On syntomic cohomology and regulators for p-adic rigid-analytic varieties

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Abstract

We construct for rigid-analytic varieties over a p-adic local field a natural syntomic descent spectral sequence compatible with the Hochschild-Serre spectral sequence. We also define a motivic analytic K-theory for smooth rigid-analytic varieties together with syntomic and étale higher Chern class maps on K-groups. We deduce from the above compatible spectral sequences that the rigid-analytic étale regulator maps factors through the geometric Selmer groups of Bloch-Kato if the rigid-analytic variety is proper.

Contents

0 Introduction

In this article, we (re)consider the arithmetic syntomic cohomology for rigid-analytic varieties over p-adic local fields, define a rigid-analytic analogue of Soulé's étale regulators, and, in the proper case, study their images.

0.1 Main results

Let K be a p-adic local field of mixed characteristic $(0,p)$ with ring of integers $\mathcal{O}_K$, algebraic closure $\overline{K}$ and absolute Galois group $\mathscr{G}_K := \mathrm{Gal}(\overline{K}/K)$.

0.1.1. Motivation: algebraic setting. Recall that syntomic cohomology for proper and smooth schemes over $\mathcal{O}_K$ was introduced by Fontaine and Messing [27] in their proof of the crystalline-étale comparison theorem as a natural bridge between crystalline cohomology and étale cohomology. It was generalised later by Kato [36] to log-syntomic cohomology for semistable schemes over $\mathcal{O}_K$ (allowing horizontal divisors), and further extended by Nekovář and Nizioł [46] to a syntomic cohomology theory for arbitrary varieties over K or $\overline{K}$. Very roughly speaking, for $r \in \mathbf{N}$ and for schemes $\mathscr{X}$ semistable over $\mathcal{O}_K$, the r-th (log-)syntomic cohomology is the filtered Frobenius eigenspace (up to certain power of p) of the absolute (log-)crystalline cohomology associated with the eigenvalue p^r; then it is rationalised and globalised to arbitrary varieties over K or $\overline{K}$ using alteration techniques. The syntomic cohomology could be thought of as a p-adic analogue of Deligne-Beilinson cohomology for complex manifolds X, which is defined as the cohomology of the Deligne complex

$$\mathbf{Z}(r)_{\mathscr{D}} : 0 \to \mathbf{Z}(r) \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{r-1}_X.$$

Indeed, since its introduction, log-syntomic cohomology has proved to be useful in the study of special values of p-adic L-functions and in formulating p-adic Beilinson conjectures.

Let us recall Nekovář and Nizioł's result [46, Theorem A].

Theorem (Nekovář-Nizioł). *For any varieties over K, there is a canonical graded commutative dg $\mathbf{Q}_p$-algebra $R\Gamma_{\mathrm{syn}}(X_h, *)$ such that*

(i) It is the unique extension of log-syntomic cohomology to varieties over K that satisfies h-descent.

(ii) It is a Bloch-Ogus cohomology theory.

(iii) For $X = \mathrm{Spec}\, K$, we have $H^*_{\mathrm{syn}}(X_h, r) \simeq H^*_g(\mathscr{G}_K, \mathbf{Q}_p(r))$, where $H^i_g(\mathscr{G}_K, -)$ denotes the Ext*-group* $\mathrm{Ext}^i(\mathbf{Q}_p, -)$ *in the category of potentially semistable Galois representations of $\mathscr{G}_K$.*

(iv) There are functorial syntomic-étale period maps

$$\rho^{\mathrm{arith}}_{\mathrm{syn}} : R\Gamma_{\mathrm{syn}}(X_h, r) \to R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r)), \quad \rho^{\mathrm{geom}}_{\mathrm{syn}} : R\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r) \to R\Gamma_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Q}_p(r))$$

compatible with product structures and inducing quasi-isomorphisms after taking the canonical truncation $\tau^{\leq r}$.

(v) The Hochschild-Serre spectral sequence for étale cohomology

$${}^{\mathrm{HS}}E_2^{i,j} = H^i_{\mathrm{cont}}(\mathscr{G}_K, H^j_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r))$$

has a syntomic analogue, the syntomic descent spectral sequence

$$^{\mathrm{syn}}E_2^{i,j} = H_g^i(\mathscr{G}_K, H_{\text{ét}}^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\mathrm{syn}}^{i+j}(X, \mathbf{Q}_p(r)).$$

(vi) There is a canonical morphism of spectral sequences $^{\mathrm{syn}}E_t^{i,j} \Rightarrow {}^{\text{ét}}E_t^{i,j}$ *compatible with the syntomic-étale period map.*

(vii) There are syntomic Chern class maps

$$c_{i,j}^{\mathrm{syn}} : K_j(X) \to H_{\mathrm{syn}}^{2i-j}(X_h, i)$$

compatible with Chern classes via the syntomic-étale period map.

For finite-dimensional $\mathbf{Q}_p$-representations V of $\mathscr{G}_K$, the extension groups $H_g^i(\mathscr{G}_K, V)$ that appear in (iii), so-called *geometric Selmer groups*, were introduced by Bloch and Kato [6] as part of the local tools for the Tamagawa Number Conjecture and the Tate-Shafarevich group of a motive. These extension groups are crucial in many questions in modern algebraic number theory due to the fact that there is a natural injection from $H_g^1(\mathscr{G}_K, V)$ into $H^1(\mathscr{G}_K, V)$ which is often strict. In light of (v) and (vi), syntomic cohomology groups can be viewd as a higher dimensional (or geometric) generalisation of these extension groups. They are used as an approximation of p-adic étale motivic cohomology (a refinement of p-adic étale cohomology capturing classes coming from geometry), and enter into the study of special values of p-adic L-functions, more precisely the p-adic regulators.

0.1.2. Rigid-analytic syntomic cohomology. For rigid-analytic varieties, which are deemed to be a suitable non-archimedean analogue of complex analytic spaces, the syntomic cohomology still serves as a useful tool for proving p-adic comparison theorems and even beyond them. We wish to extend the above theorem (0.1.1) to the rigid-analytic context.

Our principal object of interest has been well-defined: for smooth rigid-analytic varieties, it can be obtained by η-étale hyperdescent from the (log-)syntomic cohomology for models due to Fontaine and Messing (and Kato) [17, 19]; it can be further defined for general (singular) rigid-analytic varieties by further éh-hyperdescent [9] thanks to the nice local smoothness of the éh-topology studied by Haoyang Guo [32]. The rigid-analytic syntomic cohomology was employed firstly by Colmez and Nizioł to prove a (potentially) semistable comparison theorem for smooth and proper semistable formal schemes over $\mathcal{O}_K$ (allowing horizontal divisors) [16], later generalised to the case of smooth proper rigid-analytic varieties [20], which has another proof by Bosco [9] using period sheaves and the Fargues-Fontaine curve. The syntomic method is important in the study of the Stein rigid-analytic varieties by Colmez, Dospinescu and Nizioł [13] in the semistable case, and continued by their other works [17, 19, 20] and by Bosco [8, 9].

0.1.3. Syntomic descent spectral sequence. What come next in the rigid-analytic analogue of Theorem (0.1.1) are syntomic-proétale period maps and a syntomic analogue of the Hochschild-Serre spectral sequence (see (3.1.3), (4.2.12), (3.1.8) *resp.* (3.2.17), (3.2.12), (3.3.7) for details).

Theorem (Syntomic descent spectral sequence). *Let X be a (quasi-separated, finite-dimensional and paracompact) proper or smooth Stein rigid-analytic variety or smooth dagger affinoid rigid-analytic variety over K. Let $r \geq 0$.*

(i) There are functorial syntomic-proétale period maps

$$\rho_{\mathrm{syn}}^{\mathrm{arith}} : R\Gamma_{\mathrm{syn}}(X, r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$$

compatible with product structures and inducing quasi-isomorphisms after the canonical truncation $\tau^{\leq r}$.

(ii) The Hochschild-Serre spectral sequence for proétale cohomology

$$^{\mathrm{HS}}E_2^{i,j} = H_{\mathrm{cont}}^i(\mathscr{G}_K, H_{\mathrm{pro\acute{e}t}}^j(X_C, \mathbf{Q}_p(r))) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}(X, \mathbf{Q}_p(r))$$

admits a natural map from the syntomic descent spectral sequence

$$ {}^{\mathrm{syn}}E_2^{i,j} \Rightarrow H^{i+j}_{\mathrm{syn}}(X,r) $$

compatible with the syntomic-proétale period maps.

(iii) If X is proper over K, then the syntomic descent spectral sequence is identified with

$$ {}^{\mathrm{syn}}E_2^{i,j} = H^i_g(\mathscr{G}_K, H^j_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{syn}}(X,r) $$

compatible with the natural maps $H^i_g(\mathscr{G}_K,-) \to H^i_{\mathrm{cont}}(\mathscr{G}_K,-)$.

Here, all the cohomology groups are upgraded to the condensed world.

0.1.4 - Remark. The assumption on X relies essentially on our known instances of C_{st}-conjecture, and would be further relaxed to partial properness if the latter was proved in general (see the remarks (3.2.13) and (3.2.6)).

0.1.5 - Remark. There is another collection of period maps, the *Fontaine-Messing period maps* α_r^{FM} [17], defined by globalising the Fontaine-Messing period maps on semistable models [27], which induce quasi-isomorphisms after the canonical truncation $\tau^{\leq r}$ [17, Corollary 7.3]. However, *a priori*, it is unclear whether the period maps α_r^{FM} fit for (ii) and eventually (iii). On the contrary, the arithmetic and geometric period maps ρ_{syn} that we define in this article have good compatibilities, but it is unclear about their truncated quasi-isomorphisms. One of our main points is that there are natural homotopies $\alpha_r^{\mathrm{FM}} \simeq \rho_{\mathrm{syn}}$ thanks to the uniqueness of geometric period morphisms [29].

0.1.6 - Remark. The reason for the properness condition in the Theorem (0.1.3, iii) is twofold. Firstly, the $H^i_g(\mathscr{G}_K,-)$ groups were defined only for finite-dimensional $\mathbf{Q}_p$-representations of $\mathscr{G}_K$, and the proétale cohomology groups are only known to be finite-dimensional $\mathbf{Q}_p$-vector spaces for proper rigid-analytic varieties. It fails for Stein varieties in general, but it may not be the most important reason that we require properness. Second and more importantly, and probably related to the previous point, one lacks semistable comparison theorem in the non-proper case. However, the proétale-to-de Rham comparison conjecture/theorem [20, §9] could help reformulate ${}^{\mathrm{syn}}E_2^{i,j}$ in terms of proétale cohomology for small varieties satisfying the C_{st}-conjecture (namely those *with de Rham slopes* ≥ 0 in the terminology of [20, §1.2.3]).

One possible logarithmic generalisation of this result could be allowing horizontal divisors, namely considering the Kummer proétale cohomology of proper log-smooth fs log-rigid spaces, whose foundation and finiteness were established by Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, Xinwen Zhu [23, Theorem 6.2.1]. An analogue [58] of Temkin's altered local uniformisation could be helpful. The author hopes to return in the future to these aspects beyond proper cases.

0.1.7. Bloch-Kato exponential. As a corollary of Theorem (0.1.3), we obtain a description of the Bloch-Kato exponential. For this, recall that the arithmetic syntomic cohomology fits into a fibre sequence

$$ R\Gamma_{\mathrm{syn}}(X,r) \to R\Gamma_{\mathrm{HK}}(X)^{\varphi=p^r, N=0} \xrightarrow{\iota_{\mathrm{HK}}^{\mathrm{arith}}} R\Gamma_{\mathrm{dR}}(X)/F^r, $$

where $\iota_{\mathrm{HK}}^{\mathrm{arith}} : R\Gamma_{\mathrm{HK}}(X) \to R\Gamma_{\mathrm{dR}}(X/K)$ denotes the rigid-analytic arithmetic Hyodo-Kato morphism. It yields boundary maps ∂ on cohomology groups.

Corollary. *Let $i \in \mathbf{N}$. The composition*

$$ H^i_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^{i+1}_{\mathrm{syn}}(X,r) \xrightarrow{\rho_{\mathrm{syn}}^{\mathrm{arith}}} H^{i+1}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)) $$

is the zero map. The syntomic descent spectral sequence then induces a map

$$ H^i_{\mathrm{dR}}(X)/F^r \to H^1_{\mathrm{cont}}(\mathscr{G}_K, H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))). $$

If X is proper over K, then it is equal to the Bloch-Kato exponential map associated with the finite-dimensional [50] $\mathbf{Q}_p$-representation $H^i_{\text{proét}}(X_C, \mathbf{Q}_p(r))$ of $\mathscr{G}_K$.

To this end, for general $X \in \mathcal{R}\mathrm{ig}_K$, the boundary map ∂ that appears above can be considered as a higher dimensional analogue of Bloch-Kato exponential map.

0.1.8. Syntomic regulators. Dirichlet, followed by Dedekind, defined in the 19th century a regulator map (in fact, a logarithm map) from units in the ring integers $\mathcal{O}_E$ of an algebraic number field E to certain finite-dimensional real vector space; they showed then that the image forms a lattice, whose covolume, also called *regulator*, together with some other invariants of E are related to special values of ζ-functions (Dirichlet's class number formula), of which the regulator serves as the transcendental part.

The term *regulator* has since been used to denominate certain maps relating cycle class invariants, e.g. K-groups, Chow groups, to cohomological groups. One famous arhimedean example is the Beilinson regulator which takes values in Deligne-Beilinson cohomology groups. Intuitively speaking, the regulator plays the role of an abstract integration theory.

The syntomic cohomology has been considered as the p-adic analogue of Deligne-Beilinson cohomology (see for example [Nek98]). Then, the syntomic regulator being regarded as an abstract p-adic integration, the Bloch-Kato exponential map above compares certain p-adic integrals to the values of the p-adic étale regulator.

The map of spectral sequences in the Theorem (0.1.3, ii) allows us to study the image of p-adic (pro)étale regulator maps via syntomic regulators. Let us be more precise in the following.

0.1.9. Image of étale regulators: algebraic setting. Let us start with the algebraic setting. Let X be an algebraic variety over K. There are étale Chern class maps $c_0^{\text{ét}} : K_0(X) \to H^{2i}_{\text{ét}}(X, \mathbf{Q}_p(i))$ for $i \in \mathbf{N}$, where $K_0(X)$ denotes the Grothendieck group of the commutative monoid of all isomorphism classes of vector bundles with the monoid operation given by direct sum. It can be generalised to higher (connective) K-groups, so we have étale higher Chern class maps

$$c^{\text{ét}}_{i,j} : K_j(X) \to H^{2i-j}_{\text{ét}}(X, \mathbf{Q}_p(i))$$

for $i, j \in \mathbf{N}$. We may consider the subset of *homologically trivial elements of $K_j(X)$ along $c^{\text{ét}}_{i,j}$*, which is defined as

$$K_j(X)_0 := \ker(K_j(X) \xrightarrow{c^{\text{ét}}_{i,j}} H^{2i}_{\text{ét}}(X, \mathbf{Q}_p(i)) \to H^{2i}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))).$$

By Hochschild-Serre spectral sequence, the map $c^{\text{ét}}_{i,j}$ induces Soulé's étale higher regulator map

$$r^{\text{ét}}_{i,j} : K_j(X)_0 \to H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))).$$

Nekovář and Nizioł proved the following result on the image of regulators $r^{\text{ét}}_{i,j}$ [46, Theorem B], already know to Scholl, which generalised their own previous results with good or semi-stable reduction to arbitrary varieties over K.

Theorem (Scholl, Nekovář-Nizioł). *The regulators $r^{\text{ét}}_{i,j}$ factors through the subgroup*

$$H^1_g(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))) \subset H^1_{\text{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(i))).$$

This follows directly from nice properties of syntomic cohomology (e.g. projective bundle formula and $\mathbf{A}^1$-homotopy invariance), their theorem on syntomic descent spectral sequence (0.1.1) and compatibility of syntomic and étale Chern class maps with the syntomic-étale period maps.

0.1.10. Image of étale regulators: rigid-analytic setting. Similarly as in Theorem (0.1.9), our Theorem on syntomic descent sepctral sequence (0.1.3) should have a direct consequence concerning rigid-analytic étale regulators. The main object in play is certain K-theory in the rigid-analytic setting. We have two candidates: the (non-connective) analytic K-theory K^{an} à la Kerz-Saito-Tamme [37, 38] and the (non-connective) nuclear-continuous K-theory $\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}$ for analytic adic spaces à la Andreychev [1] (essentially equivalent to Morrow's continuous K-theory); the first is built from algebraic K-theory but forcing (pro-)$\mathbf{A}^{1,\mathrm{an}}$-homotopy invariance, while the latter is built from the category of (derived) nuclear modules "just as" the algebraic K-theory is built from the category of perfect complexes.

We adopt the first construction for the moment, and view K^{an} as a presheaf on $\mathcal{R}\mathrm{ig}_K$ with values in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$. We prove that there is a rigid analogue of Theorem (0.1.9), as follows (see (4.5.11) and (4.6.15) for details).

Theorem (Image of (pro)étale regulators). *Let X be a rigid-analytic variety over K. There are rigid-analytic (pro)étale regulators*

$$r^{\mathrm{éh}}_{i,j} : K^{\mathrm{an,éh}}_j(X)_0 \to H^1_{\mathrm{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\mathrm{proét}}(X_C, \mathbf{Q}_p(i)))$$

for $i, j \in \mathbf{N}$, which factor through ${}^{\mathrm{syn}}E_2^{1,2i-j-1} \to^{\mathrm{HS}} E_2^{1,2i-j-1} = H^1_{\mathrm{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\mathrm{proét}}(X_C, \mathbf{Q}_p(i)))$. In particular, if X is proper, then the regulators $r^{\mathrm{éh}}_{i,j}$ factor through the geometric Selmer groups

$$H^1_g(\mathscr{G}_K, H^{2i-j-1}_{\mathrm{ét}}(X_C, \mathbf{Q}_p(i))) \subset H^1_{\mathrm{cont}}(\mathscr{G}_K, H^{2i-j-1}_{\mathrm{ét}}(X_C, \mathbf{Q}_p(i))).$$

Here, we denote by K^{an} the non-connective analytic K-theory functor on $\mathcal{R}\mathrm{ig}_K$, which is a Nisnevich sheaf; it takes values in the ∞-category of light condensed spectra. There is a natural map $K_0^{\mathrm{naive}} \to \pi_0 K^{\mathrm{an}}$ of presheaves on $\mathcal{R}\mathrm{ig}_K$, where $K_0^{\mathrm{naive}}(X)$ is the naive Grothendieck (abelian) group (without condensed structure) of the commutative monoid of all isomorphism classes of vector bundles on X. We denote the éh-sheafification of K^{an} by $K^{\mathrm{an,éh}}$. Finally, we define the $K^{\mathrm{an,éh}}_j$ as the objectwise j-th homotopy group of $K^{\mathrm{an,éh}}$, i.e. we define

$$K^{\mathrm{an,éh}}_j := \pi_j^{\mathrm{pre}} K^{\mathrm{an,éh}} : \begin{array}{rl} \mathcal{R}\mathrm{ig}_K & \to \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b} \\ X & \mapsto \pi_j(K^{\mathrm{an,éh}}(X)). \end{array}$$

Since there are natural maps $K^{\mathrm{an}}_j \to K^{\mathrm{an,éh}}_j$, these results still hold when we restrict the Chern class maps and regulator maps to K^{an}_j.

0.2 Outline of proofs

0.2.1 (Usage of condensed mathematics). To have a better control on p-adic cohomology theories, which have huge cohomology groups and are thus difficult to handle as plain groups, one need to take their natural, if not canonical, topological structures into account. Our point of view will be upgrading objects and statements to the condensed world without much loss of information. The advantage of condensed mathematics that we take in this article is essentially the good homological algebraic properties of solid p-adic functional analysis, especially those of nuclear modules over $\mathbf{Q}_p$, such as behaviours of interactions between countable limits and solid tensor products.

We would like to express our belief that many results could have been done in the more classical language of locally convex topological $\mathbf{Q}_p$-vector spaces or in other potentially adequate models of topological structures on algebras, but we are not going to pursue this point of view.

0.2.2. Let us look at the first Theorem (0.1.3). As observed above in (0.1.5), it is easy to find a candidate, namely the Fontaine-Messing period maps α_r^{FM} to satisfy the assertion (i), however it is not clear whether they fulfill (ii) and (iii); on the other hand, using (condensed) proétale period sheaves as in [9], one may construct Galois equivariant geometric period maps $\rho^{\mathrm{geom}}_{\mathrm{syn}}$, with the help of which the statements (ii) and (iii) are more accessible; the uniqueness proved by Sally Gilles [29] reunites these two sets of period maps.

Now let us focus on (ii) and (iii). Following the path of proof of Nekovář and Nizioł, we find that the key is the compatibility between arithmetic and geometric Hyodo-Kato morphisms, not only after taking derived fixed points and monodromy-trivial elements but also before it. In other words, we should have the following Galois equivariant commutative diagram

$$
\begin{tikzcd}
R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{F} B^{+}_{\mathrm{st}} \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}} \otimes \iota_p"] \arrow[d] & R\Gamma_{\mathrm{dR}}(X/K) \otimes^{\blacksquare}_{K} B^{+}_{\mathrm{dR}} \arrow[d, "\simeq"] \\
R\Gamma_{\mathrm{HK}}(X_C) \otimes^{\blacksquare}_{\breve{F}} B^{+}_{\mathrm{st}} \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(X_C/B^{+}_{\mathrm{dR}})
\end{tikzcd}
\tag{0.2.2.1}
$$

for $X \in \mathcal{R}\mathrm{ig}_K$, where the vertical maps are obvious ones and $\iota_p : B^{+}_{\mathrm{st}} \to B^{+}_{\mathrm{dR}}$ is the canonical embedding given by $\log[p^\flat] \mapsto \log\left(\frac{[p^\flat]}{p}\right)$. Let us discuss the Hyodo-Kato morphisms appearing in the horizontal maps. Recall that Colmez and Nizioł defined globally for rigid-analytic varieties (as well as dagger varieties) an arithmetic Hyodo-Kato morphism in [17] using zig-zag construction, which we denote for the moment by $\iota^{\mathrm{arith,CN}}_{\mathrm{HK}}$, and defined a geometric Hyodo-Kato morphism in [19] using Beilinson's method, which we denote by $\iota^{\mathrm{geom,CN}}_{\mathrm{HK}}$.

It is unclear to the author whether these two Hyodo-Kato morphisms are compatible via the obvious maps; if it is the case, then our paper should have been greatly shortened on this issue; otherwise, we are at least able to construct by Galois descent a seemingly new arithmetic Hyodo-Kato morphisms $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ making the diagram (0.2.2.1) commutative and Galois equivariant. Indeed, this follows essentially from our construction.

It is then natural to ask whether there is a homotopy $\iota^{\mathrm{arith}}_{\mathrm{HK}} \simeq \iota^{\mathrm{arith,CN}}_{\mathrm{HK}}$. We prove this in the semistable reduction case, however, the homotopy that we construct does not seem to be natural (due to potential higher associativity issues), since it depends on the choice of uniformiser of a finite extension L of K. Despite this shortcoming, the syntomic cohomology defined by this new arithmetic Hyodo-Kato morphisms is naturally isomorphic to the usual one defined by Colmez and Nizioł in [17].

Once the Galois equvariant compatibility (0.2.2.1) is established, the (ii) and (iii), namely the existence of syntomic descent spectral sequence mapping naturally to the Hochschil-Serre spectral sequence and the identification of terms of its E_2-page in the proper case with Selmer groups, follow as in [46].

0.2.3. Now we turn to the second Theorem (0.1.10). We have proétale first Chern class maps for p-adic proétale cohomology, which induces by projective bundle formula Chern classes for vector bundles. Similarly, we obtain a theory of Chern classes taking values in syntomic cohomology groups. These two different Chern classes are compatible via the period maps ρ_{syn}, and even better, compatible with the map from the syntomic descent spectral sequence to the Hochschild-Serre spectral sequence, from which the Theorem (0.1.10) then follows immediately for the Grothendieck K_0-group $K_0^{\mathrm{naive}}(X)$ of the category of vector bundles on the rigid-analytic variety X over K in place of the higher éh-analytic K-group $K_j^{\mathrm{an,éh}}$ (for $j = 0$) on the left hand side of the regulator map.

Regarding the case of higher K-groups, there remains something to do on the K-theory side; for this purpose, it is enough to construct Chern class maps for higher analytic K-groups. It suffices in turn to show the following representability result of (étale) analytic K-theory (4.6.3), which is a direct corollary of [22, §5].

Theorem (Dahlhausen-Yaylali). *We have natural equivalences in* $\mathcal{S}\mathrm{hv}_{\mathrm{\acute{e}t}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc}))$

$$L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1}(\mathbf{Z} \times \mathrm{BGL}) \xrightarrow{\simeq} \Omega^\infty \tau_{\geq 0} L_{\mathrm{\acute{e}t}}(k^{\mathrm{an}})^B \xleftarrow{\simeq} \Omega^\infty \tau_{\geq 0} L_{\mathrm{\acute{e}t}} K^{\mathrm{an}} = \Omega^\infty \tau_{\geq 0} K^{\mathrm{an,\acute{e}t}}.$$

Here, the first term is the étale sheafification of the $\mathbf{A}^1$-exactification (seen as a presheaf) of the presheaf $\mathbf{Z} \times \mathrm{BGL}$ on the category of smooth rigid-analytic varieties over K, an analytic analogue of the algebraic counterpart; the third term (*resp.* the second term) is the connective cover of the étale sheafification of the non-connective analytic K-theory (*resp.* a variant of it) à la Kerz-Saito-Tamme [37, 38], which is defined as the Bass construction applied to the connective analytic K-theory (*resp.* to its connected cover).

Replacing $\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ with $\mathcal{R}\mathrm{ig}_K$ and the étale topology with the éh-topology, one obtains similar results for $K^{\mathrm{an,\acute{e}h}}$. Then the machinery of universal Chern classes runs as usual to yield the desired higher Chern class maps.

0.3 Structure of the paper

0.3.1. In the first section, we are going to collect some preliminary results from higher topos theory and condensed mathematics, recall condensed group cohomology, which generalises Tate's continuous group cohomology, and finally discuss what condensed structures we will consider on typical p-adic cohomology theories. Readers familiar with condensed mathematics may skip this on their first reading.

