Z}_p(1))[1],$$

one can define the i-th integral étale Chern class maps

$$c_i^{\text{ét}} \colon K_0^{\text{naive}}(Z) \to H^{2i}_{\text{ét}}(Z,\mathbf{Z}_p(i))(*), \quad i \in \mathbf{N}$$

extending those in (4.3.2.1).

4.3.3 - Remark. Alternatively, we might also define higher Chern classes using the following universal computation: let GL_n be the rigid-analytic general linear group of rank n over $\mathbf{Q}_p$ and $B_\bullet(-)$ be the simplicial classifying space construction for monoids; then

$$H^{\bullet}_{\text{syn}}(B_\bullet \mathrm{GL}_{n,L},\bullet) \simeq H^{\bullet}_{\text{syn}}(L,\bullet)[c_1,\ldots,c_n]$$

for $L = K$ or $L = C$ and $n \geq 1$, where $c_i \in H^{2i}_{\text{syn}}(B_\bullet \mathrm{GL}_{n,L}, i)$ is the i-th Chern class of the universal vector bundle of rank n, and similarly

$$H^{\bullet}_{\text{ét}}(B_\bullet \mathrm{GL}_{n,L},\mathbf{Z}_p(\bullet)) \simeq H^{\bullet}_{\text{ét}}(L,\mathbf{Z}_p(\bullet))[c_1,\ldots,c_n].$$

[34] We keep the naive superscript as opposed to Andreychev's nuclear-continuous K-groups for adic spaces.

Indeed, for the integral étale cohomology, this follows by standard computation using the projective bundle formula and $\mathbf{A}^1$-homotopy invariance, cf. [63, §2.A. $\mathrm{BGL}(n)$]; in the syntomic case, one can proceed as in [3, §5] by analytification.

It would require some work to pull these universal classes back along maps $X \to B_\bullet \mathrm{GL}_n$ in order to define Chern classes. By naturality, these would coincide with our original definition of Chern class maps $c_i(\mathcal{E})$ for vector bundles, cf. [71, Appendix B.2.9] for more details of this construction.

Before proving compatibility of syntomic and étale Chern classes, we review some compatibility results.

4.3.4. Some compatibility results with comparison maps. Let us be more precise about the (iso)morphisms (3.1.3.1) used to construct $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ and $\rho_{\mathrm{syn}}^{\mathrm{geom}}$.

(i) *The isomrophism* $\mathrm{Fil}^\bullet(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}}) \simeq \mathrm{Fil}^\bullet_{\mathrm{Hdg}}(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})$ *commutes with product structures and Chern class maps* c_0 *and* c_1. Indeed, arguing éh-locally, we may assume Z smooth affinoid. In this case, the isomorphism is constructed via [11, Lemma 5.16, Lemma 5.17, especially formula (5.14)], where all maps are compatible with product structures and respective Chern class maps c_0 and c_1.

(ii) *The isomorphism* $R\Gamma_{\mathrm{cris}}(\mathfrak{X}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B_I \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_I)$ *is compatible with product structures.* Indeed, the map is constructed locally for $\mathfrak{X} = \mathrm{Spf}(R)$ framed by (Σ, Λ) [11, Notation 4.12] via the Čech-Alexander computation

$$\begin{aligned}
R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A^\times_{\mathrm{cris}}) &\simeq (D_{\Sigma,\Lambda}(R)(0) \to D_{\Sigma,\Lambda}(R)(1) \to D_{\Sigma,\Lambda}(R)(2) \to \cdots) \\
&\to (D_{\Sigma,\Lambda}(R) \to \underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma_{\Sigma,\Lambda}], D_{\Sigma,\Lambda}(R)) \to \underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma^2_{\Sigma,\Lambda}], D_{\Sigma,\Lambda}(R)) \to \cdots) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, D_{\Sigma,\Lambda}(R)) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, \mathbf{B}_I(R_{\Sigma,\Lambda,\infty})) \\
&\simeq R\Gamma((\mathfrak{X}_\eta)_{\mathrm{\acute{e}t}}, \mathbf{B}_I)
\end{aligned}$$

where we followed the proof of [11, Corollary 4.16, Theorem 4.3] and used the map $D_{\Sigma,\Lambda}(R) \to \mathbf{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \to \mathbf{B}_I(R_{\Sigma,\Lambda,\infty})$. Or samely, we may use Koszul complex computation

$$\begin{aligned}
R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A^\times_{\mathrm{cris}}) &\simeq \mathrm{Kosz}_{D_{\Sigma,\Lambda}(R)}((\partial_\sigma)_{\sigma\in\Sigma}, (\partial_{\lambda,i})_{\lambda\in\Lambda, 1\le i\le d}) \\
&\to \mathrm{Kosz}_{D_{\Sigma,\Lambda}(R)}((\gamma_\sigma - 1)_{\sigma\in\Sigma}, (\gamma_{\lambda,i} - 1)_{\lambda\in\Lambda, 1\le i\le d}) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, D_{\Sigma,\Lambda}(R)) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, \mathbf{B}_I(R_{\Sigma,\Lambda,\infty})) \\
&\simeq R\Gamma((\mathfrak{X}_\eta)_{\mathrm{\acute{e}t}}, \mathbf{B}_I)
\end{aligned}$$

using [11, Lemma 4.14, Lemma 4.15]. All maps here are compatible with product structures.

(iii) *The isomorphism* $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}) \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathrm{Fil}^\bullet \mathbf{B}_{\mathrm{dR}})$ *is compatible with product structures.* Indeed, the maps is constructed locally via

$$\begin{aligned}
R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) &\simeq \mathrm{Kosz}_{D_{\Psi,\Xi,m}(A)}((\partial_u)_{u\in\Psi\amalg\Xi}) \\
&\to \mathrm{Kosz}_{D_{\Psi,\Xi,m}(A)}((\gamma_u - 1)_{u\in\Psi\amalg\Xi} \\
&\to \mathrm{Kosz}_{(\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty})}((\gamma_u - 1)_{u\in\Psi\amalg\Xi}) \\
&\simeq R\Gamma(\underline{\Gamma_{\Psi,\Xi}}, (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty})) \\
&\simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m))
\end{aligned}$$

where we used the map $D_{\Psi,\Xi,m}(A) \to (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty})$ [11, Formula (5.19)].