0.3.2. In the second section, we "redefine" the arithmetic Hyodo-Kato morphism for rigid-analytic varieties to be compatible with the geometric Hyodo-Kato morphisms defined in [19], which induces the same arithmetic syntomic cohomology as defined in [17]. We will also extend the construction to overconvergent situation by standard procedures.

0.3.3. In the third section, we will define the arithmetic syntomic-proétale period map, compare it with the Fontaine-Messing period maps, and construct morphisms of spectral sequences under specific conditions, namely the C_{st}-conjecture.

0.3.4. In the last section, we define and study étale higher Chern class maps for rigid analytic varieties. Along the way, we provide details of the projective bundle formula and $\mathbf{A}^1$-homotopy invariance for syntomic and integral p-adic (pro)étale cohomology, and produce the higher Chern class maps by standard methods. The representability of étale analytic K-theory will finally be established before we complete the proof of the main result on regulators.

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1 Preliminaries

1.0.1. Notation. Fix a prime number p. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$, with ring of integral $\mathcal{O}_K$ and perfect residue field k. Let $\overline{K}$ be the algebraic closure of K and $C=\widehat{\overline{K}}$ its p-adic completion, which is still algebraically complete. Let $\mathscr{G}_K=\mathrm{Gal}(\overline{K}/K)$ be the absolute Galois groupe of K, which is a profinite group. Let $\mathcal{O}_{K_0}=W(k)$ be the ring of Witt vectors over k, and $F=\mathcal{O}_F[\frac{1}{p}]$, which is the maximal unramified subfield of K. Let F^{nr} be the maximal unramified extension of F and $\breve{F}$ be its p-adic completion.

Let L be any other complete nonarchimedean field of mixed characteristic $(0,p)$. We denote by k_L its residue field, by F_L its maximal unramified subfield with ring of integral elements $\mathcal{O}_{F_L}$. We have three log-structures: the trivial log-structure $\mathcal{O}_L^{\mathrm{triv}}=(\mathcal{O}_L,\mathcal{O}_L^\times)$, the canonical log-structure $\mathcal{O}_L^\times=(\mathcal{O}_L,\mathcal{O}_L\backslash\{0\})$ defined by its closed point, and the fat or hollow log-structure $\mathcal{O}_{F_L}^0$ associated to the pre-log-structure $(\mathcal{O}_{F_L},\mathcal{O}_L\backslash\{0\})$ sending $0\neq a\in\mathcal{O}_L$ to $[\bar{a}]\in W(k_L)$. We have reductions $\mathcal{O}_{L,n}:=\mathcal{O}_L/p^n$ for $n\geq 1$ and $\mathcal{O}_{L,0}:=\mathcal{O}_L/\mathfrak{m}_L\simeq\mathcal{O}_{F_L,1}$, which could be decorated to designate corresponding induced log-structures.

1.0.2. We will principally work with ∞-categories, though computations could be done in ordinary categories.

1.0.3. Rigid spaces. Let L be a complete nonarchimdean field of mixed characteristic. All the rigid-analytic varieties over L that we will consider are supposed to be quasi-separated, of finite dimension, and *paracompact*, i.e. admitting an admissible locally finite affinoid covering; in particular, they are admissible disjoint unions of *connected* paracompact rigid-analytic varieties *of countable type*, i.e. having a countable admissible affinoid covering. When we regard them as adic spaces, we may use the term *rigid spaces*.

Let $\mathcal{R}\mathrm{ig}_L$ (*resp.* $\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_L$) denote the category of rigid spaces (*resp.* smooth rigid spaces) over L.

1.0.4. Cohomological indices are denoted by superscripts, while homological indices are denoted by subscripts. They could be quite confusing when we mix homotopy theories with cohomological notation of derived categories.

1.1 Sheaves and derived categories

1.1.1. Stabilisation. Let $\mathscr{V}$ be a presentable ∞-category with finite limits. Let $*$ denote its final object. Write $\mathscr{V}_* := \mathscr{V}_{*/}$ for the ∞-category of its pointed objects. There is the loop space functor $\Omega : \mathscr{V}_* \to \mathscr{V}_*$ defined as sending X to $\Omega X := \mathrm{fib}(* \to X)$; it preserves limits and thus admit a left adjoint $\Sigma : \mathscr{V}_* \to \mathscr{V}_*$. The functors Σ and Ω are equivalences if $\mathscr{V}$ is a stable ∞-category.

In general, $\mathscr{V}$ is not necessary stable. The *stabilisation* of $\mathscr{V}$ is the ∞-category

$$\mathcal{S}\mathrm{p}(\mathscr{V}_*) := \lim(\cdots \to \mathscr{V}_* \xrightarrow{\Omega} \mathscr{V}_* \xrightarrow{\Omega} \mathscr{V}_*),$$

which is universal among stable ∞-categories with a functor from $\mathscr{V}$ that sends Ω to an equivalence. Its objects can be described as a sequence $X = (X_0, X_1, \dots)$ together with structural equivalences $X_n \xrightarrow{\simeq} \Omega X_{n+1}$. There is an adjunction of functors $\Sigma^\infty \dashv \Omega^\infty$, called respectively the *infinite suspension spectrum functor* and the *infinite loop space functor*, between ∞-categories $\mathscr{V}$ and $\mathcal{S}\mathrm{p}(\mathscr{V}_*)$. Concretely, we have $\Omega^\infty X = X_0$, and $\Sigma^\infty Y$ is given by $(\Sigma^\infty Y)_n := \mathrm{colim}_m\, \Omega^m \Sigma^{m+n} Y_+$ together with evident structural morphisms.

1.1.2 - Example (Spaces and spectra). Let $\mathcal{S}\mathrm{pc}$ be the ∞-category of spaces (or anima). It is presentable and admits all limits. Its stablisation $\mathcal{S}\mathrm{p} := \mathcal{S}\mathrm{p}(\mathcal{S}\mathrm{pc}_*)$ is the ∞-category of spectra. As a right adjoint functor, the infinite loop space functor $\Omega^\infty : \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{pc}$ preserves limits; and $\Omega^\infty|_{\mathcal{S}\mathrm{p}_{\geq 0}}$ also preserves sifted colimits [43, Proposition 1.4.3.9], in particular filtered colimtis and geometric realisations. The key to the proof of the preservation of sifted colimts is that the formation of sifted colimits in $\mathcal{S}\mathrm{pc}$ commutes with finite products.

1.1.3 - Lemma. *Let R be an ordinary ring. Let $\mathrm{LMod}_R := \mathrm{LMod}_{\mathrm{H}R}(\mathcal{S}\mathrm{p})$ be the stable ∞-category of left R-module spectra, or simply called left R-modules.*

(i) The t-structure on the stable ∞-category $\mathcal{S}\mathrm{p}$ induces naturally the canonical t-structure on LMod_R. The forgetful functor $\mathrm{LMod}_R \to \mathcal{S}\mathrm{p}$ is conservative and t-exact. The functor π_0 induces an equivalence

$$\mathrm{LMod}_R^\heartsuit \xrightarrow{\simeq} \mathrm{Mod}_R,$$

where the latter is (the nerve of) the ordinary category of (discrete) R-modules. The subcategories $\mathrm{LMod}_{R,\geq 0}, \mathrm{LMod}_{R,\leq 0} \subset \mathrm{LMod}_R$ are stable under all (small) products and (small) filtered colimits.

(ii) There are canonical equivalences of ∞-categories

$$\mathscr{D}^-(R) \xrightarrow{\simeq} \mathrm{LMod}_R^-, \quad \mathscr{D}^+(R) \xrightarrow{\simeq} \mathrm{LMod}_R^+, \quad \mathscr{D}(R) \xrightarrow{\simeq} \mathrm{LMod}_R.$$

Proof. The (i) is the content of [43, Proposition 7.1.1.13], and the (ii) is due to [43, Proposition 7.1.1.15, Remark 7.1.1.16]. □

1.1.4. Sheaves and hypersheaves. Let $\mathcal{C}$ be a site and $\mathscr{V}$ an ∞-category. A presheaf $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathscr{V})$ with values in $\mathscr{V}$ is a *sheaf* (*resp.* *hypersheaf*) if it satisfies descent for Čech coverings (*resp.* for hypercoverings).

Inside the ∞-category of presheaves $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathscr{V}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathscr{V})$, we define the full ∞-category of sheaves (*resp.* hypersheaves) with values in $\mathscr{V}$, denoted by $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathscr{V})$ (*resp.* $\mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{C},\mathscr{V})$).

If $\mathcal{V}$ admits all limits, then the inclusion $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathscr{V}) \subset \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathscr{V})$ admits a left adjoint $L = L_\tau$ called ***sheafification***, with subscript τ indicating the topology on $\mathcal{C}$, and the inclusion $\mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{C},\mathscr{V}) \subset \mathcal{S}\mathrm{hv}(\mathcal{C},\mathscr{V})$ admits a left adjoint $(-)^{\mathrm{hyp}}$ called ***hypersheafification***. They preserve finite limits. If $\mathscr{V}$ admits all limits and filtered colimits, then L (*resp.* $(-)^{\mathrm{hyp}}$) can be described as the transfinite iteration of the operation

$$F \mapsto (F^\dagger : U \mapsto \varinjlim_{\mathcal{U}} \lim_{\Delta} F(\check{C}(\mathcal{U},F)_\bullet) \in \mathscr{V}) \tag{1.1.4.1}$$

for $F \in \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathscr{V})$, where $\mathcal{U}$ runs through the filtered sets of all coverings (*resp.* hypercoverings) of U and $\check{C}(\mathcal{U},F)_\bullet$ denotes the Čech cosimplicial nerve associated to $\mathcal{U}$.

If $\mathscr{V}$ is a presentable ∞-category, then $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathscr{V}) \simeq \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) \otimes \mathscr{V}$ is a presentable ∞-category [44, Remark 1.3.1.6] with tensor product being the Lurie's tensor product of presentable ∞-categories (which is colimit preserving) introduced in [43, §4.8.1].

One can produce hypersheaves by taking global sections, for example namely for any bounded below complexes of sheaves F on $\mathcal{C}$ with values in an abelian category $\mathcal{A}$, the associated global section functor $R\Gamma(-,F) : U \mapsto R\Gamma(U,F)$ is a hypersheaf on $\mathcal{C}$ with values in $\mathscr{D}(\mathcal{A})$ [42, Lemma 6.5.2.9].

1.1.5 - Example. Let $\mathcal{C}$ be a site. We consider the presentable ∞-categories $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}_{(*)})$, $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ as well as $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}_{(*)})$ and $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$. Then the adjunction $\Sigma^\infty \dashv \Omega^\infty$ between $\mathcal{S}\mathrm{pc}$ and Sp induces adjunctions $\Sigma^{\infty,\mathrm{pre}} \dashv \Omega^{\infty,\mathrm{pre}}$ between $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc})$ and $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$, and $\Sigma^\infty \dashv \Omega^\infty$ between $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc})$ and $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$. For presheaves, $\Sigma^{\infty,\mathrm{pre}}$ and $\Omega^{\infty,\mathrm{pre}}$ are given simply by composition with the functors on $\mathcal{C}^{\mathrm{op}}$. For sheaves, Σ^∞ is given by the sheafification of $\Sigma^{\infty,\mathrm{pre}}$, while Ω^∞ is given by $\Omega^{\infty,\mathrm{pre}}$ since it preserves sheaves.

In fact, these constructions work more generally for any ∞-topoi $\mathcal{X}$, and there is a canonical adjunction $\Sigma^\infty \dashv \Omega^\infty$ between the ∞-categories $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{X},\mathcal{S}\mathrm{pc})$ and $\mathcal{S}\mathrm{hv}(\mathcal{X},\mathcal{S}\mathrm{p})$, see [44, Remark 1.3.2.2].

By the explicit description of the sheafification functor L (1.1.4.1), and the fact that $\Omega^\infty|_{\mathcal{S}\mathrm{p}_{\geq 0}}$ commutes with sifted colimits (1.1.2), we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) \arrow[d, "L"] & \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}) \arrow[l, "\Omega^{\infty,\mathrm{pre}}"'] \arrow[d, "L"] \\ \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}). \arrow[l, "\Omega^\infty"'] \end{tikzcd} \tag{1.1.5.1}$$

1.1.6. Canonical t-structure on sheaves of spectra. Let $\mathcal{C}$ be a site. Let $n \in \mathbf{Z}$. There is a canonical t-structure on $\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ given by the pair $(\mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}), \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\leq 0}))$. Let $\pi_n^{\mathrm{pre}} : \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}) \to \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{A}\mathrm{b})$ be the n-th homotopy group functor for presheaves.

There are also well-defined n-th homotopy group functor for sheaves $\pi_n : \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{A}\mathrm{b})$, which can be identified with the sheafification of π_n^{pre}. The functor pi_n commutes with finite limits. Then, we have notions of n-connective objects and n-coconnective objects. They span respectively the full subcategories $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})_{\geq n}$ and $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})_{\geq n}$ of $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$, which determine a canonical t-structure on $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$.

This t-structure is compatible with filtered colimits, that is, the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})_{\leq 0} \subset \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ is closed under filtered colimits. Besides, the full subcategory $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})_{\geq 0} \subset \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ is automatically closed under all colimits, as this inclusion has a right adjoint.

The composition with the truncation functor $\tau_{\geq 0} : \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_{\geq 0}$, which preserves sheaves since it is the right adjoint to the inclusion of connective spectra $\mathcal{S}\mathrm{p}_{\geq 0} \hookrightarrow \mathcal{S}\mathrm{p}$, induces an equivalences of ∞-categories

$$\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})_{\geq 0} \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}).$$

The commutative diagram (1.1.5.1) is then refined to

$$\begin{tikzcd} \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) \arrow[d, "L"] & \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}) \arrow[l, "\Omega^{\infty,\mathrm{pre}}"'] \arrow[r, hook] \arrow[d, "L"] & \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}) \arrow[d, "L"] \\ \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}) \arrow[l, "\Omega^{\infty}"'] \arrow[r, hook] & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}). \end{tikzcd} \tag{1.1.6.1}$$

The sheafification functor $L : \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ is t-exact and symmetric monoidal. By t-exactness we have isomorphisms

$$L\tau^{\mathrm{pre}}_{\geq n} \xrightarrow{\simeq} \tau_{\geq n}L, \quad L\tau^{\mathrm{pre}}_{\leq n} \xrightarrow{\simeq} \tau_{\leq n}L, \quad L\pi^{\mathrm{pre}}_0 \simeq \pi_0 L$$

for $n \in \mathbf{Z}$, as well as a commutative diagram

$$\begin{tikzcd} \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) \arrow[d, "L"] & \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}) \arrow[l, "\Omega^{\infty,\mathrm{pre}}"'] \arrow[d, "L"] & \mathcal{PS}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}) \arrow[l, "\tau^{\mathrm{pre}}_{\geq 0}"'] \arrow[d, "L"] \\ \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc}) & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}_{\geq 0}) \arrow[l, "\Omega^{\infty}"'] & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}). \arrow[l, "\tau_{\geq 0}"'] \end{tikzcd} \tag{1.1.6.2}$$

These also generalise to sheaves of spectra on any ∞-topoi [44, Proposition 1.3.2.7, Proposition 1.3.4.7, Proposition 1.3.5.7].

Here is a sheafified generalised version of (1.1.3).

1.1.7 - Lemma. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{CA}\mathrm{lg}(\mathcal{S}\mathrm{p})) \simeq \mathcal{CA}\mathrm{lg}(\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}))$ [44, §1.3.5] be a sheaf of connective commutative ring spectrum on $\mathcal{C}$, i.e. the underlying sheaf of spectra is connective; and let $\mathrm{Mod}_{\mathcal{O}} := \mathrm{Mod}_{\mathcal{O}}(\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p}))$ denote the symmetric monoidal ∞-category of $\mathcal{O}$-module objects of $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ [44, Definition 2.1.0.1].*

(i) The forgetful functor $\mathrm{Mod}_{\mathcal{O}} \to \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ is conservative and preserves (small) limits and colimits.

(ii) The t-structure on $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ induces naturally the canonical t-structure on $\mathrm{Mod}_{\mathcal{O}}$. The forgetful functor $\mathrm{Mod}_{\mathcal{O}} \to \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ is t-exact. The subcategories $\mathrm{Mod}_{\mathcal{O},\geq 0}, \mathrm{Mod}_{\mathcal{O},\leq 0} \subset \mathrm{Mod}_{\mathcal{O}}$ are stable under (small) filtered colimits.

(iii) If the structure sheaf $\mathcal{O}$ is discrete, then the functor π_0 induces an equivalence

$$\mathrm{Mod}^{\heartsuit}_{\mathcal{O}} \xrightarrow{\simeq} \mathrm{Mod}_{\mathcal{O}}(\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{et})),$$

where the latter is (the nerve of) the ordinary category of (discrete) $\mathcal{O}$-modules.

(iv) If the structure sheaf $\mathcal{O}$ is discrete, then there is a canonical colimit-preserving and t-exact fully faithful embedding of ∞-categories

$$\iota : \mathscr{D}(\mathrm{Mod}^{\heartsuit}_{\mathcal{O}}) \hookrightarrow \mathrm{Mod}_{\mathcal{O}},$$

whose image is the full subcategory of those objects of $\mathrm{Mod}_{\mathcal{O}}$ whose underlying object in $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{p})$ is hypercomplete.

Proof. The (i) is from [44, Proposition 2.1.0.3 (iii)]. The (ii) follows from [44, Proposition 2.1.1.1] and (i). The (iii) is clear by definition, cf. [44, Remark 2.1.2.1]. The (iv) is the content of [44, Corollary 2.1.2.3][1]. □

[1]It is a special yet frequent case of *loc. cit.* Actually, in the case of the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc})$, the condition *loc. cit.* (b), namely that for any $F \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc})$, there exists an effective epimorphism $F' \to F$ where $F' \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{et})$ is a discrete sheaf, turns out to be a consequence of the condition *loc. cit.* (a) that the structure sheaf $\mathcal{O}$ is discrete. Indeed, for any such F, we can take, similarly in the proof of [44, Proposition 2.1.2.5], $F' \in \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{pc})$ to be the sheafification of the discrete presheaf $F'^{\mathrm{pre}} := \coprod_{c\in\mathcal{C},\eta\in\pi_0 F(c)} h_C$; then F' is discrete and the canonical map $F' \to F$ is an effective epimorphism by [42, Lemma 6.2.4.5]. We remark that the existence of such $F' \to F$ may fail for more general ∞-topoi than ∞-categories of sheaves.

1.1.8. Let $\mathcal{C}$ be a site and $\mathscr{V}$ be a presentable ∞-category with finite limits. Taking sheaves on $\mathcal{C}$ commutes with certain operations:

(i) For another site $\mathcal{D}$, we have canonical equivalences $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathscr{V})) \simeq \mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{S}\mathrm{hv}(\mathcal{C},\mathscr{V}))$ and $\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{S}\mathrm{hv}(\mathcal{D},\mathscr{V})) \simeq \mathcal{S}\mathrm{hv}(\mathcal{D},\mathcal{S}\mathrm{hv}(\mathcal{C},\mathscr{V})) \simeq \mathcal{S}\mathrm{hv}(\mathcal{C}\times\mathcal{D},\mathscr{V})$ sending $F : U \mapsto F_U$ to $V \mapsto (U \mapsto F_U(V))$ (then to $U \times V \mapsto F_U(V)$).

(ii) We have $\mathcal{S}\mathrm{h}(\mathcal{C},\mathcal{S}\mathrm{p}(\mathscr{V}_*)) \simeq \mathcal{S}\mathrm{p}(\mathcal{S}\mathrm{h}(\mathcal{C},\mathscr{V}_*))$.

1.2 Condensed mathematics

1.2.1. Condensed mathematics. Let us recall some definitions in condensed mathematics:

(i) Let $\mathcal{C}$ be an ∞-category that admits small limits. For any uncountable strong limit cardinal κ, one defines the *∞-category of κ-condensed objects* of $\mathcal{C}$ as the ∞-category of sheaves on the site of light condensed sets valued in $\mathcal{C}$, i.e.

$$\mathcal{C}\mathrm{ond}_\kappa(\mathcal{C}) := \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa},\mathcal{C}).$$

It is the same as the ∞-category of contravariant functors from κ-small extremally disconnected profinite sets to $\mathcal{C}$ that take finite coproducts to products, denoted by

$$\mathcal{C}\mathrm{ond}_\kappa(\mathcal{C}) \simeq \mathrm{Fun}^\times(\mathrm{EDS}^{\mathrm{op}}_{<\kappa},\mathcal{C}).$$

(ii) Light condensed theory, introduced by Dustin Clausen and Peter Scholze in their lectures on Analytic Stacks jointly held in IHES and Bonn, are of certain interest, since most objects that concern us lives in the light setting. A profinite set is called *light* if it can be written as a countable inverse limit of finite sets. One defines the *∞-category of light condensed objects* of $\mathcal{C}$ as

$$\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{C}) := \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}},\mathcal{C}).$$

(iii) Both $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{C})$ and $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{C})$ are stable (*resp.* presentable) ∞-categories if $\mathcal{C}$ is stable (*resp.* presentable).

(iv) The κ-condensed and light condensed theories can be related via the adjunctions $L \dashv \mathrm{Res}^{\mathrm{light}} \dashv R$, where $L_\kappa : \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}},\mathcal{C}) \to \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa},\mathcal{C})$ is the left Kan extension $L_\kappa X(S) := \varinjlim_{S\to T\in\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}}} X(T)$, and $\mathrm{Res}^{\mathrm{light}} : \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa},\mathcal{C}) \to \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}},\mathcal{C})$ is the restriction of the functor to the subcategory $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}} \subset \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa}$, and $R_\kappa : \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}},\mathcal{C}) \to \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa},\mathcal{C})$ is the right Kan extension (or "sheafification") $R_\kappa X(S) := \lim_{\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}}\ni T\to S} X(T)$. The formula shows that $\mathrm{Res}^{\mathrm{light}} \circ L_\kappa \simeq \mathrm{id}$, so L_κ is fully faithful; and $\mathrm{Res}^{\mathrm{light}} \circ R_\kappa \simeq \mathrm{id}$, so R_κ is also fully faithful; Moreoever, both $\mathrm{Res}^{\mathrm{light}}$ and R_κ preserves sheaves, so restricts to an adjunction $\mathrm{Res}^{\mathrm{light}} \dashv R_\kappa$ between $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{C})$ and $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{C})$.

As opposed to the κ-condensed case, colimits of light condensed objects may not be computed pointwisely. Nevertheless, most statements about $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{C})$ can be transferred to the $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{C})$, sometimes requiring extra countability control on index sets.

(v) There is a functor $\mathrm{const} : \mathcal{C} \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{C}), X \mapsto \underline{X}$ as the composition of the constant functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa},\mathcal{C})$ with sheafification. There is an adjunction $\mathrm{const} \dashv \mathrm{ev}_*$ where $\mathrm{ev}_* : \mathcal{C}\mathrm{ond}_{<\kappa}(\mathcal{C}) \to \mathcal{C}$ is the functor taking underlying objects $\mathrm{ev}_* X = X(*)$. Similarly for the light setting.

(vi) There is a comparison functor $\gamma_\kappa : \mathrm{pro}(\mathcal{C}) \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{C}), \text{"}\lim\text{"}_{i\in I} X_i \mapsto \lim_{i\in I} \underline{X_i}$ which preserves small limit, similarly we have γ^{light} in the light setting; they are related by $\gamma^{\mathrm{light}} = \mathrm{Res}^{\mathrm{light}} \circ \gamma_\kappa$[2]. This could be thought of as "putting discrete topology on objects of $\mathcal{C}$ and then profinite topology on pro-objects". For

[2]However, it is not true in general that we have $L \circ \gamma^{\mathrm{light}} = \gamma_\kappa$, even when restricted to the subcategory of light pro setting and $\mathcal{C} = \mathcal{A}\mathrm{b}$.

our purposes, it will be indifferent to choose between κ- and light condensed settings. But since light condensed setting tends to have its own interest, we pretend to keep these two in parallel status in our paper. All statements will be true both in κ- and light condensed setting, unless otherwise mentioned.

(vii) The restriction functor to light pro-spectra $\gamma^{\mathrm{light}} : \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}^+) \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$ is conservative by Clausen-Scholze [22, Theorem A.12][3]. Here $\mathcal{S}\mathrm{p}^+$ denotes the category of bounded above spectra.

1.2.2 - Remark. We are not sure about the proof of [22, Theorem A.12] (Clausen-Scholze), due to my doubts on whether their functor $\gamma_\kappa^{\mathrm{DY}} := R \circ \gamma^{\mathrm{light}}$ is really equal to the functor $\gamma_\kappa : \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}^+) \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$ (with a superscript now) defined as $"\lim"_i X_i \mapsto \lim_i \underline{X_i}$.

Their proof actually proves that γ_κ is conservative.

As for the conservativity of γ^{light}, we have not found a way to deduce it from that of γ_κ because we do not know the relation between $\gamma_\kappa^{\mathrm{DY}}$ and γ_κ. Nevertheless, their proof works *verbatim* in the light condensed setting, by the following lemma, analogue of [22, Lemma A.15] but in the light condensed setting:

1.2.2.1 - Lemma. *For a tower $(M_n)_{n\in\mathbf{N}}$ of abelian groups the following are equivalent:*

(i) The tower is Mittag-Leffler and $\lim_n M_n = 0$.

(ii) The tower is pro-zero, i.e. $"\lim"_n M_n = 0$ *in* $\mathrm{pro}(\mathcal{A}\mathrm{b})$.

(iii) We have $\lim_n \underline{M_n} = \lim^1_n \underline{M_n} = 0$ *in* $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$.

Proof. The proof of (i) $\Rightarrow$ (ii) is totally the same.

The proof of (iii) $\Rightarrow$ (i) is also the same, except that we have to establish the isomorphisms of abelian groups

$$(\clubsuit) \qquad \lim_n \Gamma(S, \underline{M_n}) \simeq \Gamma(S, \lim_n \underline{M_n}), \quad \lim^1_n \Gamma(S, \underline{M_n}) \simeq \Gamma(S, \lim^1_n \underline{M_n})$$

for *the* light profinite set $S := \mathbf{N} \cup \{\infty\}$. For this, noticing that for this special profinite set S, the object $\mathbf{Z}[S] \in \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ is an (internally) projective object, so that

$$R\Gamma(S, \underline{M_n}) \simeq \Gamma(S, \underline{M_n})[0].$$

Using the general isomorphisms $R\lim_n R\Gamma(-, \underline{M_n}) \simeq R\Gamma(-, R\lim_n \underline{M_n})$ on $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{light}}$ for general countable projective system in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$, we obtain

$$R\lim_n(\Gamma(S, \underline{M_n})[0]) \simeq R\lim_n R\Gamma(S, \underline{M_n}) \simeq R\Gamma(S, R\lim_n \underline{M_n}) \simeq \Gamma(S, R\lim_n \underline{M_n}).$$

Taking cohomology groups, using again the projectivity of $\mathbf{Z}[S]$, one finally obtains ($\clubsuit$).