(iv) *The isomorphisms in (ii) and (iii) are compatible.* This follows from the commutativity of [11, Proof Proposition 5.11]

$$\begin{tikzcd}
D_{\Sigma,\Lambda}(R) \arrow[r] \arrow[d] & \mathbf{B}_I(R_{\Sigma,\Lambda,\infty}) \arrow[d] \\
D_{\Psi,\Xi,m}(A) \arrow[r] & (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty}).
\end{tikzcd}$$

4.3.5 - Theorem. *We have* $\rho^{\mathrm{arith}}_{\mathrm{syn}} \circ c_1^{\mathrm{syn}} = c_1^{\text{ét}}$ *as morphisms* $H^1_{\text{ét}}(Z, \mathcal{O}_Z^\times) \to H^2_{\text{proét}}(Z, \mathbf{Q}_p(1))$.

Proof. Let us elaborate the proof of the statement evaluated at a point, i.e. we will prove that for any line bundle $\mathcal{L}$ on $Z_{\text{ét}}$, we have the identity

$$\rho^{\mathrm{arith}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(\mathcal{L})) = c_1^{\text{ét}}(\mathcal{L}) \in H^2_{\text{proét}}(Z, \mathbf{Q}_p(1))(*).$$

The proof consists of fastidious reductions to the annulus case where an elementary computation is then done (4.3.5.17), and can be skipped on first reading. For the condensed statement, the arguments are the same.

4.3.5.1. Let us first reduced to its geometric counterpart, and meanwhile give another characterisation of the first Chern class $c_1(\mathcal{L})$ by using the simplicial classifying stack $B_\bullet \mathbf{G}_m$. For this, let us denote by $\mathbf{G}_m = \mathbf{G}_{m,K}$ the *rigid-analytic torus* over K (i.e. analytification of the algebraic torus over K, to be distinguished from the unit circle torus $\mathbf{T}^1_K \coloneqq \mathrm{Spa}(K\langle T^{\pm 1}\rangle, \mathcal{O}_K\langle T^{\pm 1}\rangle)$). The étale line bundle $\mathcal{L}$ determines a morphism $f_{\mathcal{L}}\colon X \to B_\bullet \mathbf{G}_m$ (up to translation by $\mathbf{G}_m$). Let $\mathcal{F}$ be a big étale sheaf over $\mathrm{Spec}\, K$, and denote

$$R\Gamma_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{F}) \coloneqq \lim_{\Delta} R\Gamma_{\text{ét}}(\mathbf{G}_m^{\times \bullet}, \mathcal{F}).$$

By the spectral sequence $E_1^{i,j} \coloneqq H^j_{\text{ét}}(\mathbf{G}_m^{\times i}, \mathcal{F}) \mapsto H^{i+j}_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{F})$, we see that

(i) We have $H^1_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{O}^\times) \simeq H^0(\mathbf{G}_m, \mathcal{O}^\times)$ with a canonical element the coordinate function $T \in H^0(\mathbf{G}_m, \mathcal{O}^\times)$; similarly in the geometric setup, we have $H^1_{\text{ét}}(B_\bullet \mathbf{G}_{m,C}, \mathcal{O}^\times) \simeq H^0(\mathbf{G}_{m,C}, \mathcal{O}^\times)$ with a canonical element the coordinate function $T \in H^0(\mathbf{G}_{m,C}, \mathcal{O}^\times)$.

(ii) We have

$$H^2_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathbf{Z}_p(1)) \simeq H^1_{\text{ét}}(\mathbf{G}_m, \mathbf{Z}_p(1)) \xrightarrow[\simeq]{(e^*, \nu^*)} H^1_{\text{ét}}(K, \mathbf{Z}_p(1)) \oplus H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1)) \tag{4.3.5.2}$$

where two component maps are induced by base change respectively along the unit section $e\colon \mathrm{Spa}(K, \mathcal{O}_K) \to \mathbf{G}_m$ and the projection from the *geometric rigid-analytic torus* $\nu\colon \mathbf{G}_{m,C} \to \mathbf{G}_m$. Similarly, we have

$$H^2_{\text{ét}}(B_\bullet \mathbf{G}_{m,C}, \mathbf{Z}_p(1)) \simeq H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1)).$$

Let us denote by κ be the canonical pro-(finite étale) $\mathbf{Z}_p(1)$-torsor

$$\widetilde{\mathbf{G}}_{m,C} \coloneqq \lim_{T \mapsto T^p} \mathbf{G}_{m,C} \to \mathbf{G}_{m,C};$$

then $[\kappa]$ is the canonical topological generator of $H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1)) \simeq \mathbf{Z}_p$. By an explicit calculation, we have

$$c_1^{\text{ét}}(T) = [\kappa] \in H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1))$$

for $T \in H^0(\mathbf{G}_{m,C}, \mathcal{O}^\times)$.

(iii) $H^2_{\mathrm{syn}}(B_\bullet \mathbf{G}_m, \mathbf{Z}_p(1)) \simeq H^1_{\mathrm{syn}}(\mathbf{G}_m, \mathbf{Z}_p(1)) \to H^1_{\mathrm{HK}}(\mathbf{G}_m)$ with a canonical element $\mathrm{dlog}\, t$.

The morphism $f_{\mathcal{L}}$ is naturally chosen (up to translation by $\mathbf{G}_m$) so that $f_{\mathcal{L}}^*\colon H^0(\mathbf{G}_m, \mathcal{O}^\times) \simeq H^1_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{O}^\times) \to H^1_{\text{ét}}(X, \mathcal{O}^\times)$ is such that $[T] \mapsto [\mathcal{L}]$. By the naturality of $c_1^?$, we have $c_1^?(\mathcal{L}) = c_1^?(f_{\mathcal{L}}^*(T)) = f_{\mathcal{L}}^*(c_1^?(T))$ for $? \in \{\mathrm{syn}, \text{ét}\}$. Also, $\rho^{\mathrm{arith}}_{\mathrm{syn}}$ commutes with $f_{\mathcal{L}}^*$, so that $\rho^{\mathrm{arith}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(\mathcal{L})) = \rho^{\mathrm{arith}}_{\mathrm{syn}} f_{\mathcal{L}}^*(c_1^{\mathrm{syn}}(T)) = f_{\mathcal{L}}^* \rho^{\mathrm{arith}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T))$; therefore, we are reduced to showing that

$$\rho^{\mathrm{arith}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T)) = c_1^{\text{ét}}(T) \in H^1_{\text{ét}}(\mathbf{G}_m, \mathbf{Q}_p(1)). \tag{4.3.5.3}$$

Similarly as the direct sum decomposition (4.3.5.2), we have

$$H^2_{\text{proét}}(B_\bullet \mathbf{G}_m, \mathbf{Q}_p(1)) \simeq H^1_{\text{proét}}(\mathbf{G}_m, \mathbf{Q}_p(1)) \xrightarrow[\simeq]{(e^*, \nu^*)} H^1_{\text{proét}}(K, \mathbf{Q}_p(1)) \oplus H^0(\underline{\mathcal{G}_K}, H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1))). \tag{4.3.5.4}$$

Therefore, (4.3.5.3) is further reduced to the identities

$$e^*\rho_{\mathrm{syn}}^{\mathrm{arith}}(c_1^{\mathrm{syn}}(T)) = e^*c_1^{\text{ét}}(T) \in H^1_{\text{proét}}(K, \mathbf{Q}_p(1)) \tag{4.3.5.5}$$

$$\nu^*\rho_{\mathrm{syn}}^{\mathrm{arith}}(c_1^{\mathrm{syn}}(T)) = \nu^*c_1^{\text{ét}}(T) \in H^0(\underline{\mathscr{G}_K}, H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1))). \tag{4.3.5.6}$$

4.3.5.7. For (4.3.5.5), we notice that both sides are zero: on the one hand, on the other hand, by naturality of $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ and c_1^{syn}, they commutes with e^*, so we have $e^*\rho_{\mathrm{syn}}^{\mathrm{arith}}(c_1^{\mathrm{syn}}(T)) = \rho_{\mathrm{syn}}^{\mathrm{arith}}(c_1^{\text{ét}}(e^*T))$ and $e^*T = 1 \in H^0(K, \mathcal{O}^\times)$; similarly, by naturality of $c_1^{\text{ét}}$, we obtain $e^*c_1^{\text{ét}}(T) = c_1^{\text{ét}}(e^*T) = 0$.

4.3.5.8. Now, we are left with (4.3.5.6), which is the geometric counterpart of our theorem. Again by naturality, since $\nu^*T = T$, (4.3.5.6) amounts to the equality

$$\rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T)) = c_1^{\text{ét}}(T) \in H^0(\underline{\mathscr{G}_K}, H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1))). \tag{4.3.5.9}$$

We may discard taking Galois invariants here as it will not help prove the identity.

The rigid-analytic torus $\mathbf{G}_{m,C}$ has a non quasi-compact semistable formal model over $\mathcal{O}_C$. One common semistable formal model is obtained by gluing a countable chain of copies of $\mathbf{P}^1_{\mathcal{O}_C}$ by successively identifying the points 0 and ∞ on neighbouring copies. On the other hand, if we allow horizontal divisors, $\mathbf{G}_{m,C}$ also admits a *projective* semistable model by gluing two copies of $\mathrm{Spf}(\mathcal{O}_C\langle T\rangle)$ (equipped also with the horizontal divisor $\{T = 0\}$) along $\mathrm{Spf}(\mathcal{O}_C\langle T^{\pm 1}\rangle)$. According to [20, Corollary 1.10] and [17, 4.3.2], we have computations

(i) $H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) = \breve{F} \cdot c_1^{\mathrm{HK}}(T)$,

(ii) $H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}_{\log})^{N=0} \leftarrow (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{N=0} = B \cdot c_1^{\mathrm{HK}}(T)$,

(iii) $H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}_{\log})^{\varphi=p, N=0} \leftarrow (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=0, N=0} = \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T)$,

(iv) $H^1_{\mathrm{inf}}(\mathbf{G}_{m,C}/B^+_{\mathrm{dR}}) \simeq H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}} \cdot c_1^{\mathrm{dR}}(T)$, where

$$c_1^{\mathrm{dR}}(T) = [\mathrm{dlog}\, T]$$

by construction of (compatible) de Rham Chern class maps c_1^{inf} and c_1^{dR} (4.1.7).

(v) There is a short exact sequence [10, Formula (6.7)]

$$0 \to H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \to H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \to \Omega^1_{\mathbf{G}_m/K}(\mathbf{G}_m)^{d=0} \otimes^{\blacksquare}_K C(-1) \to 0$$

obtained by taking the first cohomology group of $R\Gamma_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) = \mathrm{Fil}^0(R\Gamma_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})$ [10, Theorem 6.5].

(vi) There is a natural map of short exact sequences

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \arrow[r, equal] \arrow[d, "\mathrm{can}"] & \mathbf{Q}_p \cdot [\kappa] \arrow[r] \arrow[d, dashed, "\rho'"] & 0 \\
0 \arrow[r] & \mathcal{O}(\mathbf{G}_{m,C})/\mathbf{Q}_p \arrow[r] & H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \arrow[r, "\beta"] & \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T) \arrow[r] & 0.
\end{tikzcd}$$

Here ρ' is supposed to behave like $\rho_{\mathrm{syn}}^{\mathrm{geom},-1}$.