As for the proof of (ii) $\Rightarrow$ (iii), we apply [22, Lemma A.15], which shows that $R\lim_n \underline{M_n} = 0$ in $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{D}(\mathcal{A}\mathrm{b}))$. But we have a restriction functor $\mathrm{Res}^{\mathrm{light}} : \mathcal{C}\mathrm{ond}_\kappa(\mathcal{D}(\mathcal{A}\mathrm{b})) \to \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{D}(\mathcal{A}\mathrm{b}))$ which preserves limits, cf. [22, Remark A.7], and which commutes with the "underline" functor $M \mapsto \underline{M}$; so $R\lim_n \underline{M_n} = 0$ remains true in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{D}(\mathcal{A}\mathrm{b}))$. □

1.2.3. (Pre)sheaves of condensed spectra. Let κ be an uncountable strong limit cardinal. According to (1.1.5), there are adjunction $\Sigma^{\infty,\mathrm{pre}} \dashv \Omega^{\infty,\mathrm{pre}}$ between $\mathcal{PS}\mathrm{hv}(\mathcal{C}, \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc}))$ and $\mathcal{PS}\mathrm{hv}(\mathcal{C}, \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}))$, and adjunction $\Sigma^\infty \dashv \Omega^\infty$ between $\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc}))$ and $\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}))$.

The same proof as [43, Proposition 1.4.3.9], using results from [42, §7.2.2], shows that the functor $\Omega^\infty : \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})_{\geq 0} \simeq \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}_{\geq 0}) \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})$ preserves sifted colimits[4]; as a right adjoint, it also preserves

[3] The proof *loc. cit.* is not correct. Please refer to (1.2.2) for one correct proof.

[4] The key to its proof is the fact that sifted colimits in $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})$ commute with finite products; it can be checked using κ-small extremally disconnected sets.

limits. Hence, similarly as (1.1.6), there are commutative diagrams

(1.2.3.1)
$$\begin{tikzcd}
\mathcal{P}\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})) \arrow[d, "L"] & \mathcal{P}\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}_{\geq 0})) \arrow[l, "\Omega^{\infty,\mathrm{pre}}"'] \arrow[r, hook] \arrow[d, "L"] & \mathcal{P}\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})) \arrow[d, "L"] \\
\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})) & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}_{\geq 0})) \arrow[l, "\Omega^{\infty}"'] \arrow[r, hook] & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}))
\end{tikzcd}$$

(1.2.3.2)
$$\begin{tikzcd}
\mathcal{P}\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})) \arrow[d, "L"] & \mathcal{P}\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}_{\geq 0})) \arrow[l, "\Omega^{\infty,\mathrm{pre}}"'] \arrow[d, "L"] & \mathcal{P}\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})) \arrow[l, "\tau_{\geq 0}^{\mathrm{pre}}"'] \arrow[d, "L"] \\
\mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})) & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}_{\geq 0})) \arrow[l, "\Omega^{\infty}"'] & \mathcal{S}\mathrm{hv}(\mathcal{C},\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})). \arrow[l, "\tau_{\geq 0}"']
\end{tikzcd}$$

Here is a special case of (1.1.7).

1.2.4 - Lemma. *Let $R \in \mathcal{C}\mathrm{ond}_\kappa(\mathcal{C}\mathcal{A}\mathrm{lg})_{\geq 0}$ be a connective condensed ring spectrum, and let $\mathrm{Mod}_R^{\mathrm{cond},\kappa} := \mathrm{Mod}_R(\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}))$ denote the symmetric monoidal ∞-category of R-module objects of $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$.*

(i) The forgetful functor $\mathrm{Mod}_R^{\mathrm{cond},\kappa} \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$ is conservative and preserves (small) limits and colimits.

(ii) The t-structure on $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$ induces naturally the canonical t-structure on $\mathrm{Mod}_R^{\mathrm{cond},\kappa}$. The forgetful functor $\mathrm{Mod}_R^{\mathrm{cond},\kappa} \to \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$ is t-exact. The subcategories $\mathrm{Mod}_{R,\geq 0}^{\mathrm{cond},\kappa}, \mathrm{Mod}_{R,\leq 0}^{\mathrm{cond},\kappa} \subset \mathrm{Mod}_R^{\mathrm{cond},\kappa}$ are stable under (small) filtered colimits.

(iii) If R is discrete, then the functor π_0 induces an equivalence

$$\mathrm{Mod}_R^{\mathrm{cond},\kappa,\heartsuit} \xrightarrow{\simeq} \mathrm{Mod}_R(\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{et})),$$

where the latter is (the nerve of) the ordinary category of (discrete) condensed R-modules.

(iv) If the condensed ring spectrum R is discrete, then there is a canonical colimit-preserving and t-exact equivalences of ∞-categories

$$\iota : \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond},\kappa,\heartsuit}) \xrightarrow{\simeq} \mathrm{Mod}_R^{\mathrm{cond},\kappa}.$$

The same holds for the light condensed setting.

Proof. This follows from applying (1.1.7) to $\mathcal{C} = \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}_{<\kappa}$. For (iv), it suffices to notice that the ∞-topos $\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})$ is hypercomplete by definition. The same works in the light condensed setting. □

1.2.5. Solid mathematics. It will be particularly useful to employ "complete" objects, under the name of *solid* objects.

(i) There is a subcategory $\mathcal{S}\mathrm{olid} \subset \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ consisting of *solid abelian groups*, which is a localisation with left adjoint $(-)^{\blacksquare}$ the solidification functor. It is stable under all limits and colimtis, equipped with a tensor product $- \otimes^{\blacksquare} - := (- \otimes -)^{\blacksquare}$, compactly generated under colimits by $\mathbf{Z}[S]^{\blacksquare}$ for κ-small extremally disconnected sets S, hence compactly generated under colimits by their retracts $\prod_I \mathbf{Z}$ (in particular $\mathbf{Z}[S]^{\blacksquare}$, isomorphic to a product of $\mathbf{Z}$, is a compact projective for any profinite set S). For any $M \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ and $N \in \mathcal{S}\mathrm{olid}$, we have $\underline{\mathrm{Hom}}_{\mathcal{C}}\,\mathrm{ond}\mathcal{A}\mathrm{b}(M,N) \in \mathcal{S}\mathrm{olid}$; similarly on the derived level.

Moreoever, this preservation remains true in the light condensed setting. However, in the light condensed setting, the extremally disconnected sets do not form a basis for the topology, so we cannot argue by checking in the same way. Nevertheless, we know that sifted colimits in any ∞-topos commute with finite products [42, Remark 5.5.8.12], so in particular for the ∞-topos $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc})$.

Alternatively, one can use the restriction $\mathrm{Res}^{\mathrm{light}}$ (1.2.1, iv) to deduce the result in the light setting from the κ-condensed setting. Namely, for any sifted category I and any $X \in \mathrm{Fun}(I,\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq 0}))$, there exists $\widetilde{X}_\kappa \in \mathrm{Fun}(I,\mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p}_{\geq 0}))$ such that $\mathrm{Res}^{\mathrm{light}}\,\widetilde{X}_\kappa = X$ (for example applying L_κ); then we have

$$\Omega^\infty \operatorname{colim}_I X = \Omega^\infty \operatorname{colim}_I \mathrm{Res}^{\mathrm{light}}\,\widetilde{X}_\kappa \simeq \mathrm{Res}^{\mathrm{light}}\,\Omega^\infty \operatorname{colim}_I \widetilde{X}_\kappa \simeq \mathrm{Res}^{\mathrm{light}} \operatorname{colim}_I \Omega^\infty \widetilde{X}_\kappa \simeq \operatorname{colim}_I \Omega^\infty \mathrm{Res}^{\mathrm{light}}\,\widetilde{X}_\kappa = \operatorname{colim}_I \Omega^\infty X,$$

where $\mathrm{Res}^{\mathrm{light}}$ passes through colim_I since it is a left adjoint and passes through Ω^∞ by construction, and the middle commutation is what we have proven in the κ-condensed setting.

(ii) For any (noncommutative) condensed ring R, we denote by $\mathrm{Mod}_R^{\mathrm{cond}} \subset \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ the full subcategory of R-modules, and $\mathrm{Mod}_R^{\blacksquare} := \mathrm{Mod}_R^{\mathrm{cond}} \cap \mathcal{S}\mathrm{olid}$ the category of R-modules that are solid, which amount to solid $R^{\blacksquare}$-modules. Similarly as $\mathcal{S}\mathrm{olid}$, the category $\mathrm{Mod}_R^{\blacksquare}$ carries a tensor product $- \otimes_R^{\blacksquare} - := (- \otimes_R -)^{\blacksquare}$, and is compactly generated under colimits by $(\prod_I \mathbf{Z}) \otimes_{\mathbf{Z}}^{\blacksquare} R$.

(iii) Derived solidification analogue holds, and the derived solidification and derived solid tensor products coincide with respective derived functors. We have $\mathbf{Z}[S]^{L\blacksquare} \simeq \mathbf{Z}[S]^{\blacksquare}[0]$ by [12, Proposition 5.6].

(iv) For example, there is a canonical isomorphism $(\prod_I \mathbf{Z}) \otimes_{\mathbf{Z}}^{L\blacksquare} \underline{M} \xrightarrow{\simeq} \prod_I \underline{M}$ for any profinite abelian group M [8, Lemma A.19]. In particular, we obtain $(\prod_I \mathbf{Z}) \otimes_{\mathbf{Z}}^{L\blacksquare} \underline{\mathcal{O}_K} \xrightarrow{\simeq} \prod_I \underline{\mathcal{O}_K}$ and then $(\prod_I \mathbf{Z}) \otimes_{\mathbf{Z}}^{L\blacksquare} \underline{K} \xrightarrow{\simeq} (\prod_I \underline{\mathcal{O}_K})[\frac{1}{p}] \simeq (\prod_I \underline{\mathcal{O}_K})[\frac{1}{p}]$.

(v) When the situation is clear, for example for usual topological abelian groups M such as $\mathbf{Z}$, $\mathcal{O}_K$, K, C, etc., we will not distinguish the notations $\underline{M}$ and M, unless we want to stress some results in the classical topological setting.

1.2.6. Let $\mathcal{C}$ be a site. We record two consequences of [44, Corollary 2.1.2.3]:

(i) For any ordinary ring R, there is an equivalence of ∞-categories

$$\mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}}) \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(*_{\mathrm{pro\acute{e}t}}, \mathscr{D}(\mathrm{Mod}_R))$$

sending M to $S \mapsto R\Gamma(S, \Gamma)$ on profinite sets S, which simplifies to $M(S)$ on extremally disconnected sets S.

(ii) For any condensed ring R, we have equivalences of ∞-categories

$$\mathscr{D}(\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Mod}_R^{\mathrm{cond}})) \to \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{C}, \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})), \quad \mathscr{D}(\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Mod}_R^{\blacksquare})) \to \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{C}, \mathscr{D}(\mathrm{Mod}_R^{\blacksquare}))$$

induced by sending a sheaf $F \in \mathcal{S}\mathrm{hv}(\mathcal{C}, \mathrm{Mod}_R^{\mathrm{cond}})$ to its global section functor $R\Gamma(-, F)$.

1.2.7. Convention. For simplicity, we will denote by $\mathrm{Hom}(-,-)$ the internal Hom (bi)functor $\mathrm{Hom}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})}(-,-)$ or $\mathrm{Hom}_{\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}}(-,-)$ depending on the context, and also denote $\mathrm{Hom}_R(-,-) := \mathrm{Hom}_{\mathrm{Mod}_R^{\mathrm{cond}}}(-,-)$ for any condensed ring R; similarly for the derived internal Hom. However, the internal Hom in categories of solid modules will keep the full notation $\mathrm{Hom}_{\mathrm{Mod}_R^{\blacksquare}}(-,-)$.

1.2.8. Topological structures on p-adic cohomology theories: a review. We may understand the p-adic proétale cohomology of rigid-analytic varieties over K or C by comparisons with other (integral or rational) p-adic cohomology theories such as Hyodo-Kato cohomology and de Rham cohomology. However, in order to control maps between their cohomology groups, which are huge in general with no naive hope of finite-dimensionality (except in the dagger qcq case), hence it would be useful to put certain topological structure on them into consideration in order to control the maps between them.

Firstly, Colmez, Nizioł et Dospinescu [13] have considered the category C_K of locally convex topological K-vector spaces, which is a quasi-abelian category. Its left bounded derived ∞-category $\mathscr{D}(C_K)$ admits a t-structure whose left heart $LH(C_K)$ are represented (up to equivalence) by a monomorphism $f : E \to F$, where F sits in degree 0. The cohomology groups of an object $X \in \mathscr{D}(C_K)$ are given by

$$\widetilde{H}^n(X) := \tau^{\leq n}\tau^{\geq n}(X) = (\mathrm{coim}\, d^{n-1} \to \ker d^n) \in LH(C_K),$$

while there are also naive cohomology groups

$$H^n(X) := (\ker d^n / \mathrm{coim}\, d^{n-1}) \in C_K$$

endowed with the quotient topology. An object $(E \to F) \in LH(C_K)$ is called *classical* if the natural morphism $(E \to F) \to F/E = H^0(E \to F)$ is an equivalence.

More recently, Guido Bosco has used in his PhD thesis [8, 9] condensed mathematics to study rational p-adic Hodge theory, which is related to the previous perspective by the *condensification* functor [15, Section 4]

$$C_K^{\mathrm{Hcg}} \to \mathrm{Mod}_K^{\mathrm{cond}}, \quad V \mapsto \underline{V}$$

as well as its natural extension to derived categories, where $C_K^{\mathrm{Hcg}} \subset C_K$ denotes the full subcategory of spaces that are Hausdorff and compactly generated, and $\mathrm{Mod}_K^{\mathrm{cond}}$.

The condensification functor sends strict exact sequences of K-Fréchet spaces (*resp.* of spaces of compact type) to exact sequences in $\mathrm{Mod}_K^{\blacksquare}$ [15, Lemma 2.18]. This applies for example to rational p-adic cohomology theories such as (pro)étale cohomology, de Rham cohomology, Hyodo-Kato cohomology, since $R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(i))$ can be represented by a complex of $\mathbf{Q}_p$-Banach spaces if $X \in \mathcal{R}\mathrm{ig}_K^{\mathrm{qcqs}}$ [13, §3.3.2], by a complex of $\mathbf{Q}_p$-vector spaces of compact type if $X \in \mathcal{R}\mathrm{ig}_K^{\dagger,\mathrm{qcqs}}$ [15, Proposition 4.23]; similarly for the geometric case, and for de Rham cohomology and Hyodo-Kato cohomology.

Moreover, for Fréchet spaces, the condensification functor transforms projective tensor products in C_K to solid tensor products [8, Proposition A.68].

1.2.9. Solid p-adic functional analysis. It seems that the condensed, especially solid, mathematics has better homological behaviors than the classical p-adic functional analysis. For this reason, we will always stick to the condensed point of view unless arguments demand intervention of the classical one.

We gather some nice features of solid p-adic functional analysis as follows.

(i) There is a particular class of *nuclear K-vector spaces.* To avoid conceptual confusion with classical nuclear K-vector spaces, we call the former *solid-nuclear K-vector spaces.* The classcial nuclear K-vector spaces are somewhat orthogonal to the concept of K-Banach spaces, as their common objects are finite-dimensional. On the contrary, any K-Banach space is solid-nuclear.

(ii) Let $\mathrm{Mod}_K^{\mathrm{nuc}} \subset \mathrm{Mod}_K^{\blacksquare}$ denote the full subcategory of solid-nuclear K-vector spaces. It is stable under finite limits, countable products (hence countable limits), all colimits, and the solid tensor product [8, Theorem A.43]. It contains all K-Banach spaces, and is generated under colimits by these, which are flat objects for the solid tensor product [8, Corollary A.61]. It contains all K-Fréchet spaces, which can be written as filtered colimits of K-Banach spaces [8, Proposition A.64].

(iii) Let $(V_n)_{n\in\mathbf{N}}$ be a countable projective system of solid-nuclear K-vector spaces, and let W be a K-Fréchet space; then [8, Corollary A.67 (i)]

$$\varprojlim_n (V_n \otimes_K^{\blacksquare} W) \simeq (\varprojlim_n V_n) \otimes_K^{\blacksquare} W.$$

This can be refined into the following lemma (1.2.9.1). As a corollary [8, Corollary A.67 (ii)], if $(V_n)_{n\in\mathbf{N}}$ be a countable projective system of objects in $\mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$ such that each V_n is represented by a complex of solid-nuclear K-vector spaces, and if $W \in \mathscr{D}(\mathrm{Mod}_K^{\blacksquare})$ is represented by a bounded above complex of K-Fréchet spaces, then

$$R\varprojlim_n (V_n \otimes_K^{L\blacksquare} W) \simeq (R\varprojlim_n V_n) \otimes_K^{L\blacksquare} W.$$

1.2.9.1 - Lemma. *Let $(V_n)_{n\in\mathbf{N}}$ be a countable projective system in $\mathrm{Mod}_L^{\blacksquare}$. Consider the full subcategory $C \subset \mathrm{Mod}_K^{\blacksquare}$ consisting of W such that*

$$\varprojlim_n (V_n \otimes_K^{\blacksquare} W) \simeq (\varprojlim_n V_n) \otimes_K^{\blacksquare} W.$$

(i) C is closed under finite colimits.

(ii) If V_n and $\varprojlim_n V_n$ are all flat for $\otimes_L^{\blacksquare}$, then C is closed under finite limits.

(iii) If V_n are all solid-nuclear objects of $\mathrm{Mod}_K^{\blacksquare}$, then C contains all K-Fréchet spaces.

Proof. (i) This is because $\otimes_K^{\blacksquare}$ commutes with all colimits in each variable and $\varprojlim_n$ commutes with finite colimits.

(ii) By flatness, $\otimes_K^{\blacksquare}$ preserves finite limits; and $\varprojlim_n$ commutes with all limits.

(iii) This is [8, Corollary A.67]. $\square$

As a naive example, let us illustrate in a more elementary way what happens when we apply this lemma to the end of the previous proof.

1.2.9.2 - Example. Here is an elementary example of the lemma (1.2.9.1). Let $V_1 \to V_2$ be a morphism between two K-Banach spaces with cokernel N in $\mathrm{Mod}_K^{\blacksquare}$. Let I be a countable set.

(i) The natural map $N \otimes_L^{\blacksquare} L\langle I\rangle \to \prod_I N$ is injective.

(ii) The limit $\varprojlim_J N \otimes_L^{\blacksquare} L\langle J\rangle$ vanishes, where J runs over all cofinite subsets of I.

Proof. (i) We have an injection $N \otimes_L^{\blacksquare} L\langle I\rangle \hookrightarrow N \otimes_L^{\blacksquare} \prod_I L$ by the injectivity of $L\langle I\rangle \to \prod_I L$ and flatness of N for $\otimes_L^{\blacksquare}$ [8, Corollary A.61]. Consider the following commutative diagram

$$\begin{tikzcd}
V_1 \otimes_L^{\blacksquare} \prod_I L \arrow[r] \arrow[d, "i_1"', "\simeq"] & V_2 \otimes_L^{\blacksquare} \prod_I L \arrow[r] \arrow[d, "i_2"', "\simeq"] & N \otimes_L^{\blacksquare} \prod_I L \arrow[r] \arrow[d, "i"'] & 0 \\
\prod_I V_1 \arrow[r] & \prod_I V_2 \arrow[r] & \prod_I N \arrow[r] & 0
\end{tikzcd}$$

where the maps i_1 and i_2 are isomorphisms by [8, Corollary A.59] since I is countable, the first row is exact by flatness of $\prod_I L$ for $-\otimes_L^{\blacksquare}$ [8, Lemma A.58], the second row is exact by exactness of countable products (AB4*) [12, Theorem 2.2]. From this, one deduces that i is also an isomorphism. Thus follows (i) immediately.

(ii) Consider the short exact sequence of projective systems indexed by cofinite subsets $J \subset I$

$$0 \to N \otimes_L^{\blacksquare} L\langle J\rangle \to N \otimes_L^{\blacksquare} L\langle I\rangle \to N \otimes_L^{\blacksquare} L\langle I\backslash J\rangle \to 0$$

where again we are using flatness of N. Using left exactness of limit, we obtain a left exact sequence

$$0 \to \varprojlim_J (N \otimes_L^{\blacksquare} L\langle J\rangle) \to N \otimes_L^{\blacksquare} L\langle I\rangle \to \varprojlim_J (N \otimes_L^{\blacksquare} L\langle I\backslash J\rangle)$$

whose rightmost term is identified with $\prod_I N$. Then (ii) follows from (i). $\square$

1.3 Condensed group actions

1.3.1. Condensed group algebra. Let G be a condensed group. Let ρ be a G-action on a condensed ring R. We define the R-algebra $R[G]_\rho$, called the *skew group algebra of G over R*[5], to be the $R[G] \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ with multpilication law $[g]\cdot[g'] = [gg']$ and $[g]\cdot x = g(x)\cdot[g]$ for $g \in G$ and $x \in R$. We define the category of R-modules with semilinear G-action by $\mathrm{Mod}_{R[G]_\rho}^{\mathrm{cond}}$, and the derived ∞-category of R-modules with semilinear G-action by $\mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})^G := \mathscr{D}(\mathrm{Mod}_{R[G]_\rho}^{\mathrm{cond}})$; similarly for solid objects by replacing $\mathrm{Mod}^{\mathrm{cond}}$ with $\mathrm{Mod}^{\blacksquare}$, or more generally for analytic rings. We may simply denote $R[G] := R[G]_\rho$ with ρ understood.

1.3.2. Let G be a condensed group on a condensed ring R.

(i) The forgetful functor $\mathscr{D}(\mathrm{Mod}_{R[G]}^{\mathrm{cond}}) \to \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})$ induced by the structural ring homomorphism $R \to R[G]$ (i.e. forgetting the G-action) is conservative and commutes with all limits and colimits. It admits as right adjoint the *orbit* functor $R\,\underline{\mathrm{Hom}}_R(R[G]_\rho, -)$, and a left adjoint $-\otimes_R R[G]_\rho$.

(ii) Assume the G-action on R to be trivial. Then there is a natural trivial action functor $(-)^{\mathrm{triv}}$: $\mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}}) \to \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})^G$ induced by the R-algebra homomorphism $R[G] \to R, [g] \mapsto 1$ (i.e.

[5] In some literature, it is denoted by $R\#G$ or $R \rtimes_\rho G$, etc. hence there is no standard notation.

putting trivial G-action on complexes), which is fully faithful and commutes with all limits and colimits. It admits as right adjoint the *derived G-fixed points* functor $(-)^G : \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})^G \to \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})$, the which is computed by $M^G \simeq R\Gamma(G,M) := R\,\underline{\mathrm{Hom}}_{R[G]}(R,M) \simeq R\,\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z},M) \in \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}})$, namely the *condensed group cohomology of G with coefficients in M*. On the other hand, $(-)^{\mathrm{triv}}$ admits as left adjoint the *G-coinvariance functor* $- \otimes_{R[G]} R$.

A standard way to compute the condensed group cohomology of profinite group is to consider the condensed cochain complex, cf. (1.3.4.1).

1.3.3 - Lemma. *For any condensed group G et $M \in \mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})^G$, we have*

$$R\Gamma(G,M) \simeq \underline{\mathrm{RHom}}((\cdots \to \mathbf{Z}[G\times G] \to \mathbf{Z}[G] \to \mathbf{Z} \to 0), M).$$

Proof. The key is the standard bar resolution $\mathbf{Z}[G^{\bullet+1}] \to \mathbf{Z} \to 0$ in $\mathrm{Mod}_{\mathbf{Z}[G]}^{\mathrm{cond}}$, and the natural equivalence $R\,\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z}[G^{\bullet+1}],-) \simeq R\,\underline{\mathrm{Hom}}(\mathbf{Z}[G^\bullet],-)$ killing the first component G; for the latter, we have $\mathbf{Z}[G^{\bullet+1}] \simeq \mathbf{Z}[G] \otimes_{\mathbf{Z}} \mathbf{Z}[G^\bullet] \simeq \mathbf{Z}[G] \otimes_{\mathbf{Z}}^L \mathbf{Z}[G^\bullet]$ where the last isomorphism follows from the flatness of $\mathbf{Z}[G]$ over $\mathbf{Z}$. □

1.3.4 - Proposition. *Let G be a profinite group. For any $M \in \mathscr{D}(\mathcal{S}\mathrm{olid})^{\underline{G}} \subset \mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})^{\underline{G}}$, the condensed group cohomology is computed by the condensed cochain complex*

$$R\Gamma(\underline{G},M) \simeq \left(M \to \underline{\mathrm{Hom}}(\mathbf{Z}[G],M) \to \underline{\mathrm{Hom}}(\mathbf{Z}[G\times G],M) \to \cdots\right). \tag{1.3.4.1}$$

Moreover, $R\Gamma(\underline{G},-)$ commutes with filtered colimits on $\mathscr{D}(\mathcal{S}\mathrm{olid})^{\underline{G}}$.

Proof. This follows from the facts that we have a natural equivalence of functors $\underline{\mathrm{RHom}}(\mathbf{Z}[S],-) \simeq \underline{\mathrm{RHom}}_{\mathcal{S}}\,\mathrm{olid}(\mathbf{Z}[S]^{\blacksquare},-)$ on $D(\mathcal{S}\mathrm{olid})$ and that the functor $\underline{\mathrm{Hom}}_{\mathcal{S}}\,\mathrm{olid}(\mathbf{Z}[S]^{\blacksquare}-,)$ is an exact functor on $\mathcal{S}\mathrm{olid}$, with $\mathbf{Z}[S]^{\blacksquare}$ being an internally compact projective object in $\mathcal{S}\mathrm{olid}$ for any profinite set S. □

The following lemma is a special case of condensed group cohomology with solid-nuclear coefficients.