(vii) Consider the composition of natural maps

$$\begin{aligned}
\iota'_{\mathrm{HK}} \colon \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T) &= (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p, N=0} \\
&\to H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}_{\log}) \\
&\to H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \simeq H^1\big(\mathrm{Fil}^1(R\Gamma_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})\big) \\
&\supset H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \\
&= B^+_{\mathrm{dR}} \cdot c_1^{\mathrm{dR}}(T).
\end{aligned} \tag{4.3.5.10}$$

It matches $c_1^{\mathrm{HK}}(T)$ with $c_1^{\mathrm{dR}}(T) = [\mathrm{dlog}\, T]$ by its compatibility with the geometric Hyodo-Kato morphism [11, Theorem 5.3 (ii)], hence it is injective. Moreover, the composition $\iota'_{\mathrm{HK}}\rho'$ coincides with the composition map

$$\rho'' \colon H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \to H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \tag{4.3.5.11}$$

induced by $\mathbf{Q}_p(1) \to B^+_{\mathrm{dR}}$ sending $\varepsilon = (\zeta_{p^n})_n \mapsto \log[\varepsilon]$.

4.3.5.12. We claim that

$$\rho'(\kappa) = c_1^{\mathrm{HK}}(T) \in (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=0,N=0} = \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T). \tag{4.3.5.13}$$

Admitting this, by construction of $\rho^{\mathrm{geom}}_{\mathrm{syn}}$ (3.1.3) and compatibility between differntial symbols, we have $\beta\rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}T) = c_1^{\mathrm{HK}}(T)$, so $\beta\left(c_1^{\text{ét}}(T) - \rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T))\right) = 0$, or equivalently

$$\delta(T) \coloneqq c_1^{\text{ét}}(T) - \rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T)) \in \ker\left(H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \to \mathbf{Q}_p\right) = \mathcal{O}(\mathbf{G}_{m,C})/\mathbf{Q}_p.$$

we need to show that the difference function $\delta(T) = 0$. For this, consider the squaring morphism $\sigma \coloneqq (-)^2 \colon \mathbf{G}_{m,C} \to \mathbf{G}_{m,C}$ over C given by $T^2 \leftarrow\!\shortmid T$. On the one hand, we have

$$\sigma^*(c_1^{\text{ét}}(T) - \rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T))) = c_1^{\text{ét}}(T^2) - \rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T^2)) = 2(c_1^{\text{ét}}(T) - \rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T))) = 2\delta(T)$$

by naturality of $\rho^{\mathrm{geom}}_{\mathrm{syn}}$ and $c_1^?$; on the other hand, by naturality of $\mathcal{O}(\mathbf{G}_{m,C})/\mathbf{Q}_p$,

$$\sigma^*(c_1^{\text{ét}}(T) - \rho^{\mathrm{geom}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(T))) = \sigma^*\delta(T) = \delta(T^2).$$

Therefore, $2\delta(T) = \delta(T^2)$ whence $\delta(T) = 0$.

4.3.5.14. It now remains to show (4.3.5.13). The injectivity of ι'_{HK} (4.3.5.10) reduces it to

$$\rho''(\kappa) = [\operatorname{dlog} T] \in H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \hookleftarrow H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}, \tag{4.3.5.15}$$

where we used $\iota'_{\mathrm{HK}}\rho' = \rho''$ (4.3.5.11) and $\iota'_{\mathrm{HK}}(c_1^{\mathrm{HK}}(T)) = c_1^{\mathrm{dR}}(T) = [\operatorname{dlog} T]$ (see the point (vii) above for these identities).

Before the proof, we introduce certain perfectoid covering of $\mathbf{G}_m$ for our computation. We denote by $X_n \coloneqq \operatorname{Spa}(K\langle p^nT, p^nT^{-1}\rangle, \mathcal{O}_K\langle p^nT, p^nT^{-1}\rangle) = \{|p^n| \le |T| \le |p^{-n}|\} \subset \mathbf{G}_m$ the arithmetic annuli over K and by $X_{n,C}$ their base change to C. We do not use the pro-(finite étale) $\mathbf{Z}_p(1)$-torsor $\kappa \colon \widetilde{\mathbf{G}}_{m,C} \to \mathbf{G}_{m,C}$. Instead, consider the closed embeddings

$$X_n \hookrightarrow Y_n \coloneqq \operatorname{Spa}(K\langle U_n, V_n\rangle, \mathcal{O}_K\langle U_n, V_n\rangle)$$

given by $U_n \mapsto p^nT, V_n \mapsto p^nT^{-1}$. Consider $\widetilde{X}_{n,C}$ the pullback along this closed embedding of the canonical $\mathbf{Z}_p(1)^2$-torsor affinoid perfectoid cover

$$\widetilde{Y}_{n,C} \coloneqq \operatorname{Spa}(C\left\langle U_n^{1/p^\infty}, V_n^{1/p^\infty}\right\rangle, \mathcal{O}_C\left\langle U_n^{1/p^\infty}, V_n^{1/p^\infty}\right\rangle) \to Y_n.$$

Then we have compatible strict inclusions of perfectoid space $\widetilde{X}_{n,C} \subset^\dagger \widetilde{X}_{n+1,C}$ induced by $pU_n \leftarrow\!\shortmid U_{n+1}, pV_n \leftarrow\!\shortmid V_{n+1}$. Their union $\widetilde{X}_C$ is the canonical $\mathbf{Z}_p(1)^2$-torsor perfectoid cover of $\mathbf{G}_{m,C}$, with a *Stein affinoid perfectoid covering* by $\widetilde{X}_{n,C}$; the system $\{H^i_{\text{proét}}(\widetilde{X}_{n,C}, \mathbf{B}^+_{\mathrm{dR}})\}_{n\in\mathbf{N}}$ is Mittag-Leffer [18, lemme 3.10], and vanishes if $i > 0$ [64, Theorem 6.5]. Therefore, $H^i_{\text{proét}}(\widetilde{\mathbf{G}}_{m,C}, \mathbf{B}^+_{\mathrm{dR}})$ is equal to $B^+_{\mathrm{dR}}(\widetilde{\mathbf{G}}_{m,C})$ if $i = 0$ and vanishes if $i > 0$. Then the Hochschild–Serre spectral sequence and the $\lim^1$-sequence give

$$H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) = H^1(\underline{\mathbf{Z}_p(1)}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_C)) \xrightarrow{\simeq} \lim_n H^1(\underline{\mathbf{Z}_p(1)}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{m,C})).$$

Now the proof reduces to identifying

$$\rho''(\kappa)|_{X_{n,C}} = [\operatorname{dlog} T] \in H^1(\underline{\mathbf{Z}_p(1)}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{m,C})). \tag{4.3.5.16}$$

4.3.5.17. Let us prove (4.3.5.16). By applying Hochschild–Serre spectral sequence and then by (1.2.4), we obtain isomorphisms $H^1_{\text{proét}}(X_{n,C}, \mathbf{B}^+_{\text{dR}}) \simeq H^1(\mathbf{Z}_p(1)^2, \mathbf{B}^+_{\text{dR}}(\widetilde{X}_{n,C})) \simeq H^1\big(\underline{\text{Hom}}((\underline{\mathbf{Z}_p(1)^2})^{\times\bullet}, \mathbf{B}^+_{\text{dR}}(\widetilde{\mathbf{G}}_{m,C}))\big)$. The image of $\rho''(\kappa)|_{X_{n,C}} \in H^1_{\text{proét}}(X_{n,C}, \mathbf{B}^+_{\text{dR}})$ under these isomorphisms is represented by the continuous cocycle

$$(\gamma_u, \gamma_v) \mapsto \log[\gamma_u] - \log[\gamma_v] \tag{4.3.5.18}$$

in $\underline{\text{Hom}}((\underline{\mathbf{Z}_p(1)^2})^{\times 2}, \mathbf{B}^+_{\text{dR}}(\widetilde{\mathbf{G}}_{m,C}))$.

Now we will go through the construction of the comparison map $R\Gamma_{\text{dR}}(X/K) \otimes^{\blacksquare}_K B^+_{\text{dR}} \to R\Gamma_{\text{proét}}(X_C, \mathbf{B}^+_{\text{dR}})$ [11, Proof of Theorem 5.9] (see the point (iii) above) in order to identify the image of $[\operatorname{dlog} T]$ in $H^1_{\text{proét}}(X_{n,C}, \mathbf{B}^+_{\text{dR}}) \simeq H^1(\mathbf{Z}_p(1)^2, \mathbf{B}^+_{\text{dR}}(\widetilde{X}_{n,C}))$. Let $D_n \coloneqq D_{X_{n,C}}(Y_{n,C})$ be the ring of functions of the B^+_{dR}-envelope of $X_{n,C}$ in $Y_{n,B^+_{\text{dR}}}$ as in [11, Lemma 5.16] (cf. [41, §2.2]). Then $R\Gamma_{\text{dR}}(X_n/K) \otimes^{\blacksquare}_K B^+_{\text{dR}} \simeq \text{Kosz}_{D_n}(\partial_u, \partial_v)$ where $\partial_u = u\frac{\partial}{\partial u}, \partial_v = v\frac{\partial}{\partial v}$ are log derivations, and the class $[\operatorname{dlog} T] = [\operatorname{dlog}(p^n T)] = -[\operatorname{dlog}(p^n T^{-1})]$ corresponds to

$$[(1,-1)] \in H^1(\text{Kosz}_{D_n}(\partial_u, \partial_v)).$$

The map $\text{Kosz}_{D_n}(\partial_u, \partial_v) \to \text{Kosz}_{D_n}(\gamma_u - 1, \gamma_v - 1)$ of the point (iii) is induced by [13, Proposition 5.34]

$$\begin{tikzcd}
(D_n \arrow[r, "\partial_u"] \arrow[d, "\text{id}"] & D_n) \arrow[d, "h_{u,\varepsilon}"] \\
(D_n \arrow[r, "\gamma_{u,\varepsilon}-1"] & D_n)
\end{tikzcd}$$

where $\varepsilon \in \mathbf{Z}_p(1)$ is a chosen generator, $h_\varepsilon \coloneqq \sum_{i \geq 1} \frac{(\log[\varepsilon])^i}{i!} \partial_u^{i-1}$, and the u-component element $\gamma_{u,\varepsilon} \in \mathbf{Z}_p(1)$, alias of ε, acts on the B^+_{dR}-algebra D_n by $\gamma(U_n) = [\varepsilon]U_n$; similarly for the variable v part. Under this map, the class $[\operatorname{dlog} T]$ is mapped to

$$[(\log[\varepsilon], -\log[\varepsilon])] \in H^1\text{Kosz}_{D_n}(\gamma_{u,\varepsilon} - 1, \gamma_{v,\varepsilon} - 1).$$

Finally, using the map $D_n \to \mathbf{B}_{\text{dR}}(\widetilde{X}_{n,C}), U_n \mapsto [(p^n T)^\flat], V_n \mapsto [(p^n T^{-1})^\flat]$, we obtain the class

$$[(\log[\varepsilon], -\log[\varepsilon])] \in H^1\big(\text{Kosz}_{\mathbf{B}^+_{\text{dR}}(\widetilde{X}_{n,C})}(\gamma_{u,\varepsilon} - 1, \gamma_{v,\varepsilon} - 1)\big). \tag{4.3.5.19}$$

To conclude, we claim that (4.3.5.19) recovers the formula (4.3.5.18) for $\rho''(\kappa)|_{X_{n,C}}$ under the identification $H^1\big(\text{Kosz}_{\mathbf{B}^+_{\text{dR}}(\widetilde{X}_{n,C})}(\gamma_u - 1, \gamma_v - 1)\big) \simeq H^1(\underline{\mathbf{Z}_p(1)^2}, \mathbf{B}^+_{\text{dR}}(\widetilde{X}_{n,C})) \simeq H^1\big(\underline{\text{Hom}}((\underline{\mathbf{Z}_p(1)^2})^{\times\bullet}, \mathbf{B}^+_{\text{dR}}(\widetilde{\mathbf{G}}_{m,C}))\big)$, which we summarised in Lemma (1.2.9).