1.3.5 - Lemma. *Let G be a profinite group. Let $K_0 \subset K$ be a p-adic local subfield. For any $V \in \mathrm{Mod}_{K_0}^{\mathrm{nuc}}$ with trivial $\underline{G}$-action and any $B \in \mathrm{Mod}_{K_0}^{\mathrm{nuc}} \cap \mathrm{Mod}_{K_0[G]}^{\mathrm{cond}}$, there is a natural isomorphism*

$$V \otimes_{K_0}^{\blacksquare} R\Gamma(\underline{\mathscr{G}_K},B) \xrightarrow{\simeq} R\Gamma(\underline{\mathscr{G}_K}, V \otimes_{K_0}^{\blacksquare} B).$$

We remark that this remains true in a more genral setting: we still have this isomorphism for solid-nuclear $V \in \mathscr{D}(\mathrm{Mod}_{K_0}^{\blacksquare})$, which are characterised by the property $H^i(V) \in \mathrm{Mod}_{K_0}^{\mathrm{nuc}}$ for all $i \in \mathbf{Z}$, and for general $B \in \mathscr{D}(\mathrm{Mod}_{K_0[G]}^{\blacksquare})$ [11, Proposition 13.14][6].

Proof. Since V is flat for $\otimes_F^{\blacksquare}$, it suffices to see that the natural morphism suffit de voir que le morphisme naturel

$$V \otimes_F^{\blacksquare} \underline{\mathrm{Hom}}(\mathbf{Z}[G^\bullet],B) \to \underline{\mathrm{Hom}}(\mathbf{Z}[G^\bullet], V \otimes_F^{\blacksquare} B)$$

is an isomorphism. According to [8, Theorem A.43 (i)], $V \otimes_F^{\blacksquare} W$ is still solid-nuclear, so by characterisation of nuclear objects as trace-class functors [8, Proposition A.55 (i)], we have both natural vertical isomorphisms in the following commutative diagram

$$\begin{tikzcd}
V \otimes_F^{\blacksquare} \underline{\mathrm{Hom}}(\mathbf{Z}[\mathscr{G}_K^\bullet],B) \arrow[r] & \underline{\mathrm{Hom}}(\mathbf{Z}[\mathscr{G}_K^\bullet], V \otimes_F^{\blacksquare} B) \\
V \otimes_F^{\blacksquare} \underline{\mathrm{Hom}}(\mathbf{Z}[\mathscr{G}_K^\bullet],F) \otimes_F^{\blacksquare} B \arrow[u, "\simeq"] \arrow[r, "\simeq"] & \underline{\mathrm{Hom}}(\mathbf{Z}[\mathscr{G}_K^\bullet],F) \otimes_F^{\blacksquare} V \otimes_F^{\blacksquare} B. \arrow[u, "\simeq"']
\end{tikzcd}$$

[6]The proof of *loc. cit.* actually works for $S = \mathscr{G}_K$ by the compact projectivity of $\mathbf{Z}[\mathscr{G}_K]^\bullet$.

Now, we can conclude. □

In the particular case of profinite groups $\iota : G \simeq \mathbf{Z}_p^n$ (often non canonically isomorphic), its condensed group cohomology can be computed alternatively by the Koszul complex, determined naturally though not canonically by the isomorphism ι.

1.3.6. Let $M \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ and $f_1, \ldots, f_n \in \mathrm{End}_{\mathcal{C}}\,\mathrm{ond}\mathcal{A}\mathrm{b}(M)$. Its *Koszul complex* is the complex

$$\mathrm{Kosz}_M(f_1, \ldots, f_n) := \mathrm{Tot}\left(M \otimes_{f,\mathbf{Z}[X_1,\ldots,X_n]} \bigotimes_{i=1}^{n} (\mathbf{Z}[X_1, \ldots, X_n] \xrightarrow{\cdot X_i} \mathbf{Z}[X_1, \ldots, X_n])\right)$$

with first R sitting in degree 0 and the second sitting in degree 1, where M is regarded as a $\mathbf{Z}[X_1, \ldots, X_n]$-module via the map $\mathbf{Z}[X_1, \ldots, X_n] \to \mathrm{End}_{\mathcal{C}}\,\mathrm{ond}\mathcal{A}\mathrm{b}(M), X_i \mapsto f_i$.

1.3.7 - Proposition. *Let G be a profinite group with an isomorphism $\iota : G \simeq \mathbf{Z}_p^n$, and let $\gamma_1, \ldots, \gamma_n$ denote the associated canonical (topological) generators of G, transported via i from those of $\mathbf{Z}_p^n$. For any $M \in \mathscr{D}(\mathrm{Mod}_{\mathbf{Z}_p}^{\blacksquare})^{\underline{G}}$, the condensed group cohomology is computed by the following Koszul complex*

(1.3.7.1) $$R\Gamma(\underline{G}, M) \simeq \mathrm{Kosz}_M(\gamma_1 - 1, \ldots, \gamma_n - 1).$$

The identification depends on ι, and is compatible with changing the isomorphism ι (which amounts to changing generators).

Proof. We have a projective resolution $\mathrm{Tot}\left(\bigotimes_{\mathbf{Z}_p\,i=1}^{\blacksquare\;n} (\mathbf{Z}_p[G]^{\blacksquare} \xrightarrow{\cdot[\gamma_i]-\mathrm{id}} \mathbf{Z}_p[G]^{\blacksquare})\right) \to \mathbf{Z}_p \to 0$ in $\mathrm{Mod}_{\mathbf{Z}_p}^{\blacksquare}$, where the first $\mathbf{Z}_p[G]^{\blacksquare}$ sits in the degree -1 and the second sits in the degree 0. Taking $R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathbf{Z}_p[G]}^{\blacksquare}}(-, M)$, using that $R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathbf{Z}_p[G]}^{\blacksquare}}(\mathbf{Z}_p[G]^{\blacksquare}, -) \simeq R\,\underline{\mathrm{Hom}}_{\mathbf{Z}_p[G]}(\mathbf{Z}_p[G], -) \simeq \mathrm{id}$ on $\mathrm{Mod}_{\mathbf{Z}_p[G]}^{\blacksquare}$, we obtain the statement. □

We have the following relation between the two identifications.

1.3.8 - Lemma. *Let $\Gamma \simeq \mathbf{Z}_p^n$ be a profinite group with associated generator $\gamma_1, \ldots, \gamma_n$, and $M \in \mathrm{Mod}_{\mathbf{Z}_p[\Gamma]}^{\blacksquare}$. Then the composite identification $H^1\,\underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma^\bullet], M) \simeq H^1(\underline{\Gamma}, M) \simeq H^1\mathrm{Kosz}_M(\gamma_1 - 1, \ldots, \gamma_n - 1)$ agrees with the map $(\gamma_1^*, \ldots, \gamma_n^*) : \underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma], M) \to M^{\oplus n}$ restricted to cocycles, where γ_i^* is induced by $\mathbf{Z}[*] \to \mathbf{Z}[\Gamma]$ sending the point to γ_i. More precisely, evaluated at a point, the class of a cocycle $\bar{c}$ gets identified with the class of the element $m_{\bar{c}} := (\bar{c}(\gamma_1), \ldots, \bar{c}(\gamma_d))$.*

Proof. Since $M(S) \simeq \underline{\mathrm{Hom}}(\mathbf{Z}[S], M)(*)$ for any profinite set S, we only need to prove the statement evaluated at a point.

Denote by Γ' a copy of Γ for the Koszul construction, and F (read as "digamma") another copy for the continuous cochain complex construction. We have quasi-isomorphisms of genuine complexes

$$\begin{array}{ccc} \mathrm{Kosz}_{\mathbf{Z}_p[\Gamma']^{\blacksquare}}(\gamma_1 - 1, \ldots, \gamma_n - 1) & \xrightarrow{\simeq} & \mathbf{Z}_p \\ \uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} \\ \mathrm{Tot}\left(\mathrm{Kosz}_{\mathbf{Z}_p[\Gamma' \times F^{\bullet+1}]^{\blacksquare}}(\gamma_1 - 1, \ldots, \gamma_n - 1)\right) & \xrightarrow{\simeq} & \mathbf{Z}[F^{\bullet+1}]^{\blacksquare} \end{array}$$

in $\mathrm{Mod}_{\mathbf{Z}_p[\Gamma]}^{\blacksquare}$, inducing after $R\,\underline{\mathrm{Hom}}_{\mathrm{Mod}_{\mathbf{Z}_p[\Gamma]}^{\blacksquare}}(-, M)$ the quasi-isomorphisms genuine complexes

(1.3.8.1) $$\begin{array}{ccc} \mathrm{Kosz}_M(\gamma_1 - 1, \ldots, \gamma_n - 1) & \xleftarrow{\quad\simeq\quad} & R\Gamma(\underline{\Gamma}, M) \\ {\scriptstyle\alpha'}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ \mathrm{Tot}\left(\mathrm{Kosz}_{\underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma' \times F^{\bullet+1}], M)}(\gamma_1' - 1, \ldots, \gamma_n' - 1)\right) & \xleftarrow[\simeq]{\alpha''} & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[F^{\bullet+1}], M) \end{array}$$

in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p}$. We are going to show that $\alpha'(m_{\bar{c}}) - \alpha''(\bar{c})$ is a coboundary in the total complex

Consider the following upper right pieces

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'],M)^{\oplus n} \arrow[d, "\alpha':=p'^*"] & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'],M) \arrow[l, "(\gamma_i^*-1)_{i=1}^n"'] \arrow[d, "p'^*"] & \\
\underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'\times F],M)^{\oplus n} & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'\times F],M) \arrow[l, "d':=(\gamma_i'^*-1)_{i=1}^n"'] \arrow[d, "d''=p_1^*-p_0^*"] & \\
\iddots & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma'\times F^2],M) & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[F^2],M) \arrow[l, "\alpha'':=p''^*"'] \arrow[d, "\delta"] \\
& & \underline{\mathrm{Hom}}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[F^3],M)
\end{tikzcd} \tag{1.3.8.2}$$

of (1.3.8.1), or equivalently

$$\begin{tikzcd}
M^{\oplus n} \arrow[d, "\alpha':=\mathrm{const}"] & M \arrow[l, "(\gamma_i-1)_{i=1}^n"'] \arrow[d, "\mathrm{const}"] & \\
M(F)^{\oplus n} & M(F) \arrow[l, "d':=(\gamma_i\circ\gamma_i^{-1^*}-1)_{i=1}^n"'] \arrow[d, "d''=p_1^*-p_0^*"] & \\
\iddots & M(F^2) & M(F) \arrow[l, "\alpha''(\bar{c})=c"'] \arrow[d, "\delta"] \arrow[ul, dashed, bend right, "\mathrm{id}"'] \\
& & M(F^2)
\end{tikzcd} \tag{1.3.8.3}$$

if we kill the Γ-action by reducing the respective first components to the unit $1 \in \Gamma$. We write with a bar $\bar{f} \in M(\Gamma^\bullet)$ of the de-Γ-equivaried function of a Γ-equivariant function $f \in \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[\Gamma^{\bullet+1}], M)$.

Consider the identity map of $M(F)$ fitting as the dashed arrow into the diagram (1.3.8.3). It is a direct computation to check that

$$\alpha'' = d'' + \delta$$

on $M(F)$, and

$$d'\bar{f}(g) = (-\bar{f}(\gamma_i) + \delta\bar{f}(\gamma_i, \gamma_i^{-1}g))_{i=1}^n$$

for $\bar{f} \in M(F)$. In particular, for any cocycle $\bar{c} \in Z^1 M(F^\bullet) = \ker\delta \subset M(F)$, if we view it in degree 0 (upper right corner) of the total complex and apply the differential $d = d' + d''$, we obtain

$$d\bar{c} = \left(-(\mathrm{const}_{\bar{c}(\gamma_i)})_{i=1}^n + (\delta\bar{c}(\gamma_i, \gamma_i^{-1}g))_{i=1}^n, \alpha''(\bar{c}) - \delta\bar{c}\right) = (-\alpha'(m_{\bar{c}}), \alpha''(\bar{c})) \in M(F)^{\oplus n} \oplus M(F^2),$$

thus proving the identification. □

1.3.9 - Remark. In fact, conversely, for any $m = (m_1, \ldots, m_n) \in M(*)^{\oplus n}$ such that $(\gamma_i - 1)m_j = (\gamma_j - 1)m_i$, there exists a cocycle $\bar{c}_m$ such that $\bar{c}(\gamma_i) = m_i$ for $i = 1, \ldots, n$. One needs to prove certain explicit formula defines really an element $\bar{c} \in M(\Gamma)$. For this, using compact projective generation of $\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p[G]}$, one can reduce to the case where M is of the form $M = \mathbf{Z}_p[\Gamma]^{\blacksquare} \otimes^{\blacksquare}_{\mathbf{Z}} \mathbf{Z}[S]^{\blacksquare} \simeq \mathbf{Z}_p[\Gamma \times S]^{\blacksquare}$, in which the verification is direct. Alternatively, admitting the identification of the lemma, one can argue that there exists an element $m_0 \in M(*)$ such that $m + dm_0$ comes from a cocycle $\bar{c}$, i.e. $m_i + \gamma_i m_0 - m_0 = \bar{c}(\gamma_i)$ for $i = 1, \ldots, d$, but now $m_i = (\bar{c} - \delta m_0)(\gamma_i)$ where δ is the zeroth differential of the cochain complex $M(\Gamma^\bullet)$.

1.3.10. Galois cohomology. Now we study examples of condensed Galois cohomology, whose computation has essentially been done in the classical continuous cohomology context. We record these results and interpretate them in the condensed language.

Let $M \in D(\mathrm{Mod}_K^{\mathrm{cond}})$. Consider $C = \widehat{\overline{K}}$ with its natural continuous $\mathscr{G}_K$-action as an object of $D(\mathrm{Mod}_K^{\mathrm{cond}})^{\underline{\mathscr{G}_K}}$ concentrated in degree 0. We have a canonical morphism $M \to M \otimes_K^L C$ in $D(\mathrm{Mod}_K^{\mathrm{cond}})^{\underline{\mathscr{G}_K}}$, inducing the morphism $D(\mathrm{Mod}_K^{\mathrm{cond}})$

$$M \to R\Gamma(\underline{\mathscr{G}_K}, M \otimes_K^L C) = (M \otimes_K^L C)^{\underline{\mathscr{G}_K}}$$

functorial in M. We may also replace C by other objects dans $D(\mathrm{Mod}_K^{\mathrm{cond}})^{\underline{\mathscr{G}_K}}$.

Here is a solid variant. The advantage of working in the solid world $D(\mathrm{Mod}_K^{\blacksquare})$ is that C becomes now a flat object for $\otimes_K^{\blacksquare}$, whence the natural isomorphism $M \otimes_K^{L\blacksquare} C \simeq M \otimes_K^{\blacksquare} C$. Then we get the natural transformation

$$D(\mathrm{Mod}_K^{\blacksquare}) \to D(\mathrm{Mod}_K^{\blacksquare}), \quad - \to (- \otimes_K^{\blacksquare} C)^{\underline{\mathscr{G}_K}}.$$

Here is a noncompleted version. The object $\overline{K}$ is flat already for nonsolid $\otimes_K$, hence $M \otimes_K^L \overline{K} \simeq M \otimes_K \overline{K}$ and we get the natural transformation

$$D(\mathrm{Mod}_K^{\mathrm{cond}}) \to D(\mathrm{Mod}_K^{\mathrm{cond}}), \quad - \to (- \otimes_K \overline{K})^{\underline{\mathscr{G}_K}}.$$

1.3.11 - Example. Let's review some examples of Galois cohomology. Let $i \in \mathbf{N}, j \in \mathbf{Z}$.

(i) Tate calculated

$$H^i_{\mathrm{cont}}(\mathscr{G}_K, C(j)) \simeq \begin{cases} K, & j = 0, i = 0, \\ K \log \chi_{\mathrm{cyc}}, & j = 0, i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

from which we deduce the same formula for $H^i_{\mathrm{cont}}(\mathscr{G}_K, B^+_{\mathrm{dR}}(j))$. There is a generalised version calculating $H^i(\underline{\mathscr{G}_K}, W \otimes_K^{\blacksquare} C(j))$ for $W \in \mathrm{Mod}_K^{\mathrm{cond}}$ a K-Banach space, a (classical) nuclear Fréchet K-vector space, or a K-vector space of compact type equipped with a trivial $\mathscr{G}_K$-action.

(ii) By adding a variable $\log t$ with action $g(\log t) = \log t + \log(\chi_{\mathrm{cyc}}(\sigma))$, one obtains the computation [?]

$$H^i(\underline{\mathscr{G}_K}, W \otimes_K^{\blacksquare} C[\log t](j)) \simeq \begin{cases} W, & j = 0, i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

for $W \in \mathrm{Mod}_K^{\mathrm{cond}}$ a K-Banach space or a (classical) nuclear Fréchet K-vector space with trivial $\mathscr{G}_K$-action; similarly, one has

$$H^i(\underline{\mathscr{G}_K}, W \otimes_K^{\blacksquare} B^+_{\mathrm{dR}}[\log t](j)) \simeq H^i(\underline{\mathscr{G}_K}, W \otimes_K^{\blacksquare} B_{\mathrm{dR}}[\log t](j)) \simeq \begin{cases} W, & j = 0, i = 0, \\ 0, & \text{otherwise,} \end{cases}$$

for the same $W \in \mathrm{Mod}_K^{\mathrm{cond}}$.

We denote

$$B^+_{\mathrm{pdR}} := B^+_{\mathrm{dR}}[\log t], \quad B_{\mathrm{pdR}} := B_{\mathrm{dR}}[\log t] \simeq B^+_{\mathrm{pdR}} \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$$

with $g(\log t) = \log t + \log \chi_{\mathrm{cyc}}(g)$.

1.3.12. Discrete condensed objects. For $M \in \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$, we denote by $M^\delta := \underline{M(*)_{\mathrm{disc}}}$, or equivalently $M^\delta(S) := \varinjlim_{S \twoheadrightarrow S_i} M(S_i)$. There is a natural continuous map $M(*)_{\mathrm{disc}} \to M(*)_{\mathrm{top}}$ underlying the morphism $M^\delta \to M$; we say that M is *discrete* if it is an isomorphism. The functor $(-)^\delta : \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et}) \to \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$ preserves all limits and colimits.

1.3.13 - Definition. Let S be a profinite set. Let $M \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$. We define

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(S, M) = \underline{\mathrm{Hom}}(S, \mathbf{Z}) \otimes_{\mathbf{Z}} M \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b} \tag{1.3.13.1}$$

as the *condensed group of locally constant functions from S to M*. We denote $\mathrm{Hom}^{\mathrm{sm}}(S,M) := \underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M)(*)$.

1.3.14 - Lemma. *Let S be a profinite set and $M \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$.*

(i) There is a natural map $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M) \to \underline{\mathrm{Hom}}(S,M)$.

(ii) We have $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M) = \varinjlim_i M^{S_i}$, in particular for any extremally disconnected set T, we have $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M)(T) = \varinjlim_i \mathrm{Hom}(S_i, M(T))$; in other words, we have $\mathrm{Hom}^{\mathrm{sm}}(S,M) = M^{\delta}(S)$, or more generally, $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M)(T) = \underline{\mathrm{Hom}}(T,M)^{\delta}(S)$. The map in (i) corresponds to the colimit of natural maps $\mathrm{Hom}(S_i, M(T)) = M(S_i \times T) \to M(S \times T)$; in particular, it is injective.

(iii) If M is discrete, then $\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(S,M)$.

Proof. (i) The natural map is clear by remarking that $M = \underline{\mathrm{Hom}}(\mathbf{Z}, M)$.

(ii) For extremally disconnected sets T, we have $\underline{\mathrm{Hom}}(S,\mathbf{Z})(T) = \mathbf{Z}(S \times T) = \varinjlim_{i,j} \mathbf{Z}(S_i \times T_j)$, hence

$$\underline{\mathrm{Hom}}(S,\mathbf{Z}) = \varinjlim_i \mathbf{Z}^{S_i}$$

and by tensoring with M, we obtain

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M) = \varinjlim_i \mathbf{Z}^{S_i} \otimes_{\mathbf{Z}} M = \varinjlim_i M^{S_i}.$$

Evalued on extremally totally disconnected sets T, this is

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(S,M)(T) = \varinjlim_i M^{S_i}(T) = \varinjlim_i \mathrm{Hom}(S_i, M(T)).$$

We know that $\underline{\mathrm{Hom}}(S,M)(T) = M(S \times T)$, hence there are maps $M(S_i \times T) \to M(S \times T)$, and the above map is identified its colimit.

(iii) If M is discrete, then $M(S \times T) = \varinjlim_{i,j} M(S_i \times T_j) = \varinjlim_i M(S_i \times T)$. $\square$

1.3.15. Smooth group action. Let G be a profinite group. Let M be an object of $\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$. We say that the $\underline{G}$-action on M is *smooth* if the map $M \to \underline{\mathrm{Hom}}(G,M)$ factors through

$$M \to \underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M) \hookrightarrow \underline{\mathrm{Hom}}(G,M) \tag{1.3.15.1}$$

where the injectivity is due to (1.3.14, ii). We denote by $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ the full subcategory of $\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$ consisting of condensed abelian groups with smooth $\underline{G}$-action and $\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) := \mathscr{D}(\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]})$. In the solid situation, we denote $\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]} := \mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]} \cap \mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ and $\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}) := \mathscr{D}(\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]})$. The categories $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ and $\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]}$ are Grothendieck abelian categories with generators $\mathbf{Z}[S \times G/H]$ for profinite sets S and open subgroups $H \leq G$.

By design, $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]} \subset \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$ is stable under all colimits and solidification. Indeed, it is clear from (1.3.15.1) and (1.3.14, ii). Deriving it, we obtain a strictly commutative diagram of functors

$$\begin{tikzcd} \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \arrow[r] \arrow[d, "(-)^{L\blacksquare}"] & \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \arrow[d, "(-)^{L\blacksquare}"] \\ \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}) \arrow[r] & \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}). \end{tikzcd} \tag{1.3.15.2}$$

1.3.16 - Lemma. *Let G be a profinite group. The fully faithful embedding functor $\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]} \subset \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$ has a right adjoint, which is given by*

$$M \mapsto M^{G-\mathrm{sm}} := M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M). \tag{1.3.16.1}$$

More concretely, for any profinite set S, the set $M^{G-\mathrm{sm}}(S)$ consists of $m \in M(S)$ such that its orbit $\mathrm{orb}_m \in M(G \times S)$ *comes from the subset $M(G_i \times S)$ for some finite quotient G_i of G.*

Proof. The natural projection map $\iota : M^{G-\mathrm{sm}} \to M$ is injective by (1.3.14, ii). The right translation action $r : G \times G \to G, (g_1, g_2) \mapsto g_2 g_1$ induces a $\underline{G}$-action on $\underline{\mathrm{Hom}}(G, M)$ and on $\underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M)$ compatible with the G-action on M; indeed, for the latter, we need to check that the subobject $\underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \subset \underline{\mathrm{Hom}}(G, M)$ is stable under this G-action, but this follows from the expression (1.3.13.1). As a result, there is a natural induced G-action on $M^{G-\mathrm{sm}}$ making ι a $\underline{G}$-equivariant map. Now let us show that the $\underline{G}$-action on M is smooth. For this, the coaction map $M^{G-\mathrm{sm}} \to \underline{\mathrm{Hom}}(G, M^{G-\mathrm{sm}})$ lands in $\underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \cap \underline{\mathrm{Hom}}(G, M^{G-\mathrm{sm}}) = \underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M^{G-\mathrm{sm}})$.

Finally about the adjunction statement: for any $N_1 \in \mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}$ and $N_2 \in \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, any $\underline{G}$-equivariant map $N_1 \to N_2$ factors uniquely as $N_1 \to N_2^{G-\mathrm{sm}} \subset N_2$ by factorisation property in definition (1.3.15) and the above injectivity. □

1.3.17 - Lemma. *Let G be a profinite group. For any $M \in \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, we have*

$$M^{G-\mathrm{sm}} \simeq \varinjlim_H H^0(\underline{H}, M)$$

where H runs over the filtered system of all open normal subgroups of G.

Proof. Let H be an open normal subgroup of G acting on M via restriction $\rho_H : M \to \underline{\mathrm{Hom}}(H, M)$. We have $\underline{\mathrm{Hom}}(G/H, M) \simeq \ker(m^* - \mathrm{pr}_1^* : \underline{\mathrm{Hom}}(G, M) \to \underline{\mathrm{Hom}}(G \times H, M))$ over $\underline{\mathrm{Hom}}(G, M)$, induced from the colimt diagram $\mathrm{colim}(m, \mathrm{pr}_1 : G \times H \to G) \xrightarrow{\simeq} G/H$ with compatible maps from G (e.g. $(\mathrm{id}, e_G) : G \to G \times H$), hence

$$M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}(G/H, M) \simeq \ker(\rho_H - \mathrm{const} : M \to \underline{\mathrm{Hom}}(H, M)) \simeq H^0(\underline{H}, M).$$

Here, we used that

$$m^*, \mathrm{pr}_1^* : M \simeq M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}(G, M) \to M \times_{\underline{\mathrm{Hom}}(G,M)} \underline{\mathrm{Hom}}(G \times H, M) \simeq \underline{\mathrm{Hom}}(\frac{G \times H}{G \times \{e_G\}}, M)$$

where $\frac{G \times H}{G \times \{e_G\}}$ is the quotient topological space, and that they both factors through the subobject $\underline{\mathrm{Hom}}(H, M)$. We conclude by taking colimits over H (1.3.16.1). □

1.3.18. Let $(-)^{RG-\mathrm{sm}} : \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \to \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]})$ denote the right derived functor of $(-)^{G-\mathrm{sm}}$. Then (1.3.17) implies that

$$M^{RG-\mathrm{sm}} \simeq \varinjlim_H R\Gamma(\underline{H}, M)$$

for $M \in \mathscr{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]})$, where H runs over the filtered system of all open normal subgroups of G.

Taking right adjoints of the commutative diagram (1.3.15.2), we find that $(-)^{G-\mathrm{sm}}$ and $(-)^{RG-\mathrm{sm}}$ restrict to functors on corresponding solid objects.