□

4.3.6 - Theorem. *The syntomic and étale Chern classes are compatible, i.e., for $Z \in \text{Rig}_K$ and $i \in \mathbf{N}$, the following diagram commutes:*

$$\begin{tikzcd}
 & H^{2i}_{\text{syn}}(Z, i) \arrow[dd, "\rho^{\text{arith}}_{\text{syn}}"] \\
K_0^{\text{naive}}(Z) \arrow[ur, "c_i^{\text{syn}}"] \arrow[dr, "c_i^{\text{ét}}"'] & \\
 & H^{2i}_{\text{ét}}(Z, \mathbf{Q}_p(i))
\end{tikzcd}$$

Proof. In view of projective bundle formulae defining general Chern classes, it suffices to show that the comparison map $\rho^{\text{arith}}_{\text{syn}} \colon R\Gamma_{\text{syn}}(Z, i) \to R\Gamma_{\text{proét}}(Z, \mathbf{Q}_p(i))$ commutes with product structures, the c_0 on the zeroth cohomology and the c_1 on the first cohomology (i.e. $c_1(\mathcal{L})$ for line bundles $\mathcal{L}$). By construction of $\rho^{\text{arith}}_{\text{syn}}$ (3.1.3), it suffices to prove the corresponding statement for the composition

$$\begin{aligned} R\Gamma_{\mathrm{syn}}(Z,i) \overset{\rho^{\mathrm{arith}}_{\mathrm{syn}}}{\to} R\Gamma_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r)) &\to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \\ &\overset{\simeq}{\to} \left[R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\log}[\frac{1}{t}])^{\varphi=p^r,N=0} \to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\mathrm{dR}}/t^r\mathbf{B}^+_{\mathrm{dR}})\right]. \end{aligned}$$

But it is clear that the latter preserves product structures by (4.2.1), commutes with c_0 by construction, and commutes with c_1 for line bundles by (4.3.5). ◻

4.4 Image of étale regulators

4.4.1. Étale regulators. For $Z \in \mathrm{Rig}_K$ and $i \in \mathbf{N}$, let $\tilde{c}_i^{\mathrm{\acute{e}t}}\colon K_0^{\mathrm{naive}}(Z) \to H^0(\underline{\mathscr{G}_K}, H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i)))$ be the map induced by the i-th étale Chern class map and the Hochschild–Serre spectral sequence boundary map $\delta_0\colon H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i)) \to H^0(\underline{\mathscr{G}_K}, H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i)))$. We define

(4.4.1.1) $$K_0^{\mathrm{naive}}(Z)_0 := \ker(K_0^{\mathrm{naive}}(Z) \overset{\tilde{c}_i^{\mathrm{\acute{e}t}}}{\to} H^0(\underline{\mathscr{G}_K}, H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i))))$$

and the *i-th étale regulator map* as the map

(4.4.1.2) $$r_i^{\mathrm{\acute{e}t}}\colon K_0^{\mathrm{naive}}(Z) \to H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i)))$$

induced from $c_i^{\mathrm{\acute{e}t}}|_{K_0^{\mathrm{naive}}(Z)_0}$ and the Hochschild–Serre spectral sequence map

$$\delta_1\colon H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i))_0 := \ker\delta_0 \to H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Z}_p(i))).$$

4.4.2 - Theorem. *Let $Z \in \mathrm{Rig}_K$ be proper. The étale regulator map $r_i^{\mathrm{\acute{e}t}}$ factors through the subgroup*

$$H^1_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^{2i-1}_{\mathrm{pro\acute{e}t}}(Z_C,\mathbf{Q}_p(i))) \subset H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\mathrm{pro\acute{e}t}}(Z_C,\mathbf{Q}_p(i))).$$

Proof. By (4.3.6) and (3.2.22), we have the following commutative diagram

$$\begin{tikzcd} & H^{2i}_{\mathrm{syn}}(Z,i) \arrow[r, "\delta_0^{\mathrm{syn}}"] \arrow[d, "\rho^{\mathrm{arith}}_{\mathrm{syn}}"] & H^0_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Q}_p(i))) \arrow[d, "\simeq"', "\mathrm{can}"] \\ K_0^{\mathrm{naive}}(Z) \arrow[ur, "c_i^{\mathrm{syn}}"] \arrow[dr, "c_i^{\mathrm{\acute{e}t}}"] & & \\ & H^{2i}_{\mathrm{\acute{e}t}}(Z,\mathbf{Q}_p(i)) \arrow[r, "\delta_0"] & H^0(\underline{\mathscr{G}_K}, H^{2i}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Q}_p(i))). \end{tikzcd}$$

Hence the image of $K_0^{\mathrm{naive}}(Z)_0$ under c_i^{syn} is contained in $H^{2i}_{\mathrm{syn}}(Z,i)_0 := \ker(\delta_0\circ\rho^{\mathrm{arith}}_{\mathrm{syn}}) = \ker\delta_0^{\mathrm{syn}}$. Consequently, again by (3.2.22), we obtain the following commutative diagram

$$\begin{tikzcd} & H^{2i}_{\mathrm{syn}}(Z,i)_0 \arrow[r, "\delta_0^{\mathrm{syn}}"] \arrow[d, "\rho^{\mathrm{arith}}_{\mathrm{syn}}"] & H^1_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^{2i-1}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Q}_p(i))) \arrow[d, hook, "\mathrm{can}"] \\ K_0^{\mathrm{naive}}(Z)_0 \arrow[ur, "c_i^{\mathrm{syn}}"] \arrow[dr, "c_i^{\mathrm{\acute{e}t}}"] & & \\ & H^{2i}_{\mathrm{\acute{e}t}}(Z,\mathbf{Q}_p(i))_0 \arrow[r, "\delta_0"] & H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\mathrm{\acute{e}t}}(Z_C,\mathbf{Q}_p(i))) \end{tikzcd}$$

which shows the desired factorisation. ◻

References

[1] Grigory Andreychev. Pseudocoherent and Perfect Complexes and Vector Bundles on Analytic Adic Spaces, 2021. Preprint, available at https://arxiv.org/abs/2105.12591v1.

[2] Grigory Andreychev. *K-Theorie adischer Räume.* PhD thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, September 2023.

[3] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Atiyah duality for motivic spectra, 2024. Preprint, available at https://arxiv.org/abs/2403.01561v1.

[4] Toni Annala and Ryomei Iwasa. Motivic spectra and universality of K-theory, 2023. Preprint, available at https://arxiv.org/abs/2204.03434v3.

[5] Alexander Beilinson. p-adic periods and derived de Rham cohomology. *Journal of the American Mathematical Society*, 25(3):715–738, 2012.

[6] Alexander Beilinson. On the crystalline period map. *Camb. J. Math.*, 1(1):1–51, 2013.

[7] Federico Binda, Martin Gallauer, and Alberto Vezzani. Motivic monodromy and \$p\$-adic cohomology theories. *Journal of the European Mathematical Society*, 2025.

[8] Spencer Bloch and Kazuya Kato. L-Functions and Tamagawa Numbers of Motives. In Pierre Cartier, Luc Illusie, Nicholas M. Katz, Gérard Laumon, Yuri I. Manin, and Kenneth A. Ribet, editors, *The Grothendieck Festschrift: A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck*, pages 333–400. Birkhäuser Boston, Boston, MA, 2007.

[9] Spencer Bloch and Arthur Ogus. Gersten's conjecture and the homology of schemes. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 7(2):181–201, 1974.

[10] Guido Bosco. On the p-adic pro-étale cohomology of Drinfeld symmetric spaces, 2023. Preprint, available at https://arxiv.org/abs/2110.10683v2.

[11] Guido Bosco. Rational p-adic Hodge theory for rigid-analytic varieties, 2023. Preprint, available at https://arxiv.org/abs/2306.06100.

[12] Kaixing Cao. Comparison of Hyodo–Kato and de Rham Fargues–Fontaine Cohomology Theories, 2025. Preprint, available at https://arxiv.org/abs/2509.16726.

[13] Kęstutis Česnavičius and Teruhisa Koshikawa. The A_{inf}-cohomology in the semistable case. *Compositio Mathematica*, 155(11):20392128, 2019.

[14] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry, 2019. Preprint, available at https://www.math.uni-bonn.de/people/scholze/Analytic.pdf.

[15] Dustin Clausen and Peter Scholze. Lectures on Condensed Mathematics, 2019. Preprint, available at https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[16] Dustin Clausen and Peter Scholze. Condensed Mathematics and Complex Geometry, 2022. Preprint, available at https://people.mpim-bonn.mpg.de/scholze/Complex.pdf.

[17] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł. Cohomology of p-adic Stein spaces. *Invent. Math.*, 219(3):873–985, 2020.

[18] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł. Factorisation de la cohomologie étale p-adique de la tour de drinfeld. In *Forum of Mathematics, Pi*, volume 11, page e16. Cambridge University Press, 2023.

[19] Pierre Colmez, Sally Gilles, and Wiesława Nizioł. Arithmetic duality for p-adic pro-étale cohomology of analytic curves, 2023. Preprint, available at https://arxiv.org/abs/2308.07712v2.

[20] Pierre Colmez and Wiesława Nizioł. Syntomic complexes and p-adic nearby cycles. *Invent. Math.*, 208(1):1–108, 2017.

[21] Pierre Colmez and Wiesława Nizioł. On p-adic comparison theorems for rigid analytic varieties: I. *Münster J. Math.*, 13(2):445–507, 2020.

[22] Pierre Colmez and Wiesława Nizioł. On the cohomology of the affine space. In *p-adic Hodge theory. Proceedings of the Simons symposium, Schloss Elmau, Germany, May 7–13, 2017*, pages 71–80. Cham: Springer, 2020.

[23] Pierre Colmez and Wiesława Nizioł. On the cohomology of p-adic analytic spaces, I: The basic comparison theorem, 2023. Preprint, available at https://arxiv.org/abs/2104.13448v3.

[24] Pierre Colmez and Wiesława Nizioł. On the cohomology of p-adic analytic spaces, II: The C_{st}-conjecture, 2024. Preprint, available at https://arxiv.org/abs/2108.12785v4.

[25] Brian Conrad. Relative ampleness in rigid geometry. *Ann. Inst. Fourier (Grenoble)*, 56(4):1049–1126, 2006.

[26] Christian Dahlhausen and Can Yaylali. Towards $\mathbb{A}^1$-homotopy theory of rigid analytic spaces, 2024. Preprint, available at https://arxiv.org/abs/2407.09606v2.

[27] Johan de Jong and Maurius van der Put. Étale Cohomology of Rigid Analytic Spaces. *Documenta Mathematica*, 1:1–56, 1996.

[28] Pierre Deligne. Théorie de Hodge: III. *Publications Mathématiques de l'IHÉS*, 44:5–77, 1974.

[29] Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu. Logarithmic Adic Spaces: Some Foundational Results. In Bhargav Bhatt and Martin Olsson, editors, *p-adic Hodge Theory, Singular Varieties, and Non-Abelian Aspects*, pages 65–182, Cham, 2023. Springer International Publishing.

[30] Gabriel Dospinescu and Wiesława Nizioł. On de Rham flip-flopping in dual towers, 2026. Preprint, available at https://arxiv.org/abs/2603.07869.

[31] Renée Elkik. Solutions d'équations à coefficients dans un anneau hensélien. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 6(4):553–603, 1973.

[32] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. *Asterisque*, February 2019.

[33] Jean-Marc Fontaine. Exposé II : Le corps des périodes p-adiques. In Jean-Marc Fontaine, editor, *Périodes p-adiques - Séminaire de Bures, 1988*, number 223 in Astérisque, pages 59–101. Société mathématique de France, 1994. talk:2.

[34] Jean-Marc Fontaine and William Messing. p-adic periods and p-adic étale cohomology. Current trends in arithmetical algebraic geometry, Proc. Summer Res. Conf., Arcata/Calif. 1985, Contemp. Math. 67, 179-207 (1987)., 1987.

[35] Jean-Marc Fontaine and Yi Ouyang. Theory of p-adic Galois Representations, 2022. Book in preparation, available at http://staff.ustc.edu.cn/~yiouyang/galoisrep.pdf, dated 24 May 2022.