1.3.19 - Lemma. *Let G be a profinite group and $M \in \mathcal{C}ond\mathcal{A}b$. Let G act trivially on M and by right translation on G. Then we have a natural isomorphism*

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \simeq \underline{\mathrm{Hom}}(G, M)^{G-\mathrm{sm}}.$$

Proof. For T profinite set, the map $\underline{\mathrm{Hom}}(G, M) \to \underline{\mathrm{Hom}}(G, \underline{\mathrm{Hom}}(G, M)) \simeq \underline{\mathrm{Hom}}(G \times G, M)$ is induced by the multiplication map $m : G^{(1)} \times G^{(2)} \to G^{(2)}, (g_1, g_2) \mapsto g_2 g_1$, where $G^{(1)}$ and $G^{(2)}$ are two copies of G. We have for extremally disonnected

$$\underline{\mathrm{Hom}}^{\mathrm{sm}}(G^{(1)}, \underline{\mathrm{Hom}}(G^{(2)}, M))(T) = \varinjlim_i \mathrm{Hom}(G_i^{(1)}, \underline{\mathrm{Hom}}(G^{(2)}, M)(T)) = \varinjlim_i M(G_i^{(1)} \times G^{(2)} \times T),$$

so

$$\begin{aligned}\underline{\mathrm{Hom}}(G,M)^{G-\mathrm{sm}}(T) &= M(G^{(2)}\times T)\times_{M(G^{(1)}\times G^{(2)}\times T)} \varinjlim_i M(G_i^{(1)}\times G^{(2)}\times T)\\ &\simeq \varinjlim_i M(G_i^{(2)}\times T)\\ &= \underline{\mathrm{Hom}}^{\mathrm{sm}}(G^{(2)},M)(T).\end{aligned}$$

Here, for the second to last identification, we used the following commutative diagram

$$\begin{tikzcd} M(G^{(2)}\times_{G_i} G^{(2)}\times T) \arrow[r, hook, "m^*"] & M(G^{(2)}\times(G^{(1)}\times_{G_i} G^{(1)})\times T) \\ M(G^{(2)}\times T) \arrow[u, "\mathrm{pr}_1^*-\mathrm{pr}_2^*"] \arrow[r, hook, "m^*"] & M(G^{(2)}\times G^{(1)}\times T) \arrow[u, "\mathrm{pr}_1^*-\mathrm{pr}_2^*"'] \\ M(G_i^{(2)}\times T) \arrow[u, hook] \arrow[r, hook, "m^*"] & M(G^{(2)}\times G_i^{(1)}\times T) \arrow[u, hook] \end{tikzcd}$$

with exact columns, to get a unique map $M(G^{(2)}\times T)\times_{M(G^{(1)}\times G^{(2)}\times T)} M(G_i^{(1)}\times G^{(2)}\times T)\to M(G_i^{(2)}\times T)$ making the diagram commutative, whence it is an isomorphism. □

1.3.20. Smooth group cohomology. Let G be a profinite group. We define the *smooth group cohomology* functor of G as the right derived functor $R\Gamma_{\mathrm{sm}}(G,-):\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}_{\mathbf{Z}[G]})\to\mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})$ of $\underline{\mathrm{Hom}}_{\mathbf{Z}[G]}(\mathbf{Z},M)$, which is right adjoint to the trivial action functor; similarly in the solid situation, we define by abuse of notation $R\Gamma_{\mathrm{sm}}(G,-):\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]})\to\mathscr{D}(\mathcal{S}\mathrm{olid})$.

For $M\in\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]})$ with image M' in $\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]})$, there is no ambiguity, since $R\Gamma_{\mathrm{sm}}(G,M)$ has image $R\Gamma_{\mathrm{sm}}(G,M')$ in $\mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})$. Indeed, this is because we have a strict commutative diagram of functors

$$\begin{tikzcd} \mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}) \arrow[r, "(-)^{\mathrm{triv}}"] \arrow[d, "(-)^{L\blacksquare}"] & \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}) \arrow[d, "(-)^{L\blacksquare}"] \\ \mathscr{D}(\mathcal{S}\mathrm{olid}) \arrow[r, "(-)^{\mathrm{triv}}"] & \mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]}) \end{tikzcd}$$

where the right solidification functor is well-defined by (1.3.15).

1.3.21. Let G be a profinite group. Consider the sequence of morphisms $\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}\xrightarrow{(-)^{\mathrm{triv}}}\mathrm{Mod}^{\mathrm{sm}}_{\mathbf{Z}[G]}\hookrightarrow\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Z}[G]}$, which preserve colimits and finite limits. Taking their right adjoints, which thus preserve injective objects, we obtain

$$R\Gamma(\underline{G},-)\simeq R\Gamma_{\mathrm{sm}}(\underline{G},(-)^{RG-\mathrm{sm}}).$$

1.3.22 - Lemma. *Let G be a profinite group. For $M\in\mathscr{D}^{\mathrm{sm}}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}[G]})$ and S profinite set, $R\Gamma_{\mathrm{sm}}(G,M)(S)\simeq R\Gamma(G,M(S))$ where the latter computes the usual profinite group cohomology of the smooth G-representation $M(S)$.*

In particular, if M has a $\underline{\mathbf{Q}}$-linear structure, then $R\Gamma_{\mathrm{sm}}(G,M)(S)$ is represented by $(M_S^\bullet)^G$ for whichever $\mathbf{Q}$-linear complex $M_S^\bullet$ representing $M(S)$.

Proof. The first statement is [41, Lemma 3.4.15] (cf. [17, Remark 4.28] for a classical but more restrictive explanation) and the last is due to vanishing of higher cohomology groups of profinite group cohomology of smooth representations over $\mathbf{Q}$. □

1.3.23 - Proposition. *Let G be a profinite group. For any $M\in\mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]}$, the smooth group cohomology is computed by the smooth cochain complex*

$$R\Gamma_{\mathrm{sm}}(\underline{G},M)\simeq\left(M\to\underline{\mathrm{Hom}}^{\mathrm{sm}}(G,M)\to\underline{\mathrm{Hom}}^{\mathrm{sm}}(G\times G,M)\to\cdots\right).$$

Proof. This is [41, Corollary 3.4.17]. □

1.3.24 - Corollary. *For $M \in \mathrm{Mod}^{\blacksquare,\mathrm{sm}}_{\mathbf{Z}[G]}$, we have*

$$H^0_{\mathrm{sm}}(\underline{G}, M) = H^0(\underline{G}, M).$$

Proof. This is due to (1.3.23) and the factorisation $M \to \underline{\mathrm{Hom}}^{\mathrm{sm}}(G, M) \hookrightarrow \underline{\mathrm{Hom}}(G, M)$ for $M \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ (1.3.15). □

The following lemma explains the relation between classical continuous and condensed group actions.

1.3.25 - Lemma. *Let G be a profinite group and $M \in \mathcal{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})$ be a discrete object.*

(i) Any condensed $\underline{G}$-action on M is smooth and is uniquely determined by the underlying G-action on $M(*)$, which is smooth.

(ii) Conversely, any smooth G-action on $M(*)$ extends to a condensed $\underline{G}$-action on M.

Proof. (i) The smoothness of the $\underline{G}$-action on M is clear from (1.3.14, iii).

We want to reconstruct natural $G(S)$-actions on $M(S)$ or equivalently

$$M(S) \to M(G \times S)$$

from the G-action on $M(*)$, which can be interpretated as $M(*) \to M(G) = \varinjlim_n M(G_n)$; hence every $m \in M(*)$ has open stabliser in G, so that the G-action on $M(*)$ is smooth. We know $M(S) = \varinjlim_i M(S_i)$ by discreteness. So

$$M(S) \to M(G \times S)$$

is identifies as

$$\varinjlim_i M(S_i) \to \varinjlim_{i,n} M(G_n \times S_i)$$

which agrees with

$$\varinjlim_i \left(M(S_i) \to \varinjlim_n M(G_n \times S_i) \right) = \varinjlim_i \left(M(*) \to \varinjlim_n M(G_n) \right)^{S_i}.$$

The map in the last parenthesis is the same as $M(*) \to M(G)$ by discreteness of M.

(ii) Conversely, if the action of G on $M(*)$ is smooth, then the action is described by a morphism $M(*) \to \varinjlim_n M(*)^{G_n} = \varinjlim_n M(G_n)$. Hence we may use the formula in the proof of (i) to define its (unique) extension to an $\underline{G}$-action on M. □

1.4 Condensed cohomology theories

As pointed out in (1.2.8), we need to put condensed structures on cohomology theories in p-adic geometry. We start with the p-adic (pro)étale cohomology.

1.4.1. (Pro)étale cohomology. Let X be an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. We define the proétale site of X as $X_{\mathrm{pro\acute{e}t}} := X^{\diamond}_{\mathrm{qpro\acute{e}t}}$ the quasi-proétale site of the diamond associated to X, with canonical projection $\nu : X_{\mathrm{qpro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ to the étale site, with associated morphisms of topos (ν^*, ν_*). We are interested in (pro)étale cohomology of (complexes of) sheaves of R-modules on X, where $R \in \{\mathbf{Z}/p^n, \mathbf{Z}_p, \mathbf{Q}_p\}$.

(i) For any $\mathcal{F} \in X^{\sim}_{\mathrm{\acute{e}t}}$, we have equivalences $\mathcal{F} \xrightarrow{\simeq} R\nu_*\nu^*\mathcal{F}$ [51, Proposition 14.8], hence $R\Gamma_{\mathrm{\acute{e}t}}(X, \mathcal{F}) \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \nu^*\mathcal{F})$, which verifies proétale hyperdescent.

(ii) For any étale $\mathbf{Z}_p$-local system $\mathbf{L} = (\mathbf{L}/p^n)_{n\in\mathbf{N}}$ on X with completion $\widehat{\mathbf{L}} := \lim_n \nu^*(\mathbf{L}/p^n)$ on $X_{\mathrm{pro\acute{e}t}}$, we define

$$R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbf{L}) := R\lim_n R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbf{L}/p^n).$$

Then we have

$$
(1.4.1.1)\qquad \begin{aligned} R\Gamma_{\text{ét}}(X,\mathbf{L}) &\simeq R\lim_n R\Gamma_{\text{proét}}(X,\nu^*(\mathbf{L}/p^n)) \\ &\simeq R\Gamma_{\text{proét}}(X,R\lim_n \nu^*(\mathbf{L}/p^n)) \xleftarrow{\simeq} R\Gamma_{\text{proét}}(X,\widehat{\mathbf{L}}), \end{aligned}
$$

where the second isomorphism follows from the fact that $R\lim_n$ and $R\Gamma_{\text{proét}}$ commute, the last isomorphism follows from the vanishing $R^i\lim_n \nu^*(\mathbf{L}/p^n)=0$; indeed, by [50, Lemma 3.18], we may reduce it to checking that for totally disconnected perfectoid spaces U over X, we have $R^1\lim_n H^0(U,\mathbf{Z}/p^n)=0$ and $H^i(U,\mathbf{Z}/p^n)=0$ for $i>0$, which are clear.

(iii)

1.4.2. Two condensed structures on proétale cohomology. Let X be an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)$ with canonical projection morphism $f_{\text{proét}}: X_{\text{proét}} \to *_{\text{proét}} = \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$ to the proétale site of a point.

We have two versions of condensed proétale cohomology on X:

(i) For $\mathcal{F}\in\mathscr{D}(\mathcal{S}\mathrm{hv}(X_{\text{proét}},\mathcal{A}\mathrm{b}))$, we may define the pushforward to the proétale of a point $\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F}) := Rf_{\text{proét}*}\mathcal{F}\in\mathscr{D}(\mathcal{S}\mathrm{hv}(*_{\text{proét}},\mathcal{A}\mathrm{b}))\simeq\mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})\simeq\mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(*_{\text{proét}},\mathscr{D}(\mathcal{A}\mathrm{b}))$; more precisely, we have

$$\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F})(S) = R\Gamma(X_{\text{proét}/X\times\underline{S}},\mathcal{F}) \simeq R\Gamma_{\text{proét}}(X\times\underline{S},\mathcal{F})$$

for any profinite set S.

(ii) We have $\mathcal{S}\mathrm{hv}(X_{\text{proét}},\mathcal{A}\mathrm{b}) \to \mathcal{S}\mathrm{hv}(X_{\text{proét}},\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}) \simeq \mathcal{S}\mathrm{hv}(X_{\text{proét}}\times *_{\text{proét}},\mathcal{A}\mathrm{b})$ sending $\mathcal{F}$ to $\underline{\mathcal{F}}: U\mapsto (S\mapsto F(U\times\underline{S}))$, which by (1.1.8, i) is further identified with the pushforward $\mu_*\mathcal{F}$ along the morphism of sites

$$\mu: X_{\text{proét}} \to X_{\text{proét}}\times *_{\text{proét}}.$$

Then, for $\mathcal{F}\in\mathcal{S}\mathrm{hv}(X_{\text{proét}},\mathcal{A}\mathrm{b})$, we define the object $R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}})\in\mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})\simeq\mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(*_{\text{proét}},\mathscr{D}(\mathcal{A}\mathrm{b}))$ as the global section with condensed coefficients; more precisely, we have

$$R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}})(S)\simeq R\Gamma((X_{\text{proét}}\times *_{\text{proét}})_{/X\times\underline{S}},\mu_*\mathcal{F})$$

for any profinite set S.

The first is very general, while the second restricts to static sheaves (i.e. sheaves concentrated in degree 0) due to lack of exactness of μ_*: the pushforward μ_* for sheaves is not necessarily exact, hence $R\mu_*$ does not degenerate.

Though not equivalent to each other, two points of view are related for $\mathcal{F}\in\mathcal{S}\mathrm{hv}(X_{\text{proét}},\mathcal{A}\mathrm{b})$. Since the composite morphism of sites $X_{\text{proét}}\xrightarrow{\mu} X_{\text{proét}}\times *_{\text{proét}} \xrightarrow{\mathrm{pr}_1} *_{\text{proét}}$ agrees with $f_{\text{proét}}$, we have $R\Gamma((X_{\text{proét}}\times *_{\text{proét}})_{/X\times\underline{S}},\mu_*\mathcal{F})\simeq\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F})$, whence an evident natural map $R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}})\to\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F})$, whose obstruction of being an isomorphism lies in $R^i\mu_*\mathcal{F}$ for $i>0$.

1.4.3. Disambiguation of two condensed structures. Let X be an analytic adic space over $\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)$. We give instances of $\mathcal{F}\in\mathcal{S}\mathrm{hv}(X_{\text{proét}},\mathcal{A}\mathrm{b})$ such that $R^i\mu_*\mathcal{F}=0$ for $i>0$, so that by (1.4.2), there is a natural equivalence

$$(1.4.3.1)\qquad R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}})\xrightarrow{\simeq}\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F})$$

in $\mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})$. In the following examples, we will actually prove that

$$(1.4.3.2)\qquad H^i_{\text{proét}}(U\times\underline{S},\mathcal{F})=0,\quad i>0$$

for sufficiently "small" strictly totally disconnected perfectoid spaces U over X and all extremally disconnected sets S; then $U \times \underline{S}$ is also a strictly totally disconnected perfectoid space [51, Lemma 7.19], i.e. being a quasi-compact perfectoid space whose all étale covers split.

(i) Let $\mathcal{F} = \nu^* F$ pulled back from an étale sheaf F on X. Then (1.4.3.2) holds as we have $H^i_{\text{proét}}(U \times \underline{S}, \mathcal{F}) \simeq H^i_{\text{ét}}(U, F) = 0$ if $i > 0$ by splitting of étale covers of U.

(ii) Let $\mathcal{F}$ be a proétale $\mathbf{Z}_p$-local system on X. Then (1.4.3.2) holds for sufficiently "small" strictly totally disconnected U, in the sense that $\mathcal{F}|_U \simeq \mathbf{Z}_p^n$ is trivialised. We may assume $\mathcal{F} = \widehat{\mathbf{Z}}_p := \lim_n \nu^*(\mathbf{Z}/p^n)$. Now $H^i_{\text{proét}}(U \times \underline{S}, \nu^*(\mathbf{Z}/p^n))$ vanishes if $i > 0$, and is, if $i = 0$, equal to $\text{Cont}(|U \times \underline{S}|, \mathbf{Z}/p^n)$ the set of locally constant functions on the underlying topological space $U \times \underline{S}$ with values in $\mathbf{Z}/p^n$. So $(H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Z}/p^n))_{n \in \mathbf{N}}$ forms a Mittag-Leffler system. Therefore,

$$H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Z}_p) \simeq \lim_n H^i_{\text{proét}}(U \times \underline{S}, \nu^*(\mathbf{Z}/p^n)) \simeq \begin{cases} \text{Cont}(|U \times S|, \mathbf{Z}_p), & i = 0, \\ 0, & i > 0, \end{cases} \tag{1.4.3.3}$$

where $\mathbf{Z}_p$ is endowed with its natural p-adic topology.

(iii) Let $\mathcal{F}$ be a proétale $\mathbf{Q}_p$-local system on X. Then (1.4.3.2) holds for sufficiently "small" strictly totally disconnected U, in the sense that $\mathcal{F}|_U \simeq \mathbf{Q}_p^n$ is trivialised. Recall that $\mathbf{Q}_p := \widehat{\mathbf{Z}}_p[\frac{1}{p}] = \text{colim}_{p\cdot} \widehat{\mathbf{Z}}_p$. Then $H^i_{\text{proét}}(U \times S, \mathbf{Q}_p) \simeq \text{colim}_{p\cdot} H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Q}_p)$ since proétale cohomology on qcqs spaces commutes with filtered colimits (by coherence of the topoi $(U \times \underline{S})^{\sim}_{\text{proét}} \simeq (U \times \underline{S})^{\diamond,\sim}_{\text{qproét,qcqs}}$). We conclude by (1.4.3.3) that

$$H^i_{\text{proét}}(U \times \underline{S}, \mathbf{Q}_p) \simeq \begin{cases} \text{Cont}(|U \times \underline{S}|, \mathbf{Q}_p), & i = 0, \\ 0, & i > 0. \end{cases} \tag{1.4.3.4}$$

(iv) Let $\mathcal{F} = \widehat{\mathcal{O}}_X$ or $\mathcal{F} \in \{\mathbf{B}_I, \mathbf{B}, \mathbf{B}_{\log}, \mathbf{B}_{\log}[\frac{1}{t}], \mathbf{B}^+_{\text{dR}}/\text{Fil}^m, \mathbf{B}^+_{\text{dR}}, \mathbf{B}_{\text{dR}}\}$ be a proétale period sheaf, then (1.4.3.2) holds for sufficiently "small" U, in the sense that U is an affinoid perfectoid space over $\text{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, by [8, Proposition 4.7] and [?, Proposition 2.37].

In conclusion, let $\mathcal{F} \in \mathcal{S}\text{hv}(X_{\text{proét}}, \mathcal{A}\text{b})$ be the pullback of an étale sheaf on X, be a proétale $\mathbf{Z}_p$- or $\mathbf{Q}_p$-local system, or belong to $\{\widehat{\mathcal{O}}_X, \mathbf{B}_I, \mathbf{B}, \mathbf{B}_{\log}, \mathbf{B}_{\log}[\frac{1}{t}], \mathbf{B}^+_{\text{dR}}/\text{Fil}^m, \mathbf{B}^+_{\text{dR}}, \mathbf{B}_{\text{dR}}\}$, then:

(v) The equivalence (1.4.3.1) holds.

(vi) We have $\underline{R\Gamma}_{\text{proét}}(U, \mathcal{F}) \in \mathcal{D}(\mathcal{S}\text{olid})$ for sufficiently "small" U, even that $\underline{R\Gamma}_{\text{proét}}(U, \mathcal{F})$ concentrated in degree 0 is a $\mathbf{Q}_p$-Banach space if $\mathcal{F} \in \{\mathbf{Q}_p^n, \widehat{\mathcal{O}}_X, \mathbf{B}_I, \mathbf{B}^+_{\text{dR}}/\text{Fil}^m\}$ and is a $\mathbf{Q}_p$-Fréchet space if $\mathcal{F} \in \{\mathbf{B}, \mathbf{B}^+_{\text{dR}}\}$.

(vii) By descent and stability of solidness under all limits and colimits, we obtain $\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \in \mathcal{D}(\mathcal{S}\text{olid})$; furthermore, $\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F})$ is represented by a complex of $\mathbf{Q}_p$-Banach spaces if $\mathcal{F}$ is a $\mathbf{Q}_p$-local system or $\mathcal{F} \in \{\mathbf{Q}_p^n, \mathbf{B}_I, \mathbf{B}^+_{\text{dR}}/\text{Fil}^m\}$, by a complex of $\mathbf{Q}_p$-Fréchet spaces if $\mathcal{F} \in \{\mathbf{B}, \mathbf{B}^+_{\text{dR}}\}$, whence by a complex of solid-nuclear $\mathbf{Q}_p$-vector spaces in both cases.

1.4.4. The étale cohomology of qcqs objects is discrete. Let $\mathcal{F} = \nu^* F$ pulled back from an étale sheaf F on X. Then by [51, Proposition 11.23, Proposition 14.9], for any profinite set $S = \varprojlim_i S_i$, we have $R\Gamma_{\text{proét}}(X \times \underline{S}, \nu^* F) \simeq \varinjlim_i R\Gamma_{\text{proét}}(X \times \underline{S_i}, \nu^* F) \simeq \varinjlim_i R\Gamma_{\text{proét}}(X, \nu^* F)^{S_i} \simeq \varinjlim_i R\Gamma_{\text{ét}}(X, F)^{S_i}$, hence

$$\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \simeq \underline{R\Gamma_{\text{ét}}(X, F)_{\text{disc}}}$$

is classic and discrete, i.e. being the condensification of $R\Gamma_{\text{ét}}(X, F) \in \mathcal{D}(\mathcal{A}\text{b})$ endowed with the discrete topology. We define $\underline{R\Gamma}_{\text{ét}}(X, F) := \underline{R\Gamma}_{\text{proét}}(X, \mathcal{F})$.

1.4.5 - Proposition (Hochschild-Serre spectral sequence). *Let G be a profinite group and $\widetilde{X} \to X$ a proétale $\underline{G}$-torsor of analytic adic spaces over* $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. *Let $\mathcal{F} \in \mathcal{S}\text{hv}(X_{\text{proét}}, \mathbf{Q}_p)$ such that $\mathcal{F}|_{\widetilde{X}}$ is $\underline{G}$-equivariant. Then we have a natural equivalence*

$$\underline{R\Gamma}_{\text{proét}}(X, \mathcal{F}) \simeq R\Gamma(\underline{G}, \underline{R\Gamma}_{\text{proét}}(\widetilde{X}, \mathcal{F}))$$

in $\mathcal{D}(\mathcal{S}\text{olid})$.

Proof. First, we have $\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F}) \simeq \lim_{\Delta} \underline{R\Gamma}_{\text{proét}}(\widetilde{X}\times\underline{G}^{\times\bullet},\mathcal{F})$ by proétale descent along the ech hypercovering of the $\underline{G}$-torsor and $\underline{G}$-equivariance of $\mathcal{F}|_{\widetilde{X}}$.

There is an isomorphism

$$\underline{R\Gamma}_{\text{proét}}(\widetilde{X}\times\underline{G}^{\times n},\mathcal{F}) \simeq R\,\underline{\mathrm{Hom}}(\mathbf{Z}[G^{\times n}],\underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F})).$$

Indeed, for $n \in \mathbf{N}$ and profinite sets S, there are isomorphisms

$$\begin{aligned}\underline{R\Gamma}_{\text{proét}}(\widetilde{X}\times\underline{G}^{\times n},\mathcal{F})(S) &= R\Gamma_{\text{proét}}(\widetilde{X}\times\underline{G}^{\times n}\times\underline{S},\mathcal{F})\\ &\simeq \underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F})(G^{\times n}\times S)\\ &\simeq R\,\mathrm{Hom}(\mathbf{Z}[G^{\times n}\times S],\underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F}))\\ &\simeq R\,\mathrm{Hom}(\mathbf{Z}[S],R\,\underline{\mathrm{Hom}}(\mathbf{Z}[G^{\times n}],\underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F})),\end{aligned}$$

where we used $\mathbf{Z}[G^{\times n}\times S] \simeq \mathbf{Z}[G^{\times n}]\otimes_{\mathbf{Z}}\mathbf{Z}[S] \simeq \mathbf{Z}[G^{\times n}]\otimes^L_{\mathbf{Z}}\mathbf{Z}[S]$ concentrated in degree 0 for the last isomorphism, $\mathbf{Z}[S]$ being flat, and the second to last follows from the definition of $\mathbf{Z}[-]$[7].

Next, recall that $\underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F}) \in \mathscr{D}(\mathcal{S}\text{olid})$ (1.4.3, vii), and that $\mathbf{Z}[S]^{L\blacksquare} \simeq Z[S]^{\blacksquare}[0]$ is compact projective in $\mathscr{D}(\mathcal{S}\text{olid})$ for any profinite set S (not only the extremally disconnected ones). But we have

$$R\,\underline{\mathrm{Hom}}(\mathbf{Z}[G^{\times n}],M) \simeq R\,\underline{\mathrm{Hom}}_{\mathscr{D}(\mathcal{S}\text{olid})}(\mathbf{Z}[G^{\times n}]^{L\blacksquare},M) \simeq \underline{\mathrm{Hom}}_{\mathscr{D}(\mathcal{S}\text{olid})}(\mathbf{Z}[G^{\times n}]^{\bullet},M) \simeq \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\times n}],M)$$

for $M \in \mathscr{D}(\mathcal{S}\text{olid})$, whence

$$R\,\underline{\mathrm{Hom}}(\mathbf{Z}[G^{\times n}],\underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F})) \simeq \underline{\mathrm{Hom}}(\mathbf{Z}[G^{\times n}],\underline{R\Gamma}_{\text{proét}}(\widetilde{X},\mathcal{F})).$$

Finally, it remains to identify the differentials in the condensed group cohomology and the total complex of the Čech cosimplicial nerve. □

1.4.6. Notation. We will often remove for simplicity the underline in the notation $\underline{R\Gamma}_{\text{proét}}(X,\mathcal{F})$ if the context permits, while keeping that in $R\Gamma_{\text{proét}}(X,\underline{\mathcal{F}})$. Since in most cases that we will encounter they even agree with each other, there will be no harm of it.

Next, we pass to log-crystalline cohomology, keeping in mind the intuition that étale cohomology of (p^n-torsion) sheaves ought to take discrete values on qcqs objects (1.4.4).

1.4.7. Condensed log-crystalline cohomology. Let us recall some basics on log-crystalline cohomology; for details, cf. [4, §1].

First, let $S^\sharp = (S,\mathcal{L},I,\gamma)$ be a quasi-coherent log pd-scheme with $p \in \mathcal{O}_S$ is nilpotent, and $(\mathscr{X},M_{\mathscr{X}})$ be an integral and quasi-coherent log-scheme over $S^\sharp$. Let $((\mathscr{X},M_{\mathscr{X}})/S^\sharp)_{\text{cris}}$ be the log-crystalline site of $(\mathscr{X},M_{\mathscr{X}})$ over $S^\sharp$, whose objects are $(\mathscr{U},T)$ with $\mathscr{U}$ is an étale scheme over $\mathscr{X}$ with pullback log-structure $M_{\mathscr{U}}$, and $T=(T,M_T)$ is a pd-$S^\sharp$-thickening of $(\mathscr{U},M_{\mathscr{U}})$ with defining pd-ideal $\mathcal{I}_Y \subset \mathcal{O}_T$, and whose coverings are étale ones. We may simplify the notation to $(\mathscr{X}/S^\sharp)_{\text{cris}}$ or even $(\mathscr{X}/S)_{\text{cris}}$ if the context permits. The site $((\mathscr{X},M_{\mathscr{X}})/S^\sharp)_{\text{cris}}$ has a structure sheaf $\mathcal{O}_{\mathscr{X}/S^\sharp} : (\mathscr{U},T) \mapsto \Gamma(T,\mathcal{O}_T)$, with pd-ideal sheaf $\mathcal{I}_{\mathscr{X}/S^\sharp} : (\mathscr{U},T) \mapsto \Gamma(T,\mathcal{J}_T)$; it has also has a sheaf $\mathbf{G}_m^\natural : (\mathscr{U},T) \mapsto \Gamma(T,(1+\mathcal{I}_T,\times))$ (note that $\mathcal{I}_T \subset \mathcal{O}_T$ is a nil-ideal), and a mononid sheaf $M_{\mathscr{X}/S^\sharp} : (\mathscr{U},T) \mapsto \Gamma(T,M_T)$. These sheaves take values in the category of abelian groups (*resp.* commutative monoids), which we regard as condensed abelian groups (*resp.* condensed commutative monoids) with discrete topology. By local discreteness of sheaves, many actions on them could be promoted to

[7]Let $M^\bullet$ be a complex in $\mathscr{D}(\mathcal{C}\text{ond}\mathcal{A}\text{b})$, the corresponding condensed object $M \in \mathcal{S}\text{hv}^{\text{hyp}}(*_{\text{proét}},\mathscr{D}(\mathcal{A}\text{b}))$ is given by $M(S) := R\Gamma(S,M^\bullet) \in \mathscr{D}(\mathcal{A}\text{b})$, which is represented by $M^\bullet(S)$ if S is extremally disconnected. By definition of free objects $\mathbf{Z}[-]$, we have $R\,\mathrm{Hom}(\mathbf{Z}[S],M^\bullet) \simeq R\,\mathrm{Hom}(\mathbf{Z}[S],I^\bullet) = I^\bullet(S)$ for any injective resolution $M^\bullet \xrightarrow{\simeq} I^\bullet$ in $\mathscr{D}(\mathcal{C}\text{ond}\mathcal{A}\text{b})$; but this is also the definition of $R\Gamma(S,M^\bullet)$. Therefore $R\,\mathrm{Hom}(\mathbf{Z}[S],M^\bullet) \simeq M(S)$.

a condensed action. Then we define the condensed log-crystalline cohomology as

$$R\Gamma_{\mathrm{cris}}((\mathscr{X}, M_{\mathscr{X}})/S^\sharp) := R\Gamma_{\mathrm{cris}}(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}, \mathcal{O}_{\mathscr{X}/S^\sharp}) \in \mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}).$$

The cocontinuous functor $((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}} \to \mathscr{X}_{\mathrm{\acute{e}t}}, (\mathscr{U}, T) \mapsto \mathscr{U}$ induces a canonical morphism of topoi $u^{\log}_{\mathscr{X}/S^\sharp} : ((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)^{\log,\sim}_{\mathrm{cris}} \to \mathscr{X}^\sim_{\mathrm{\acute{e}t}}$ such that

$$(u^{\log *}_{\mathscr{X}/S^\sharp}\mathcal{G})(\mathscr{U}, T) = \mathcal{G}(\mathscr{U}), \quad (u^{\log}_{\mathscr{X}/S^\sharp *}\mathcal{F})(\mathscr{U}) := \Gamma((\mathscr{U}/S^\sharp)_{\mathrm{cris}}, \mathcal{F}).$$

We may shorten the notation $u^{\log}_{\mathscr{X}/S^\sharp}$ to u if there is no confusion.

Now we turn to the p-adic setting. Let $S^\sharp = (S, \mathcal{L}, I)$ be an integral and quasi-coherent p-adic formal log pd-scheme, or samely a sequence of exact closed embeddings of integral and quasi-coherent log pd-schemes $(S^\sharp_n)_{n\in\mathbf{N}}$ such that $\mathcal{O}_{S_n} = \mathcal{O}_{S_{n+1}}/p^n$ and $\mathcal{I}_n = \mathcal{I}_{n+1}\mathcal{O}_{S_n}$. For $(\mathscr{X}, M_{\mathscr{X}})$ an integral and quasi-coherent log-$S^\sharp$-scheme, i.e. over $S^\sharp_n$ for some $n \in \mathbf{N}$, we define $((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)_{\mathrm{cris}}$ as the union of fully faithful embeddings of $(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp_n)_{\mathrm{cris}})_{n\in\mathbf{N}}$. We again have sheaves $\mathcal{O}_{\mathscr{X}/S^\sharp}$, $\mathcal{I}_{\mathscr{X}/S^\sharp}$, $\mathbf{G}^\natural_m$ and $M_{\mathscr{X}/S^\sharp}$ valued in condensed abelian groups (*resp.* condensed commutative monoids) with discrete topology. We define the condensed (p-adic) log-crystalline cohomology as

$$R\Gamma_{\mathrm{cris}}((\mathscr{X}, M_{\mathscr{X}})/S^\sharp) := \lim_{n\gg 0} R\Gamma_{\mathrm{cris}}(((\mathscr{X}, M_{\mathscr{X}})/S^\sharp_n)_{\mathrm{cris}}, \mathcal{O}_{\mathscr{X}/S^\sharp_n}) \in \mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}).$$

Again, we have a canonical morphism of topoi $u^{\log}_{\mathscr{X}/S^\sharp} : ((\mathscr{X}, M_{\mathscr{X}})/S^\sharp)^{\log,\sim}_{\mathrm{cris}} \to \mathscr{X}^\sim_{\mathrm{\acute{e}t}}$.

1.4.7.1. Condensed infinitesimal cohomology. We put natural condensed structures on the infinitesimal cohomologies defined in [31].

Let us first consider the arithmetic case. Let Z be a rigid space over K and consider the infinitesimal site $(Z/K)_{\mathrm{inf}}$ whose objects are pairs (U, T) such that U is an open subset of Z with a closed nil-immersion $U \hookrightarrow T$ of rigid spaces over K, and whose coverings are open coverings. It has a structure sheaf $\mathcal{O}_{Z/K} : (U, T) \mapsto \Gamma(T, \mathcal{O}_T)$ with an ideal sheaf $\mathcal{I}_{Z/K}$, naturally promoted to a sheaf valued in C_K but landing furthermore in the subcategory of K-Fréchet spaces, which we regard as in $\mathrm{Mod}^{\blacksquare}_K$. We define the arithmetic condensed infinitesimal cohomology as

$$R\Gamma_{\mathrm{inf}}(Z/K) := R\Gamma((Z/K)_{\mathrm{inf}}, \mathcal{O}_{X/K}) \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_K).$$

It agrees with the éh-de Rham cohomology, which we denote by

$$R\Gamma_{\mathrm{dR}}(Z/K) := R\Gamma(Z_{\mathrm{\acute{e}h}}, \Omega^\bullet_{Z/K,\mathrm{\acute{e}h}}) \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_K).$$

For qcqs Z, it is represented by a bounded complex of $\mathbf{Q}_p$-Fréchet spaces (*resp.* of $\mathbf{Q}_p$-Banach spaces if Z is smooth).

Similarly in the geometric B^+_{dR}-case, for any rigid space Z over C, we have the infinitesimal site $(X/B^+_{\mathrm{dR}})_{\mathrm{inf}} = \bigcup_m (X/B^+_{\mathrm{dR},m})_{\mathrm{inf}}$, equipped with a structure sheaf $\mathcal{O}_{X/B^+_{\mathrm{dR}}}$ together with an ideal sheaf $\mathcal{I}_{X/B^+_{\mathrm{dR}}}$. They are naturally made to take values in $\mathrm{Mod}^{\blacksquare}_{B^+_{\mathrm{dR}}}$, so that we can define the geometric condensed infinitesimal cohomology as

$$R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR},m}) := R\Gamma((X/B^+_{\mathrm{dR},m})_{\mathrm{inf}}, \mathcal{O}_{X/B^+_{\mathrm{dR}}}) \in \mathscr{D}(\mathrm{Mod}^{\blacksquare}_{B^+_{\mathrm{dR},m}})$$

and

$$R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) := \lim_n R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR},m}).$$

Most results in *op. cit.* can be upgraded to condensed statements.

2 Arithmetic syntomic cohomology

It is unclear whether the original arithmetic Hyodo-Kato morphism [17, §4.2] for rigid-analytic varieties is compatible with the geometric Hyodo-Kato morphism [19, §2]. Here, we prefer to make alternatively an *ad hoc* definition of the arithmetic Hyodo-Kato morphism so that it satisfies by birth this compatibility. The key tool is the Galois cohomology computation (1.3.11, ii).

2.0.1. Semistable formal schemes.

For p-adic admissible formal schemes over $\mathcal{O}_K$ or $\mathcal{O}_C$, we denote by $(-)_\eta$ their rigid generic fibre in the sense of adic spaces.

A (p-adic) formal scheme $\mathfrak{Z}$ over K is called ***strictly semistable*** if, locally for the Zariski topology, it admits an étale morphism to a formal scheme of the form

$$\operatorname{Spf}\mathcal{O}_K\langle X_0,\dots,X_m\rangle/(X_0\dots X_l-\varpi),\quad 0\le l\le m$$

for some uniformiser ϖ of $\mathcal{O}_K$. Let $L_{\mathfrak{Z}}$ be the integral closure of K in $\Gamma(\mathfrak{Z}_\eta,\mathcal{O}_{\mathfrak{Z}_\eta})$, called the ***splitting field*** of $\mathfrak{Z}_\eta$. If $\mathfrak{Z}$ is connected, then $L_{\mathfrak{Z}}$ is a field, and $\mathcal{O}_{L_{\mathfrak{Z}}}$ is the integral closure of $\mathcal{O}_K$ in $\Gamma(\mathfrak{Z}_\eta,\mathcal{O}_{\mathfrak{Z}_\eta})$, and $\mathfrak{Z}$ is strictly semistable over $\mathcal{O}_{L_{\mathfrak{Z}}}$; in general, $\mathfrak{Z}$ is locally connected and we regard $L_{\mathfrak{Z}}$ as attaching the splitting field to each connected component. Let $\mathcal{M}^{\mathrm{ss}}$ be the full subcategory of formal schemes over K that are semistable over some finite extension L of K.

A (p-adic) formal scheme $\mathfrak{X}$ over C is called ***strictly semistable*** if, locally for the Zariski topology, it admits an étale morphism to a formal scheme of the form

$$\operatorname{Spf}\mathcal{O}_C\langle X_0,\dots,X_m\rangle/(X_0\dots X_l-\varpi),\quad 0\le l\le m$$

for some $\varpi\in\mathcal{O}_C\backslash\{0\}$. We denote by $\mathcal{M}^{\mathrm{ss}}_C$ the full subcategory of formal schemes over K that are semistable over C. Let $\mathcal{M}^{\mathrm{ss},b}_C$ be the full subcategory of formal schemes over K that are base change from a semistable formal scheme over L for some finite extension of K.

The arithmetic and geometric setting are compatible by the base change functors

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{ss}}_K & \xrightarrow{(-)_\eta} & \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K \\ \big\downarrow{\scriptstyle (-)^{\mathrm{ss}}_{\mathcal{O}_C}} & & \big\downarrow{\scriptstyle -\otimes_K C} \\ \mathcal{M}^{\mathrm{ss},b}_C & \xrightarrow{(-)_\eta} & \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_C \end{array} \tag{2.0.1.1}$$

where the left vertical map is defined as[8]

$$\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C}:=\coprod_{\sigma\in\mathrm{Hom}_K(L_{\mathfrak{Z}},C)}\mathfrak{Z}\otimes_{\mathcal{O}_{L_{\mathfrak{Z}}},\sigma}\mathcal{O}_C.$$

There is a natural $\mathscr{G}_K$-action on $\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C}$ by permuting the components $\sigma\mapsto g^{-1}\sigma$ and meanwhile acting on the coefficients $\mathcal{O}_C$; this is compatible with the $\mathscr{G}_K$-action on Z_C. One may replace $L_{\mathfrak{Z}}$ by any finite extension L/K such that $\mathfrak{Z}$ is semistable over L and obtain the same object.

Exceptionnally in this article, we will abbreviate strictly semistable to ***semistable***.

2.0.2. We refer to [32, §2] for the notion of ***éh-topology***, and to [3, 2.1] for the notion of Beilinson base. By Temkin's altered local uniformization [53], we see that the pair $(\mathcal{M}^{\mathrm{ss}}_K,(-)_\eta)$ is a Beilinson base for the

[8] Beware that $\mathfrak{Z}_{\mathcal{O}_C}:=\mathfrak{Z}\otimes_{\mathcal{O}_K}\mathcal{O}_C$ is a formal model of Z_C, but is not semistable over $\mathcal{O}_C$ in general, hence $R\Gamma_{\mathrm{HK}}(Z_C)$ is not isomorphic to $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\mathcal{O}_C,1}/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p}$.

site $\mathcal{R}\mathrm{ig}_{K,\text{éh}}$, and $(\mathcal{M}_C^{\mathrm{ss},b},(-)_\eta)$ is a Beilinson base for the site $\mathcal{R}\mathrm{ig}_{C,\text{éh}}$ [9, Proposition 3.10]; indeed, by [17, Proposition 2.8] this is true for étale topology on smooth rigid spaces, but éh-locally we have smoothness [32, Corollary 2.4.8]. Therefore, we can unfold a presheaf on the base to a hypersheaf by hyperdescent. In order to obtain a reasonable sheaf, the presheaf should satisfy hyperdescent for sufficiently refined hypercoverings.

2.0.3 - Remark. When applying Temkin's results in [53], one should be a little bit cautious: Temkin's definition of semistability is slightly more general than the usual semistability. A formal scheme over $\mathcal{O}_K$ is called *strictly semistable* (*resp. semistable*) *à la Temkin* if locally for the Zariski (*resp.* étale) topology, it admits an étale map to the standard semistable formal scheme

$$\mathfrak{S}_{n,r} := \mathrm{Spf}\left(\mathcal{O}_K\{X_0,\dots,X_n\}/(X_0\cdots X_r-\varpi^n)\right), \quad 1\le r\le n,$$

over $\mathcal{O}_K$, where ϖ is a uniformizer of K and $n\in\mathbf{N}$. By contrast, the usual (strict) semistability requires $n=1$. These are different notions because the above standard semistable formal scheme is not regular if $n\geq 2$.

Nevertheless, when we work locally for the η-étale topology, we can always assume $n=1$ by further localisation. Indeed, we may proceed as in the proof of [53, Lemma 2.4.1], in particular its step 2: consider $f_i:\mathfrak{S}_{n,r}\to\mathfrak{S}_{n-1,r}$ induced by

$$X_j\mapsto\begin{cases}\pi X_i, & j=i,\\ X_j, & j\neq i.\end{cases}$$

Temkin's argument *loc. cit.* allows to show that $\{f_0,\dots,f_n\}$ form an η-étale covering.

2.0.4. Dagger varieties. Roughly speaking, a dagger rigid space (or dagger variety) over a non-archimedean field L is a rigid space over L together with overconvergent structure sheaves, namely $X=(\widehat{X},\mathcal{O}_X^\dagger)$; cf. [30] for basic definitions and properties. A presentation of a dagger affinoid rigid space U is a prosystem $(U_h)_{h\in\mathbf{N}}$ with U_h affinoid rigid spaces such that U and U_h are rational subspaces of U_0, that $U\subset^\dagger U_{h+1}\subset^\dagger U_h$. This system is coinitial among all rational subspaces of U_0 strictly containing X. The set of such presentations is non-empty and cofiltered. For any presentation $(U_h)_{h\in\mathbf{N}}$ of a dagger affinoid U, we have $\Gamma(U,\mathcal{O}_U^\dagger)=\varinjlim_h\Gamma(U_h,\mathcal{O}_{U_h})$, which is a countable filtered colimit of Banach space.

We gather some cohomological feature of dagger rigid spaces.

2.1 Arithmetic and geometric Hyodo-Kato morphisms

2.1.1. Condensed Galois action on log-crystalline cohomology. Consider a log-scheme $\mathscr{X}^0$ over $\mathrm{Spec}\,\mathcal{O}^0_{\breve{F},1}$ (*resp.* $\mathscr{X}$ over a $\mathrm{Spec}\,\mathcal{O}^\times_{C,1}$) which is descent to a log-smooth integral map of fine log-schemes $\mathscr{Z}^0\to\mathcal{O}^0_{F_L,1}$ (*resp.* $\mathscr{Z}\to\mathcal{O}^\times_{L,1}$). By functoriality of log-crystalline cohomology, the abstract absolute Galois group $\mathscr{G}_L$ acts on $R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\overline{S}_1^0\hookrightarrow\overline{S}^0)$ and $R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^\times_{\mathrm{cris}})$. On the other hand, by base change, we have

$$R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}^0_{\breve{F}})\simeq R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}^0_{F_L})\otimes^{\blacksquare}_{\mathcal{O}_{F_L}}\mathcal{O}_{\breve{F}},\quad R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^\times_{\mathrm{cris}})\simeq R\Gamma_{\mathrm{cris}}(\mathscr{Z}^\times/\mathcal{O}^\times_L)\otimes^{\blacksquare}_{\mathcal{O}_L}A_{\mathrm{cris}}.$$

The $\mathscr{G}_L$-action comes actually from that on the coefficients $\mathcal{O}_{\breve{F}}$ (*resp.* A_{cris}), which is continuous for their natural topology, hence it is upgraded to a condensed group action.

Alternatively, we may have started by defining the $\underline{\mathscr{G}_L}$-action on log-crystalline cohomology of qcqs $\mathscr{X}$ and $\mathscr{X}^0$ as above. These $\mathscr{G}_L$-actions come from smooth $\mathscr{G}_L$-actions respectively on discrete $R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}^0_{\breve{F},n})$ and $R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^\times_{\mathrm{cris},n})$ for $n\geq 1$, which is uniquely upgraded to condensed actions by (1.3.25). Taking limits and globalising, we can pass to the p-adic setting and drop the qcqs condition. This way, we can also define condensed Galois action for more general base pd log-schemes that $\mathcal{O}^0_{\breve{F}}$ and $\mathcal{O}^\times_C$ (for example, A_{cris} and $\widehat{A}_{l,\mathrm{st}}$, see later for the notation).

2.1.2 - Lemma. *Let $\mathscr{Z}$ be a quasi-separacted, fine and saturated log-scheme log-smooth and locally of finite type over $\mathcal{O}^0_{F,1}$ of relative dimension d. Then the monodromy operator N [35, (3.6)] on $R\Gamma_{\mathrm{cris}}(\mathscr{Z}/\mathcal{O}^0_F)$ is nilpotent with order bounded above by a function depending only on d.*

Proof. This is [9, Lemma 3.3], based on resolution of singulaties for log-smooth schemes and a nilpotency result of Mokrane. $\square$

2.1.3 - Theorem (Beilinson, Colmez-Nizioł, Bosco). *Let $\mathscr{Z}$ be a qcqs fine log-smooth log-scheme over $\mathcal{O}_{L,1}^{\times}$ of Cartier type for some finite extension L/K. Let $\mathscr{X} = \mathscr{Z} \otimes_{\mathcal{O}_{L,1}^{\times}} \mathcal{O}_{C,1}^{\times}$ and $\mathscr{X}^0 = \mathscr{X} \otimes_{\mathcal{O}_{C,1}^{\times}} \mathcal{O}_{\breve{F},1}^{0}$.*

(i) There exists a natural equivalence in $D(\mathrm{Mod}_{B_{\mathrm{st}}}^{\blacksquare})$

$$\varepsilon_{\mathrm{HK}}^{\mathrm{st}} : R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_{\breve{F}}^0) \otimes_{\mathcal{O}_{\breve{F}}}^{\blacksquare} B_{\mathrm{st}}^+ \simeq R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris}}^{\times}) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} B_{\mathrm{st}}^+$$

which is independent of the choice of the descent $\mathscr{Z}$ of $\mathscr{X}$, and compatible with the $\underline{\mathscr{G}_L}$-action, the Frobenius φ and the monodromy operators N.

(ii) There exists a natural equivalence in $D(\mathrm{Mod}_{C}^{\blacksquare})$

$$\varepsilon_{\mathrm{dR}}^{\mathrm{st}} : R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}_{\breve{F}}^0) \otimes_{\mathcal{O}_{\breve{F}}}^{\blacksquare} C \simeq R\Gamma_{\mathrm{cris}}(\mathscr{X}/\mathcal{O}_{C}^{\times})_{\mathbf{Q}_p}$$

which is independent of the choice of the descent $\mathscr{Z}$ of $\mathscr{X}$, compatible with the $\underline{\mathscr{G}_L}$-action and the Frobenius φ, and compatible with the previous equivalence via Fontaine's map $\theta : A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C$.

Proof. It has been treated in [19, Theorem 2.22, Corollary 2.31] and [9, Theorem 3.2]. Only the condensed $\underline{\mathscr{G}_L}$-equivariance need explanation. For this, we need to show that the natural maps in the construction of $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$ are $\underline{\mathscr{G}_L}$-equivariant.

Let us briefly explain the construction of $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$. For sufficiently large n, we have factorisations of Frob^n:

$$\mathrm{Frob}^n : \mathscr{Z} \xrightarrow{F_n} \mathscr{Z}^0 \hookrightarrow \mathscr{Z}$$

$$\mathrm{Frob}^n : \mathscr{Z}^0 \hookrightarrow \mathscr{Z} \xrightarrow{F_n} \mathscr{Z}^0.$$

Consider the diagram commutative of log-schemes

(2.1.3.1)
$$\begin{tikzcd} \mathscr{X}^0 \arrow[r, hook] \arrow[d, "\theta^0_{\mathscr{Z}}"] & \mathscr{X}_1 \arrow[r] \arrow[d, "F_n\theta_{\mathscr{Z}}"] & \overline{S}_1^{\times} \arrow[d, "F_{L,n}\theta_L"] \\ \mathscr{Z}^0 \arrow[r, "\mathrm{Frob}^n"] & \mathscr{Z}^0 \arrow[r] & S_{L,1}^0 \end{tikzcd}$$

which we can denote by π. These data π determine a lifting $i_{l_\pi}^* : r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}_{L,1}^{\times}$ as well as a log pd-thickening $\widehat{A}_{l_\pi,\mathrm{st}} \twoheadrightarrow \mathcal{O}_{C,1}^{\times}$. Consider the commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}_{F_L,n}^0) \arrow[d, "(\mathrm{Frob}^n)^*"] & R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/r_{L,n}^{\mathrm{PD},0}) \arrow[l, "p_0^*"'] \arrow[drr, "\pi^*"] & & \\ R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}_{F_L,n}^0) & & R\Gamma_{\mathrm{cris}}(\mathscr{X}/\widehat{A}_{l_\pi,\mathrm{st}}) & R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris}}^{\times}) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} \widehat{A}_{l_\pi,\mathrm{st}}. \arrow[l, "\kappa_{l_\pi}^*"', "\simeq"] \end{tikzcd}$$

After taking limits then the isogeny category, $(\mathrm{Frob}^n)^*$ becomes invertible, and p_0^* admits a unique natural φ-equivariant $\mathcal{O}_{F_L}$-linear section in the isogeny category, called the *Hyodo-Kato section*; moreover, $(R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris}}^{\times}) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} \widehat{A}_{l_\pi,\mathrm{st}})_{\mathbf{Q}_p}^{N-\mathrm{nilp}} \simeq R\Gamma_{\mathrm{cris}}(\mathscr{X}/A_{\mathrm{cris}}^{\times}) \otimes_{A_{\mathrm{cris}}}^{\blacksquare} B_{\mathrm{st}}$. Recall also that $R\Gamma_{\mathrm{cris}}(\mathscr{Z}^0/\mathcal{O}_{F_L}^0)$ is N-nilpotent

(2.1.2). Altogether we get

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[d, "\simeq"', "(\mathrm{Frob}^n)^*"] & R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/r_L^{\mathrm{PD},0})^{N-\mathrm{nilp}}_{\mathbf{Q}_p} \arrow[l, "p_0^*"'] \arrow[drr, "\pi^*"] & & \\
R\Gamma_{\mathrm{cris}}(\mathscr{X}^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[ur, bend right, "\iota_0"'] & & R\Gamma_{\mathrm{cris}}(\mathscr{X}/\widehat{A}_{l_\pi,\mathrm{st}})^{N-\mathrm{nilp}}_{\mathbf{Q}_p} & R\Gamma_{\mathrm{cris}}(\mathscr{X}/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B_{\mathrm{st}}. \arrow[l, "\kappa^*_{l_\pi}"', "\simeq"]
\end{tikzcd}$$

The morphism $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$ is then defined as the B^+_{st}-linearisation of the composite from the lower left corner to the right end.