[36] Jean-Marc Fontaine and Bernadette Perrin-Riou. About the Bloch and Kato conjectures: Galois cohomology and values of L-functions. In *Motives. Proceedings of the summer research conference on motives, held at the University of Washington, Seattle, WA, USA, July 20-August 2, 1991*, pages 599–706. Providence, RI: American Mathematical Society, 1994.

[37] Sally Gilles. Morphismes de périodes et cohomologie syntomique. *Algebra Number Theory*, 17(3):603–666, 2023.

[38] Elmar Grosse-Klönne. Rigid analytic spaces with overconvergent structure sheaf. *Journal für die reine und angewandte Mathematik*, 2000(519):73–95, 2000.

[39] Alexander Grothendieck. Produits tensoriels topologiques et espaces nucléaires. In *Séminaire Bourbaki : années 1951/52 - 1952/53 - 1953/54, exposés 50-100*, number 2 in Séminaire Bourbaki, pages 193–200. Société mathématique de France, 1954. talk:69.

[40] Alexander Grothendieck. Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Première partie. *Publications Mathématiques de l'IHÉS*, 11:5–167, 1961.

[41] Haoyang Guo. Crystalline cohomology of rigid analytic spaces, 2021. Preprint, available at https://arxiv.org/abs/2112.14304v2.

[42] Haoyang Guo. Hodge-Tate decomposition for non-smooth spaces. *J. Eur. Math. Soc. (JEMS)*, 25(4):1553–1625, 2023.

[43] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*, volume E30. Wiesbaden: Vieweg, 1996.

[44] Osamu Hyodo and Kazuya Kato. Exposé V : Semi-stable reduction and crystalline cohomology with logarithmic poles. In Jean-Marc Fontaine, editor, *Périodes p-adiques - Séminaire de Bures, 1988*, number 223 in Astérisque, pages 221–268. Société mathématique de France, 1994. talk:5.

[45] Kazuya Kato. Logarithmic structures of Fontaine–Illusie. In *Algebraic analysis, geometry, and number theory: proceedings of the JAMI inaugural conference, held at Baltimore, MD, USA, May 16-19, 1988*, pages 191–224. Baltimore: Johns Hopkins University Press, 1989.

[46] Kazuya Kato. Exposé VI : Semi-stable reduction and p-adic etale cohomology. In Jean-Marc Fontaine, editor, *Périodes p-adiques - Séminaire de Bures, 1988*, number 223 in Astérisque, pages 269–293. Société mathématique de France, 1994. talk:6.

[47] Moritz Kerz, Shuji Saito, and Georg Tamme. K-theory of non-Archimedean rings. I. *Doc. Math.*, 24:1365–1411, 2019.

[48] Moritz Kerz, Shuji Saito, and Georg Tamme. K-Theory of Non-Archimedean Rings II. *Nagoya Mathematical Journal*, 251:669685, 2023.

[49] Ursula Köpf. Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. *Schr. Math. Inst. Univ. Münster (2)*, pages iv+72, 1974.

[50] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. PhD thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, August 2022.

[51] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[52] Jacob Lurie. Higher Algebra, 2016. Preprint, available at https://www.math.ias.edu/~lurie/papers/HA.pdf, version dated 18 September 2017.

[53] Jacob Lurie. Spectral Algebraic Geometry, 2018. Preprint, available at https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf, version dated 3 February 2018.

[54] A. Mokrane. La suite spectrale des poids en cohomologie de Hyodo–Kato. *Duke Mathematical Journal*, 72(2):301 – 337, 1993.

[55] Jan Nekovář. Syntomic cohomology and p-adic regulators, 1998. Preprint, available at https://webusers.imj-prg.fr/~jan.nekovar/pu/syn.pdf.

[56] Jan Nekovář and Wiesława Nizioł. Syntomic cohomology and p-adic regulators for varieties over p-adic fields. *Algebra Number Theory*, 10(8):1695–1790, 2016.

[57] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren Math. Wiss.* Berlin: Springer, 2nd ed. edition, 2008.

[58] Wiesława Nizioł. Toric singularities: log-blow-ups and global resolutions. *Journal of algebraic geometry*, 15(1):1–29, 2006.

[59] Wiesława Nizioł. Semistable conjecture via K-theory. *Duke Mathematical Journal*, 141(1):151 – 178, 2008.

[60] Wiesława Nizioł. On uniqueness of p-adic period morphisms. *Pure Appl. Math. Q.*, 5(1):163–212, 2009.

[61] Wiesława Nizioł. On uniqueness of p-adic period morphisms, II. *Compositio Mathematica*, 156(9):19151964, 2020.

[62] Fabienne Prosmans. Derived categories for functional analysis. *Publ. Res. Inst. Math. Sci.*, 36(1):1983, March 2000.

[63] Vadim V. Schechtman. On the delooping of Chern character and Adams operations. In Yuri I. Manin, editor, *K-theory, Arithmetic and Geometry: Seminar, Moscow University, 1984–1986*, pages 265–319. Springer Berlin Heidelberg, Berlin, Heidelberg, 1987.

[64] Peter Scholze. p-adic Hodge theory for rigid-analytic varieties. *Forum Math. Pi*, 1:77, 2013.

[65] Peter Scholze. Etale cohomology of diamonds, 2022. Preprint, available at https://arxiv.org/abs/1709.07343v3.

[66] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

[67] Michael Temkin. Altered local uniformization of Berkovich spaces. *Israel Journal of Mathematics*, 221(2):585–603, 2017.

[68] Takeshi Tsuji. p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. *Inventiones mathematicae*, 137(2):233–411, 1999.

[69] Bogdan Zavyalov. Poincaré Duality in abstract 6-functor formalisms, 2023. Preprint, available at https://arxiv.org/abs/2301.03821v2.

[70] Yicheng Zhou. Altered uniformisation of log-rigid spaces, 2024. Notes, available at https://yczhoumath.github.io/Ecrits/LogSmModel.pdf.

[71] Yicheng Zhou. *Sur les cohomologies étale et syntomique pour les variétés analytiques p-adiques arithmétiques*. Theses, Sorbonne Université, July 2025.