Regarding condensed $\underline{\mathscr{G}_L}$-equivariance of $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$, one only needs to prove the that of each morphism above. Since the action is nontrivial only on the cohomology of $\mathscr{X}$, we only need to verify the equivariance of π^* and $\kappa^{*,-1}_{l_\pi}$. But they are either base change morphism along a $\mathscr{G}_L$-equivariant morphism or inverse of such, so we are done by the second description of the condensed $\underline{\mathscr{G}_L}$-action in (2.1.1). $\square$

2.1.4. Local arithmetic Hyodo-Kato morphism. Let $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$ be a qcqs semistable formal scheme over $\mathcal{O}_L$ for some finite extension L/K. We have the following $\underline{\mathscr{G}_L}$-equivariant commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}}) \otimes^{\blacksquare}_{\mathcal{O}_{\breve{F}}} B^+_{\mathrm{st}} \arrow[d, "\simeq"', "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}"] \arrow[dddd, bend left=60, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \\
& R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[d] \\
& R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}} \arrow[d, "\simeq"] \\
& R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}}) \\
& R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{dR}} \arrow[u, "\simeq"] \arrow[d, phantom, "\wr"] \\
& R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{dR}} \arrow[d] \\
& R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}}
\end{tikzcd}$$

where $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})$ and $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L)$ are equipped with the trivial $\mathscr{G}_L$-action.

We define the *local arithmetic Hyodo-Kato morphism* as the composite

$$\iota^{\mathrm{arith}}_{\mathrm{HK}} : R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \to (R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}})^{\underline{\mathscr{G}_L}} \xleftarrow{\simeq} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L),$$

where the last isomorphism is from (1.3.11, ii) since $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L)$ is represented by a bounded complex of L-Banach spaces, or by abuse of notation as its $\mathbf{Q}_p$-linearisation

$$\iota^{\mathrm{arith}}_{\mathrm{HK}} : R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L).$$

By design,

2.1.5 - Remark. There is certain independence of $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ on the embedding $\sigma : L \hookrightarrow C$. For this, consider two embeddings $\sigma, \tau : L \hookrightarrow C$. There exists a $\gamma \in \mathscr{G}_K$ such that $\sigma = \gamma \circ \tau$. Consider $\mathfrak{X}_\sigma := \mathfrak{Z} \otimes_{L,\sigma} L$ and

$\mathfrak{X}_\tau := \mathfrak{Z} \otimes_{L,\tau} C$. We have a commutative diagram

$$\begin{tikzcd}
\mathfrak{X}_\sigma \arrow[r, "f_\gamma"] \arrow[d] & \mathfrak{X}_\tau \arrow[d] \\
\operatorname{Spf} \mathcal{O}_C \arrow[r, "\gamma"] & \operatorname{Spf} \mathcal{O}_C
\end{tikzcd}$$

hence a commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\sigma"] \arrow[d, equal] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{F_L,\sigma} \breve{F} \arrow[r, "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\sigma,1}^0/\mathcal{O}_{\breve{F}^0}) \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_{\sigma,\eta}/B_{\mathrm{dR}}^+) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{pdR}}^+ \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\tau"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{F_L,\sigma} \breve{F} \arrow[r, "\simeq"] \arrow[u, "\mathrm{id}\otimes\gamma", "\simeq"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\tau,1}^0/\mathcal{O}_{\breve{F}^0}) \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \arrow[u, "f_\gamma^*", "\simeq"'] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_{\tau,\eta}/B_{\mathrm{dR}}^+) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{pdR}}^+ . \arrow[u, "f_\gamma^*\otimes\mathrm{id}", "\simeq"']
\end{tikzcd}$$

which is $\underline{\mathscr{G}_L}$-equivariant (with respect to the $\underline{\mathscr{G}_{\sigma(L)}}$-action on the top row and the $\underline{\mathscr{G}_{\tau(L)}}$-action on the bottom row; noticing that $\mathscr{G}_{\sigma(L)} = \gamma\mathscr{G}_{\tau(L)}\gamma^{-1}$, so we have $\gamma\mathscr{G}_{\tau(L)} = \mathscr{G}_{\sigma(L)}\gamma$ showing equivariance). Taking $\underline{\mathscr{G}_L}$-invariants respectively, we obtain a commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\iota_{\mathrm{HK},\sigma}^{\mathrm{arith}}"] \arrow[d, equal] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\iota_{\mathrm{HK},\tau}^{\mathrm{arith}}"] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L). \arrow[u, "f_\gamma^*"', "\simeq"]
\end{tikzcd}$$

which depends only on the restriction of γ on the Galois closure of the extension L/K.

2.1.6 - Lemma. *The local arithmetic Hyodo-Kato morphism $\iota_{\mathrm{HK}}^{\mathrm{arith}}$ is natural for $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$.*

Proof. One goes formally through the construction of $\iota_{\mathrm{HK}}^{\mathrm{arith}}$, eventually using (2.1.5). □

2.1.7 - Proposition. *For $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ a qcqs semistable formal scheme over $\mathcal{O}_L$ and $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C$, there is a natural $\underline{\mathscr{G}_L}$-equivariant commutative diagram*

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{arith}}"] \arrow[d] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \arrow[d] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}_{\breve{F}}^0) \otimes^{\blacksquare}_{\mathcal{O}_{\breve{F}}} B_{\mathrm{st}}^+ \arrow[r, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+)
\end{tikzcd}$$

exhibiting the compatibility between local arithmetic and geometric Hyodo-Kato morphisms.

Proof. This follows from the construction. □

2.1.8 - Proposition. *If $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ be qcqs and semistable of relative dimension d over $\mathcal{O}_L$, then $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ is represented by a bounded complex of F_L-Banach spaces, and lies in $\mathscr{D}^{[0,2d]}(\mathrm{Mod}^{\blacksquare}_{F_L})$.*

Proof. It is essentially the same proof as in [9, Theorem 3.15 (ii)]. Let us sketch it. The isomorphism $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$[9] induces by base change to $\mathbf{B}_{\mathrm{dR}}^+$ an $\underline{\mathscr{G}_L}$-equivariant isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \otimes^{\blacksquare}_{\mathcal{O}_{F_L}} B_{\mathrm{dR}}^+ \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B_{\mathrm{dR}}^+ .$$

[9] Alternatively, we can also use the original arithmetic Hyodo-Kato isomorphism [17, 4.2.3 (i)] for the proof.

The right hand side is represented by a complex of F_L-Banach spaces and lies in $\mathscr{D}^{[0,2d]}(\mathrm{Mod}^{\blacksquare}_{F_L})$, while on the left we have a splitting via $B^+_{\mathrm{dR}} = F_L \oplus M$ for some F_L-Banach space M. Therefore $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$, being a direct summand of the right hand side, is also represented by a complex of F_L-Banach spaces and belongs to $\mathscr{D}^{[0,2d]}(\mathrm{Mod}^{\blacksquare}_{F_L})$. □

2.1.9 - Proposition. *The local arithmetic Hyodo-Kato morphism $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ induced under L-linearisation an isomorphism*

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes_{F_L} L \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L).$$

Proof. The isomorphism $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$ induces an $\underline{\mathscr{G}_L}$-equivariant isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \otimes^{L\blacksquare}_{\mathcal{O}_{F_L}} B^+_{\mathrm{pdR}} \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \otimes^{L\blacksquare}_{L} B^+_{\mathrm{pdR}}.$$

Since $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$ is represented by a bounded complex of F_L-Banach spaces (2.1.8), we may take $\underline{\mathscr{G}_L}$-invariants and apply (1.3.11) to conclude. □

2.1.10 - Remark. For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$ semistable over $\mathcal{O}_L$, one has the following commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes_{F_L} L \arrow[r, "\simeq"] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/L) \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes_{F} K \arrow[r] \arrow[u, hook, "\oplus"] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K). \arrow[u, "\simeq"']
\end{tikzcd}$$

The left vertical map is induced by $L_{0,K} := F_L \otimes_F K \hookrightarrow F_L \otimes_{F_L} L$, which is the inclusion of the maximal unramified subextension of L/K into L; the latter admits a canonical retract being the normalised trace map $r^{\mathrm{Gal}}_{L/K} := \overline{\mathrm{Tr}}_{L/L_{0,K}} := \frac{1}{|\mathrm{Gal}(L/L_{0,K})|}\sum_{g\in\mathrm{Gal}(L/L_{0,K})} g$; hence it is a direct factor, and is an isomorphism if and only if L/K is an unramified extension.

2.1.11. Arithmetic Hyodo-Kato cohomology. Let $R\Gamma_{\mathrm{HK}} : \mathcal{R}\mathrm{ig}_K \to \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$ be the η-éh hypersheafification of the presheaf $R\Gamma^{\mathrm{pre}}_{\mathrm{HK}} : \mathfrak{Z} \mapsto R\Gamma(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_{L_\mathfrak{Z}}})_{\mathbf{Q}_p}$ on $\mathcal{M}^{\mathrm{ss}}_K$.

2.1.12 - Proposition (Local-global compatibility). *For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$, the natural morphism*

$$R\Gamma_{\mathrm{HK}}(\mathfrak{Z}_\eta) \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})$$

is an isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$.

Proof. Let us recall the proof of [17, Proposition 4.11] (cf. [46, Proposition 3.18]). One needs to show that $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_{L_\mathfrak{Z}}})$ satisfies η-éh-hyperdescent for $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$ (up to refinement of the hypercovering), admitting the fact that $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)$ does so. For this, we may assume $\mathfrak{Z}$ is qcqs with splitting field K, and suppose that $\mathfrak{Z}_\bullet$ is an affine η-éh-hypercovering of $\mathfrak{Z}$, with respective splitting field $L_\bullet$, all finite Galois over K.

Let $k \in \mathbf{N}$. Let E/K be a finite Galois extension containing all $L_{\bullet\leq k+1}$, and let $G = \mathrm{Gal}(E/K)$. Consider $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_{\bullet\leq k+1}}}_{\bullet\leq k+1,\mathcal{O}_E}$; this is a $(k+1)$-truncated η-éh-hypercovering of $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_K}_{\mathcal{O}_E}$, whose generic fibre is Z_E, with each member having splitting field $\mathcal{O}_E$; moreover there is a natural G-action on $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_{\bullet\leq k+1}}}_{\bullet\leq k+1,\mathcal{O}_E}$ making this hypercovering G-equivariant and compatible with the Galois action on Z_E. Here, we denoted

$$\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_L}_{\mathcal{O}_E} := \coprod_{\sigma\in\mathrm{Hom}_K(L,E)} \mathfrak{Z} \otimes_{\mathcal{O}_L,\sigma} \mathcal{O}_E, \tag{2.1.12.1}$$

and let $g \in G$ act on it by permuting indices $\sigma \mapsto g^{-1}\circ\sigma$ and at the same time acting on the coefficients $\mathcal{O}_E$. It can be extended to an entire G-equivariant hypercovering $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_\bullet}}_{\bullet,\mathcal{O}_E}$. We have a bisimplicial object $\mathfrak{Z}'_{\bullet,\circ} := \coprod_{G^{\times\circ}} \mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_{L_\bullet}}_{\bullet,\mathcal{O}_E}$, the faces maps being evidently forgetting components of G except one being acting on $\mathfrak{Z}^{\mathrm{ss}/\mathcal{O}_L}_{\mathcal{O}_E}$ via $\sigma \mapsto g^{-1} \circ \sigma$ on the index set (2.1.5)[10]. Its diagonal is an affine η-éh-hypercovering of $\mathfrak{Z}_\eta$ by semistables with splitting field $\mathcal{O}_E$ refining $\mathfrak{Z}_\bullet$. We have compatible isomorphisms

$$R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet\le k+1,\circ})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} \otimes_{F_E} E \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}((\mathfrak{Z}'_{\bullet\le k+1,\circ})_\eta/E)$$

by local arithmetic Hyodo-Kato isomorphism (2.1.9) since every formal schemes here are semistable over $\mathcal{O}_E$. Taking limit over the index $\bullet$, since $\mathfrak{Z}'_{\bullet,\circ}$ forms an η-éh-hypercovering of $\coprod_{G^{\times\circ}} \mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{O}_E$, we obtain

$$\begin{aligned}
\tau^{\le k} \lim_{[n]\in\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{[n],\circ})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} \otimes_{F_E} E &\simeq \tau^{\le k} \lim_{[n]\in\Delta_{k+1}} R\Gamma_{\mathrm{dR}}((\mathfrak{Z}'_{[n],\circ})_\eta/E) \\
&\simeq \tau^{\le k} \prod_{G^{\times\circ}} R\Gamma_{\mathrm{dR}}((\mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{O}_E)_\eta/E) \\
&\simeq \tau^{\le k} \prod_{G^{\times\circ}} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K) \otimes_K E \\
&\simeq \tau^{\le k} \prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{O}_E)^0_1/\mathcal{O}^0_E)_{\mathbf{Q}_p} \otimes_{F_E} E,
\end{aligned}$$

inside which we identify the F_E-linear isomorphism

$$\begin{aligned}
\tau^{\le k} \lim_{[n]\in\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{[n],\circ})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} &\simeq \tau^{\le k} \prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z} \otimes_{\mathcal{O}_K} \mathcal{O}_E)^0_1/\mathcal{O}^0_E)_{\mathbf{Q}_p} \\
&\simeq \tau^{\le k} \prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_F)_{\mathbf{Q}_p} \otimes_F F_E
\end{aligned}$$

compatible with the residual simplicial structures.

Next, since the diagonal of a bisimplicial set calculates the total realisation, we have

$$\begin{aligned}
\tau^{\le k} \lim_{\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet\le k+1,\bullet\le k+1})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} \otimes_F K &\xleftarrow{\simeq} \tau^{\le k} \lim_{\Delta_{k+1}\times\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet\le k+1,\circ\le k+1})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p} \otimes_F K \\
&\simeq \tau^{\le k} \lim_{\Delta_{k+1}} \prod_{G^{\times\circ}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p} \otimes_F F_E \otimes_F K \\
&\simeq \tau^{\le k} R\Gamma(G, R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p} \otimes_F F_E \otimes_F K) \\
&\xleftarrow{\simeq} \tau^{\le k} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p} \otimes_F K
\end{aligned}$$

The last isomorphism follows from the observation that due to finiteness of G, the condensed group cohomology agrees with the smooth one, and can be computed pointwise; since these groups are $\mathbf{Q}$-vector spaces, this simplifies to pointwise and termwise genuine G-fixed points. Finally, we extract from it the isomorphism

$$\tau^{\le k} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_K)_{\mathbf{Q}_p} \xrightarrow{\simeq} \tau^{\le k} \lim_{\Delta} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\bullet,\bullet})^1_0/\mathcal{O}^0_E)_{\mathbf{Q}_p}$$

showing the k-truncated η-éh-hyperdescent for the hypercovering $\mathfrak{Z}'_{\bullet,\bullet}$ refining $\mathfrak{Z}_\bullet$. □

2.1.13. Global arithmetic Hyodo-Kato morphism. Let $Z \in \mathcal{R}\mathrm{ig}_K$ and $\mathfrak{Z}_\bullet$ be an η-éh hypercovering of Z. We define the *global arithmetic Hyodo-Kato morphism* as

$$\iota^{\mathrm{arith}}_{\mathrm{HK}} := \lim_{\Delta} \iota^{\mathrm{arith}}_{\mathrm{HK},\bullet} : R\Gamma_{\mathrm{HK}}(Z) \simeq \lim_{\Delta} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\bullet,1}/\mathcal{O}^0_{F_{L_\bullet}}) \to \lim_{\Delta} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_{\bullet,\eta}/L_\bullet) \xleftarrow{\simeq} R\Gamma_{\mathrm{dR}}(Z/K)$$

or by abuse of notation as the simplicial limit of $\mathbf{Q}_p$-linearised Hyodo-Kato morphisms. This can be made independent of the chosen η-étale hypercovering by taking colimit over all possible such coverings, which form a filtered system.

2.1.14 - Lemma. *For $Z \in \mathcal{R}\mathrm{ig}_K$, the K-linearised Hyodo-Kato morphism $\iota^{\mathrm{arith}}_{\mathrm{HK}} \otimes \mathrm{id}$ has a canonoical natural*

[10]The intuition comes from the decomposition $E^{\otimes_K(m+1)} \simeq \prod_{G^m} E, x_0 \otimes \cdots \otimes x_m \mapsto (x_0 \cdot g_1(x_1) \dots g_m(x_m))_{g\in G^m}$ and the interpretation of face maps.

K-linear retract

$$r_Z : R\Gamma_{\mathrm{dR}}(Z/K) \to R\Gamma_{\mathrm{HK}}(Z) \otimes_F K.$$

It is an isomorphism if Z has a η-éh-hypercovering by semistable formal schemes over $\mathcal{O}_K$.

Proof. Assume first that Z has an η-éh-hypercovering by semistable formal schemes $\mathfrak{Z}_\bullet$ over $\mathcal{O}_L$ for some finite unramified extension L/K; we may assume $\mathfrak{Z}_\bullet$ to be qcqs (even affine). By éh-hyperdescent, local-global compatibility (2.1.12) and local Hyodo-Kato morphism (2.1.9)[11], we have isomorphisms

$$\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id} : R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \simeq \lim_{\Delta} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_{\bullet,1}^0/\mathcal{O}_F^0)_{\mathbf{Q}_p} \otimes_F K \xrightarrow{\simeq} \lim_{\Delta} R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_{\bullet,\eta}/K) \xleftarrow{\simeq} R\Gamma_{\mathrm{dR}}(Z/K).$$

Hence there is a caonical retract $r_Z := (\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id})^{-1}$. Taking one step back, if $\mathfrak{Z}_\bullet$ are semistable over $\mathcal{O}_K$ only for degrees $\bullet \le k+1$, then we get a canonical k-truncated retract $r_Z^{\le k}$ of $\tau^{\le k}(\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id}) : \tau^{\le k} R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \to \tau^{\le k} R\Gamma_{\mathrm{dR}}(Z/K)$. This definition does not depend on the choice of the (truncated) éh-hypercovering, hence it is canonical; moreover, such retract is compatible with base change, i.e. for any finite Galois extension E/K, the rigid space $Z_E \in \mathcal{R}\mathrm{ig}_E$ satisfying the same reduction condition as $Z \in \mathcal{R}\mathrm{ig}_K$, there is a canonical $\mathrm{Gal}(E/K)$-equivariant commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \arrow[d] & R\Gamma_{\mathrm{dR}}(Z/K) \arrow[l, "r_Z"', "\simeq"] \arrow[d] \\
R\Gamma_{\mathrm{HK}}(Z_E) \otimes_{F_E} E & R\Gamma_{\mathrm{dR}}(Z_E/E). \arrow[l, "r_{Z_E}"', "\simeq"]
\end{tikzcd}$$

There are also normalised trace maps retracting the vertical maps by étale descent along $Z_E \to Z$, giving again a canonical $\mathrm{Gal}(E/K)$-equivariant commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{HK}}(Z) \otimes_F K & R\Gamma_{\mathrm{dR}}(Z/K) \arrow[l, "r_Z"', "\simeq"] \\
R\Gamma_{\mathrm{HK}}(Z_E) \otimes_{F_E} E \arrow[u, "\overline{\mathrm{Tr}}_{E/K}"] & R\Gamma_{\mathrm{dR}}(Z_E/E). \arrow[l, "r_{Z_E}"', "\simeq"] \arrow[u, "\overline{\mathrm{Tr}}_{E/K}"]
\end{tikzcd}$$

The constructions are natural since they are inverse to $\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id}$, which is natural. Similarly we have a truncated analogue.

More generally, let $Z \in \mathcal{R}\mathrm{ig}_K$ be qcqs. Let $\mathfrak{Z}_\bullet \in \mathcal{M}_K^{\mathrm{ss}}$ be an η-éh-hypercovering of Z by qcqs semistables splitting over $L_\bullet$. Similary as at the beginning of the proof of (2.1.12), there are increasing finite extensions E_k/L for $k \in \mathbf{N}$ such that E_k contains all $L_{\bullet \le k+1}$, so that we obtain $(k+1)$-truncated η-éh-hypercoverings $\mathfrak{Z}^{(k)}_{\bullet \le k+1}$ of Z_{E_k} in $\mathcal{M}_K^{\mathrm{ss}}$ which are compatible with base change between different k, compatible with the $\mathrm{Gal}(E_k/L)$-action, and such that $\mathfrak{Z}^{(k)}_{\bullet \le k+1}$ has splitting field E_k. For $m \ge k$, we define the k-truncated retract as the composite

$$r_Z^{\le k} : \tau^{\le k} R\Gamma_{\mathrm{dR}}(Z/K) \to \tau^{\le k} R\Gamma_{\mathrm{dR}}(Z_{E_m}/E_m) \xrightarrow{r^{\le k}_{Z_{E_m}}} \tau^{\le k} R\Gamma_{\mathrm{HK}}(Z_{E_m}) \otimes_{F_{E_m}} E_m \xrightarrow{\overline{\mathrm{Tr}_{E_m/K}}} \tau^{\le k} R\Gamma_{\mathrm{HK}}(Z) \otimes_F E,$$

which does not depend on m by the above commutative diagram, hence $r_Z^{\le}$ is well-defined; it is indeed a retract of $\tau^{\le k}(\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id})$ by diagram chasing. Taking filtered colimts, we obtain the retract $r_Z := \varinjlim_k r_Z^{\le k}$ of $\iota_{\mathrm{HK}}^{\mathrm{arith}} \otimes \mathrm{id}$. The naturality comes from that of the first special case.

Finally, the case for general Z follows formally from the qcqs case. □

2.1.15 - Proposition. *If $Z \in \mathcal{R}\mathrm{ig}_K$ be qcqs, then $R\Gamma_{\mathrm{HK}}(Z)$ is represented by a complex of solid-nuclear F-vector spaces, and lies in $\mathscr{D}^{[0,2d]}(\mathrm{Mod}_F^{\blacksquare})$, and the monodromy operator N is nilpotent with order bounded above by a function depending only on d.*

[11] Alternatively, we can also use the original arithmetic Hyodo-Kato isomorphism [17, 4.2.3 (i)] for the proof.

Proof. By local-global compatibility (2.1.12), one reduces to the case where $Z = \mathfrak{Z}_\eta$ for some qcqs $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$. Then the solid-nuclear representability and the nilpotency are clear respectively by (2.1.8) and (2.1.2). The concentration statement is clear since it is true for $R\Gamma_{\mathrm{dR}}(Z/K)$, which retracts to $R\Gamma_{\mathrm{HK}}(Z) \otimes_F K$ by (2.1.14). □

2.1.16 - Remark. Bootstrapping boundedness into the proof of solid-nuclearity, we see that

$$R\Gamma_{\mathrm{HK}}(Z) \simeq \tau^{2d}\sigma^{\leq 2d+1} R\Gamma_{\mathrm{HK}}(Z) \simeq \tau^{2d} \lim_{\Delta_{2d+1}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_{\bullet,1}/\mathcal{O}^0_{F_{L_\bullet}})_{\mathbf{Q}_p}$$

for any η-éh-hypercovering $\mathfrak{Z}_\bullet$ of Z in $\mathcal{M}_K^{\mathrm{ss}}$. Therefore $R\Gamma_{\mathrm{HK}}(Z)$ can be represented by the truncation of a *bounded* complex of F_L-Banach spaces.

2.1.17. Geometric Hyodo-Kato cohomology. We have a completed version as well as a decompeted version of geometric Hyodo-Kato cohomology.

(i) Let the (completed) geometric Hyodo-Kato cohomology $R\Gamma_{\mathrm{HK}} : \mathcal{R}\mathrm{ig}_C \to \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$ be the η-éh-hypersheafification of the presheaf $\mathfrak{X} \mapsto R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p}$ on $\mathcal{M}_C^{\mathrm{ss},b}$.
(ii) Let the decompleted geometric Hyodo-Kato cohomology $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}} : \mathcal{R}\mathrm{ig}_C \to \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}})$ be the η-éh-hypersheafification of the $R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}} : \mathfrak{X} \mapsto \operatorname{colim}_\Sigma R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$ on $\mathcal{M}_C^{\mathrm{ss},b}$, where Σ is the filtered system whose date are the reduction mod p of quadruples $(L,\mathfrak{Z},\sigma,\theta)$ such that $\sigma : L \hookrightarrow C$ is a finite extension of K, $\mathfrak{Z}$ is a semistable formal model over $\mathcal{O}_L$, and $\theta : \mathfrak{X} \xrightarrow{\simeq} \mathfrak{Z} \otimes_{\mathcal{O}_L,\sigma} \mathcal{O}_C$ is an isomorphism over $\mathcal{O}_C$, and whose morphisms are morphisms between the reduced objects $\mathfrak{Z}^0_1$ [17, 4.3.1].

One can globalise the geometric Hyodo-Kato morphism to obtain

$$\iota^{\mathrm{geom}}_{\mathrm{HK}} : R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \to R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}})$$

for $X \in \mathcal{R}\mathrm{ig}_C$; reducing mod $\ker\theta$, we obtain the Hyodo-Kato isomorphism [9, Theorem 3.15]

$$\iota^{\mathrm{geom}}_{\mathrm{HK},C} : R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} C \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}(X/C) \tag{2.1.17.1}$$

compatible with the arithmetic Hyodo-Kato morphism (2.1.13).

Recall that for $\mathfrak{X} \in \mathcal{M}_C^{\mathrm{ss},b}$, any map of quadruples $(L,\mathfrak{Z},\sigma,\theta) \to (L',\mathfrak{Z}',\sigma',\theta')$ in Σ induces a canonical base change isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes_{F_L,\sigma} F_{L'} \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}'^0_1/\mathcal{O}^0_{F_{L'}})_{\mathbf{Q}_p} \tag{2.1.17.2}$$

in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F_L})$.

2.1.18 - Proposition (Local-global compatiblity). *For $\mathfrak{X} \in \mathcal{M}_K^{\mathrm{ss},b}$, the natural maps*

$$R\Gamma_{\mathrm{HK}}(\mathfrak{X}_\eta) \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \quad \text{in } \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$$

$$R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X}_\eta) \to R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X}) \quad \text{in } \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}})$$

are isomorphisms.

Proof. For $R\Gamma_{\mathrm{HK}}$, this is [9, Theorem 3.15 (i)]. For $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}$, this is done as in [17, Proposition 4.23 (1)] using the original Hyodo-Kato morphism via convergent cohomology of *op. cit.* (4.17). □

2.1.19 - Lemma. *For $Z \in \mathcal{R}\mathrm{ig}_K$, there is canonically a natural $\underline{\mathscr{G}_K}$-action on $R\Gamma_{\mathrm{HK}}(Z_C)$, and a natural $\underline{\mathscr{G}_K}$-equivariant $\breve{F}$-linear morphism*

$$R\Gamma_{\mathrm{HK}}(Z) \otimes^{\blacksquare}_F \breve{F} \to R\Gamma_{\mathrm{HK}}(Z_C).$$

It is an isomorphism if Z has an η-éh-hypercovering by semistable formal schemes over $\mathcal{O}_K$. Moreoever, any $\breve{F}$-linear retract $r : C \to \breve{F}$ induces a natural $\breve{F}$-linear retract $r_{\mathrm{HK}} : R\Gamma_{\mathrm{HK}}(Z_C) \to R\Gamma_{\mathrm{HK}}(Z) \otimes^{\blacksquare}_F \breve{F}$.

Proof. By functoriality, $\mathscr{G}_K$ acts on $R\Gamma_{\mathrm{HK}}(Z_C)$. We want to make it condensed. By η-éh-hyperdescent, we may assume $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z}$ is a semistable formal scheme over $\mathcal{O}_L$ with splitting field a Galois extension L of K. Then $(\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C})_\eta := \coprod_{\sigma \in \mathrm{Hom}_K(L,C)} \mathfrak{Z} \otimes_{\mathcal{O}_L,\sigma} \mathcal{O}_C$ (2.0.1.1) is a semistable formal model of Z_C, the natural $\mathscr{G}_K$-action on Z_C extends canonically to one on $\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C}$ by permuting the indices $\sigma \mapsto g^{-1}\sigma$ and meanwhile acting on the coefficients $\mathcal{O}_C$. By local-global compatibilities and base change, we have

$$R\Gamma_{\mathrm{HK}}(Z) \simeq R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p}$$

and

$$\begin{aligned} R\Gamma_{\mathrm{HK}}(Z_C) \simeq R\Gamma_{\mathrm{cris}}((\mathfrak{Z}^{\mathrm{ss}}_{\mathcal{O}_C})^1_0/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} &\simeq \prod_{\sigma \in \mathrm{Hom}_K(L,C)} R\Gamma_{\mathrm{cris}}((\mathfrak{Z} \otimes_{\mathcal{O}_L,\sigma} \mathcal{O}_C)^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \\ &\simeq \prod_{\sigma \in \mathrm{Hom}_K(L,C)} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_L,\sigma} \breve{F}. \end{aligned}$$

The natural $\mathscr{G}_K$-action on the right hand side is given by permuting components $\sigma \mapsto g \circ \sigma$ and acting on the coefficients $\breve{F}$, i.e. by letting $g \cdot (x_\sigma \otimes c_\sigma)_\sigma = (x_{g^{-1}\sigma} \otimes g(x_{g^{-1}\sigma}))$; this is refined canonically to be a condensed action. The natural map $R\Gamma_{\mathrm{HK}}(Z) \to R\Gamma_{\mathrm{HK}}(X)$ corresponds to the diagonal embedding; linearising over F, we obtain the desired $\underline{\mathscr{G}_K}$-equivariant $\breve{F}$-linear morphism, via the $\mathscr{G}_K$-equivariant morphism

$$F_L \otimes_F \breve{F} \xrightarrow{\simeq} \prod_{\sigma \in \mathrm{Hom}_F(F_L, \breve{F})} F_L \otimes_{F_L,\sigma} \breve{F} \hookrightarrow \prod_{\sigma \in \mathrm{Hom}_K(L,C)} F_L \otimes_{F_L,\sigma} \breve{F},$$

where the last map is induced by the restriction surjection $\mathrm{Hom}_K(L,C) \twoheadrightarrow \mathrm{Hom}_F(F_L,\breve{F})$ on indices. This becomes an isomorphism if L/K is unramified.

As for the retract, given a $\breve{F}$-linear splitting $r : C \to \breve{F}$, we define it as the composite

$$r_{\mathrm{HK}} : R\Gamma_{\mathrm{HK}}(Z_C) \xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}} R\Gamma_{\mathrm{dR}}(Z/C) \simeq R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C \xrightarrow{r_Z \otimes r} R\Gamma_{\mathrm{HK}}(Z) \otimes^{\blacksquare}_F \breve{F}$$

using the arithmetic Hyodo-Kato retract r_Z (2.1.14), and check that it is indeed a retract. □

2.1.20 - Corollary. *For $Z \in \mathcal{R}\mathrm{ig}_K$, there is a natural $\underline{\mathscr{G}_K}$-equivariant commutative diagram*

$$\begin{array}{ccc} R\Gamma_{\mathrm{HK}}(Z) \otimes^{\blacksquare}_F B^+_{\mathrm{st}} & \xrightarrow{\iota^{\mathrm{arith}}_{\mathrm{HK}} \otimes \iota_p} & R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \\ \downarrow & & \downarrow \simeq \\ R\Gamma_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} & \xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}} & R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}) \end{array}$$

exhibiting the compatibility of arithmetic and geometric Hyodo-Kato morphisms.

Write about $\iota_p : B^+_{\mathrm{st}} \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$ by sending $\log[p^\flat] \mapsto \log \frac{[p^\flat]}{p}$.

Proof. The same proof as above applies, by reducing to semistable reduction case and using the compatibility between $\iota^{\mathrm{arith}}_{\mathrm{HK}}$ and $\iota^{\mathrm{geom}}_{\mathrm{HK}}$ (2.1.7). □

2.1.21. As a consequence of (2.1.17.2), the $\mathscr{G}_L$-action on $R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X})$ is condensed and smooth, making $R\Gamma^{\mathrm{pre}}_{\mathrm{HK},F^{\mathrm{nr}}}(\mathfrak{X}) \in \mathscr{D}^{\mathrm{sm}}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}[\mathscr{G}_L]})$. By descent and truncation argument combined with (2.0.1.1) as in the proof of (2.1.14), for $Z \in \mathcal{R}\mathrm{ig}_K$ qcqs, the $\mathscr{G}_K$-action on $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)$ is condensed and smooth, making $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C) \in \mathscr{D}^{\mathrm{sm}}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}[\mathscr{G}_K]})$; moreover, for any $k \in \mathbf{N}$, there is a finite Galois extension E/K such that $\tau^{\leq k} R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C)$ is defined over F_E, such that there exists an $(k+1)$-truncated η-éh-hypercovering $\mathfrak{Z}'_{\leq k+1}$ of Z_E compatible with $\mathrm{Gal}(E/K)$-action and such that $\mathfrak{Z}'_{\leq k+1}$ all have splitting field E; consequently, we have isomorphisms

$$\tau^{\leq k} R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C) \simeq \tau^{\leq k} \lim_{\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_E} F^{\mathrm{nr}} \simeq \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} F^{\mathrm{nr}}$$

in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{F^{\mathrm{nr}}[\mathscr{G}_K]})$. The same argument applies to the complete Hyodo-Kato cohomology, and shows, with the help of (2.1.19), that for $Z \in \mathcal{R}\mathrm{ig}_K$ qcqs and for any $k \in \mathbf{N}$, there exists a finite extension of E and an $(k+1)$-truncated η-éh-hypercovering $\mathfrak{Z}'_{\leq k+1}$ of Z_E compatible with $\mathrm{Gal}(E/K)$-action and with splitting field E, such that there are isomorphisms

$$\tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_C) \simeq \tau^{\leq k} \lim_{\Delta_{k+1}} R\Gamma_{\mathrm{cris}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{F_E} \breve{F} \simeq \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} \breve{F}$$

in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]})$. In particular, the natural map $R\Gamma_{\mathrm{HK},F^{\mathrm{nr}}}(Z_C) \otimes^{\blacksquare}_{F^{\mathrm{nr}}} \breve{F} \xrightarrow{\simeq} R\Gamma_{\mathrm{HK}}(Z_C)$ in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]})$ is an isomorphism.

2.1.22 - Proposition. *Let $Z \in \mathcal{R}\mathrm{ig}_K$ be qcqs. We have $H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{log}}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z)$.*

Proof. By (2.1.21), there exists a finite Galois extension E/K such that $H^n_{\mathrm{HK}}(Z_C) \simeq H^n_{\mathrm{HK}}(Z_E) \otimes_{F_E} \breve{F}$ in $\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]}$, hence we are reduced to showing

$$H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} B_{\mathrm{log}}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z).$$

The computation $H^0(\underline{\mathscr{G}_K}, B_{\mathrm{dR}}) = K$ and the inclusion $\iota_p : B_{\mathrm{log}}[\frac{1}{t}] \hookrightarrow B_{\mathrm{dR}}$ implies

$$K \hookrightarrow H^0(\underline{\mathscr{G}_E}, B_{\mathrm{log}}[\frac{1}{t}]) \hookrightarrow H^0(\underline{\mathscr{G}_K}, B_{\mathrm{dR}}) = K,$$

hence they are all equal. On the other hand, since $H^n_{\mathrm{HK}}(Z_E)$ is a solid-nuclear F_E-vector space (2.1.15), and so is $B_{\mathrm{log}}[\frac{1}{t}]$, we have by (1.3.5)

$$H^0(\underline{\mathscr{G}_E}, H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} B_{\mathrm{log}}[\frac{1}{t}]) \simeq H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} H^0(\underline{\mathscr{G}_K}, B_{\mathrm{log}}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z_E).$$

Moreover, since $\mathrm{Gal}(E/K)$ is a finite group, by smooth of its action, we obtain (1.3.22)

$$H^0(\underline{\mathrm{Gal}(E/K)}, H^n_{\mathrm{HK}}(Z_E)) \simeq H^n R\Gamma(\underline{\mathscr{G}_K}, R\Gamma_{\mathrm{HK}}(Z_E)) \simeq H^n_{\mathrm{HK}}(Z),$$

where the last isomorphism follows from the étale descent along $Z_E \to Z$. $\square$

2.2 Overconvergent variants

2.2.1. Cohomology for dagger rigid spaces. Now, we consider dagger varieties instead of rigid-analytic varieties. There two ways to do this [17, §5], yielding strictly quasi-isomorphic theories. On the one hand, one can consider the dagger analogue of Beilinson base $(\mathcal{M}^{\mathrm{ss},\dagger}_K, (-)_\eta)$ and $(\mathcal{M}^{\mathrm{ss},b,\dagger}_K, (-)_\eta)$ in order to exploit the finiteness property of rigid cohomology or de Rham cohomology. This way, we obtain the Hyodo-Kato cohomology à la Grosse-Klönne $R\Gamma^{\mathrm{GK}}_{\mathrm{HK}}$ as an éh-hypersheaf on $\mathcal{R}\mathrm{ig}^\dagger_K$ unfolded from the rational log-rigid cohomology [13, 3.1.2] $\mathfrak{Z} \mapsto R\Gamma_{\mathrm{rig}}(\mathfrak{Z}^0_1/\mathcal{O}^0_F)$ on $\mathcal{M}^{\mathrm{ss},\dagger}_K$ and as an éh-hypersheaf on $\mathcal{R}\mathrm{ig}^\dagger_C$ unfolded from $\mathfrak{X} \mapsto R\Gamma_{\mathrm{rig}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})$ on $\mathcal{M}^{\mathrm{ss},b,\dagger}_C$.

On the other hand, let L be K or C and let $\mathscr{V}$ be a presentable ∞-category of coefficients; there is a general procedure to produce éh-hypersheaves on $\mathcal{R}\mathrm{ig}^\dagger_L$ from étale hypersheaves on $\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_L$: for any $\mathcal{F} \in \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{R}\mathrm{ig}_{L,\acute{\mathrm{e}}\mathrm{t}}, \mathscr{V})$, we define

$$\mathcal{F}^\dagger(U) := \varinjlim_h R\Gamma(U_h, \mathcal{F})$$

for any presentation $(U_h)_h$ of a smooth affinoid U; this is well-defined and functorial, and satisfies étale hyperdescent, thus giving $\mathcal{F}^\dagger \in \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{S}\mathrm{m}\mathcal{A}\mathrm{ffd}^\dagger_{L,\text{ét}}, \mathscr{V})$; then we éh-hypersheafify it to get

$$\mathcal{F}^\dagger \in \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(\mathcal{R}\mathrm{ig}^\dagger_{L,\text{éh}}, \mathscr{V});$$

this satisfies éh-local-global compatibility for smooth affinoids cf. [17, 3.2.3] and [9, 3.24][12]. One has a natural morphism $R\Gamma(X, \mathcal{F}^\dagger) \to R\Gamma(\widehat{X}, \mathcal{F})$ for $X \in \mathcal{R}\mathrm{ig}^\dagger_L$, which is moreover an isomorphism if X is partially proper [9, Proposition 3.26]. This approach applies to for example sheaves $\mathcal{F} \in \{R\Gamma_{\mathrm{HK}}, R\Gamma_{\mathrm{dR}}(-/L)\}$, as well as $\mathcal{F} = F^\bullet R\Gamma_{\mathrm{inf}}(-/B^+_{\mathrm{dR}})$ if $L = C$, in the respective categories.

These two definitions are indeed compatible.

2.2.2 - Proposition. *Let $X \in \mathcal{R}\mathrm{ig}^\dagger_C$.*

(i) There is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$

$$R\Gamma_{\mathrm{HK}}(X) \simeq R\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X).$$

(ii) If $X = \mathfrak{X}_\eta$ for some $\mathfrak{X} \in \mathcal{R}\mathrm{ig}^{\mathrm{ss},b,\dagger}_C$, then there is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}})$

$$R\Gamma_{\mathrm{HK}}(X) \simeq R\Gamma_{\mathrm{rig}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}}).$$

(iii) There is a natural Hyodo-Kato isomorphism in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_C)$

$$\iota^{\mathrm{geom},\dagger}_{\mathrm{HK}} : R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} C \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}(X/C).$$

Proof. This is [9, Theorem 3.29]. Let us explain it briefly. The (i) is [19, 4.2.1 (iv), Lemma 4.17]. This together with local-global compatibility of $R\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(X)$ of *loc. cit.* implies (ii). For (iii), we reduce by éh-hyperdescent and [8, Corollary A.67 (ii)] to the basic semistable reduction case, we only need need to prove the statements for smooth dagger affinoids with presentation; but then this follows from the rigid-analytic case (2.1.17.1) and taking filtered colimits as in the construction of $\mathcal{F}^\dagger$ (2.2.1). □

2.2.3 - Proposition. *Let $Z \in \mathcal{R}\mathrm{ig}^\dagger_K$.*

(i) There is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_F)$

$$R\Gamma_{\mathrm{HK}}(Z) \simeq R\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(Z).$$

(ii) If $Z = \mathfrak{Z}_\eta$ for some $\mathfrak{Z} \in \mathcal{R}\mathrm{ig}^{\mathrm{ss},\dagger}_K$, then there is a natural isomorphism in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_F)$

$$R\Gamma_{\mathrm{HK}}(Z) \simeq R\Gamma_{\mathrm{rig}}(\mathfrak{Z}^0_1/\mathcal{O}^0_F).$$

(iii) There is a natural Hyodo-Kato morphism in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_K)$

$$\iota^{\mathrm{arith},\dagger}_{\mathrm{HK}} : R\Gamma_{\mathrm{HK}}(Z) \otimes_F K \xrightarrow{\simeq} R\Gamma_{\mathrm{dR}}(Z/C)$$

with a canonical natural retract $r^\dagger_Z$, which is an equivalence if Z has an η-éh-hypercovering by semistable formal schemes over $\mathcal{O}_K$.

[12] The construction 3.18 in *op. cit.* should be modified as follows: consider the fully faithful embedding $\iota : \mathcal{R}\mathrm{ig}^\dagger_L \to \mathrm{pro}(\mathcal{R}\mathrm{ig}_L)$ into the pro-system of rigid analytic analytic spaces, preserving products and étale coverings, yielding a morphism of topoi $(\iota^*, \iota_*) : \mathrm{pro}(\mathcal{R}\mathrm{ig}_L)^\sim_{\text{ét}} \to \mathcal{R}\mathrm{ig}^{\dagger,\sim}_{L,\text{ét}}$; consider the functor $l : \mathcal{R}\mathrm{ig}_L \to \mathrm{pro}(\mathcal{R}\mathrm{ig}_L)$ sending a rigid space to its constant system, also yielding a morphism of topoi $(l^*, l_*) : \mathrm{pro}(\mathcal{R}\mathrm{ig}_L)^\sim_{\text{ét}} \to \mathcal{R}\mathrm{ig}^\sim_{L,\text{ét}}$; then we have $\mathcal{F}^\dagger$ is the éh-hypercompletion of the étale sheaf (*resp.* of the étale hypersheaf) $\iota_* l^* \mathcal{F} \in \mathcal{R}\mathrm{ig}^{\dagger,\sim}_{L,\text{ét}}$.

Proof. The (i) is [19, 4.2.1 (ii), Lemma 4.14]. This together with local-global compatibility of $R\Gamma^{\mathrm{GK}}_{\mathrm{HK}}(Z)$ [17, Proposition 5.5] implies (ii). For (iii), we reduce as above to smooth dagger affinoids with presentation $Z = \mathfrak{Z}_\eta$ with $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss},\dagger}_K$; then we use (2.1.13) to construct $\iota^{\mathrm{arith},\dagger}_{\mathrm{HK}}$, and (2.1.14) to construct $r^\dagger_Z$, by passing to filtered colimits. □

2.2.4. Moreover, the identifications (2.2.2, ii) and (2.2.3, ii) are compatible with base change morphisms, as shown in their proofs. Then the same argument as in (2.1.21), now using base change properties of rational log-rigid cohomology instead of log-crystalline cohomology (2.1.17.2), we find that for $Z \in \mathcal{R}\mathrm{ig}^\dagger_K$ qcqs and for any $k \in \mathbf{N}$, there exists a finite extension of E and an $(k+1)$-truncated η-éh-hypercovering $\mathfrak{Z}'_{\leq k+1}$ of Z_E compatible with $\mathrm{Gal}(E/K)$-action and with splitting field E, such that there are isomorphisms

$$\tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_C) \simeq \tau^{\leq k} \lim_{\Delta_{k+1}} R\Gamma_{\mathrm{rig}}((\mathfrak{Z}'_{\leq k+1})^0_1/\mathcal{O}^0_{F_E}) \otimes^{\blacksquare}_{F_E} \breve{F} \simeq \tau^{\leq k} R\Gamma_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} \breve{F}$$

in $\mathscr{D}_{(\varphi,N)}(\mathrm{Mod}^{\blacksquare}_{\breve{F}[\mathscr{G}_K]})$.

2.2.5 - Remark. The advantage of using dagger version is that, for qcqs dagger variety X over C (*resp.* qcqs dagger variety Z over K), the condensed cohomology groups $H^i_{\mathrm{HK}}(X)$ (*resp.* $H^i_{\mathrm{HK}}(Z)$, $H^i_{\mathrm{dR}}(X/C)$, $H^i_{\mathrm{dR}}(Z/K)$) are finite-dimensional condensed vector spaces over $\breve{F}$ (*resp.* F, C, K) [17, Proposition 5.12, Proposition 5.6, the first paragraph à la Grosse-Klönne just before 5.1.1].

2.2.6 - Proposition. *Let $Z \in \mathcal{R}\mathrm{ig}^\dagger_K$ be qcqs. We have $H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z)$ for all $n \in \mathbf{N}$.*

Proof. The reasoning goes as (2.1.22, but there is one step which relatively much easier than (2.1.22): for any finite Galois extension E/K, one obtains

$$H^0(\underline{\mathscr{G}_E}, H^n_{\mathrm{HK}}(Z_E) \otimes^{\blacksquare}_{F_E} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z_E) \otimes_{F_E} H^0(\underline{\mathscr{G}_E}, B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z_E)$$

by classicality and finiteness of $H^n_{\mathrm{HK}}(Z_E)$ [17, Proposition 5.6 (1)] rather than using nuclearity. □

2.2.7 - Corollary. *Let $Z \in \mathcal{R}\mathrm{ig}^{(\dagger)}_K$ be partially proper. We have $H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) = H^n_{\mathrm{HK}}(Z)$.*

Proof. In the partially proper case, writing Z as the strictly increasing union of qcqs $U \in \mathcal{R}\mathrm{ig}^\dagger_K$, we have $R\Gamma_{\mathrm{HK}}(Z_C) \simeq R\varprojlim_U R\Gamma_{\mathrm{HK}}(U_C)$, hence by [9, Corollary A.67 (i)]

$$R\Gamma_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N} \simeq R\varprojlim_U (R\Gamma_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N}), \quad n \in \mathbf{N},$$

where $t^{-N} B_{\log,\leq N} \subset t^{-N} B_{\log} = t^{-N} B[U]$ denotes the subspace consisting of polynomials in U of degree $\leq N$ with coeffcients in $t^{-N} B_{\log}$, which is a $\breve{F}$-Fréchet space, being a finite direct sum of the $\breve{F}$-Fréchet space B. Now for any $i \in \mathbf{N}$, the system $\{H^i_{\mathrm{HK}}(U_C)\}_U$ is Mittag-Leffler by finite-dimensionality, hence so is the system $\{H^i_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N}\}_U$ for any $i, N \in \mathbf{N}$. Therefore, their $R^1 \varprojlim_U$ vanish, so

$$H^n_{\mathrm{HK}}(Z_C) \simeq \varprojlim_U H^n_{\mathrm{HK}}(U_C)$$
$$H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N} \simeq \varprojlim_U (H^n_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N}).$$

Applying the proposition (2.2.6) to each U, and recalling that $H^0(\underline{\mathscr{G}_K}, -)$ commutes with limits, we obtain

$$H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N}) = \varprojlim_U H^0(\underline{\mathscr{G}_K}, H^n_{\mathrm{HK}}(U_C) \otimes^{\blacksquare}_{\breve{F}} t^{-N} B_{\log,\leq N}) = \varprojlim_U H^n_{\mathrm{HK}}(U) = H^n_{\mathrm{HK}}(Z).$$

Taking the filtered colimit for $N \to +\infty$, then we are done. □

2.3 Arithmetic syntomic cohomology

Let us start with recalling a comparison result of [19] in the geometric situation, and then descend to the arithmetic case.

2.3.1. Geometric syntomic cohomology. For $X \in \mathcal{R}\mathrm{ig}_C$ and $r \in \mathbf{N}$, one defines its *(Bloch-Kato) syntomic cohomology* as

(2.3.1.1)
$$R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X,r) := \left[(R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r,N=0} \xrightarrow{\iota^{\mathrm{geom}}_{\mathrm{HK}}} R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}})/F^r \right].$$

For $\mathfrak{X} \in \mathcal{M}^{\mathrm{ss}}_C$, one has the *geometric Fontaine-Messing syntomic cohomology*

(2.3.1.2)
$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{X},r) := \left[R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \xrightarrow{\mathrm{can}} R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p}/F^r \right].$$

In this case, the two syntomic cohomology complexes are actually naturally isomorphic

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{X},r) \simeq R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\mathfrak{X}_\eta,r)$$

via the following natural commutative diagram [19, Proposition 5.3]

(2.3.1.3)
$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p}/F^r \arrow[d, "\simeq"', "\kappa\ {[19,\ \S3.3]}"] \\
(R\Gamma_{\mathrm{cris}}(\mathfrak{X}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{st}})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}) \otimes^{\blacksquare}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}})/F^r \arrow[d, "\simeq"', "{[19,\ \text{Proposition 3.27 (2)}]}"] \\
(R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F},1})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r,N=0} \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}})/F^r.
\end{tikzcd}$$

Now we deal with the arithmetic case.

2.3.2. Arithmetci syntomic cohomology. For $Z \in \mathcal{R}\mathrm{ig}_K$ and $r \in \mathbf{N}$, one defines its *arithmetic syntomic cohomology* as

(2.3.2.1)
$$R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(Z,r) := [R\Gamma_{\mathrm{HK}}(Z)^{\varphi=p^r,N=0} \xrightarrow{\iota^{\mathrm{arith}}_{\mathrm{HK}}} R\Gamma_{\mathrm{dR}}(Z/K)/F^r].$$

For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$, one has the *arithmetic Fontaine-Messing syntomic cohomology*

(2.3.2.2)
$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{Z},r) := \left[R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \xrightarrow{\mathrm{can}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}/F^r \right].$$

They are similarly defined for dagger varieties.

2.3.3 - Proposition. *For $\mathfrak{Z} \in \mathcal{M}^{\mathrm{ss}}_K$, one has a natural isomorphism*

$$R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{Z},r) \simeq R\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\mathfrak{Z}_\eta,r).$$

Proof. Let L be the splitting field of $\mathfrak{Z}$ over $\mathcal{O}_K$ (or any other field L such that $\mathfrak{Z}$ actually has a semistable structural morphism to $\mathcal{O}_L$). Consider the following natural commutative diagram

(2.3.3.1)

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{bch}"', "\simeq"] \arrow[rrddd, bend left, "c^{\urcorner}"] \arrow[rrddd, bend right, "c_{\llcorner}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[r, equal] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{bch}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}/F^r \arrow[d, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[d, "i^*"', "\simeq"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^{\times})_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[dd, "\mathrm{bch}"', "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^{\times})_{\mathbf{Q}_p}/F^r \arrow[dd, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r_L^{\mathrm{PD},0})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[d, "p_0^*"', "\simeq"] & & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L)/F^r
\end{tikzcd}$$

where the left upper vertical arrow is an isomorphism by a standard Frobenius argument [16, Proof of Lemma 5.9], the left middle vertical arrow i^* is an isomorphism by [36, Lemma 4.2], and the left lower vertical arrow p_0^* is an isomorphism because Frobenius is highly nilpotent on T ; the right upper vertical arrow is an isomorphism by Beilinson's identification [4, Theorem 1.9.2], $\mathfrak{Z}$ endowed with the log-structure induced by its special fibre being smooth over $\mathcal{O}_L^{\times}$. Every commutativity except that of the middle eye-form cavity is clear.

It remains to show that there is a natural equivalence $c^{\urcorner} \simeq c_{\llcorner}$. Since everything here lies in $\mathscr{D}(\mathrm{Mod}_L^{\blacksquare}) \hookrightarrow \mathscr{D}(\mathrm{Mod}_L^{\blacksquare})^{\mathscr{G}_L}$, by adjunction it amounts to proving the existence of natural equivalence between their post-compositions with $R\Gamma_{\mathrm{dR}}(Z/L) \to R\Gamma_{\mathrm{dR}}(Z/L) \otimes_L^{\blacksquare} B^+_{\mathrm{pdR}}$, i.e. the diagram

(2.3.3.2)

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{bch}"', "\simeq"] \arrow[rrddd, bend left, "c^{\urcorner}"] \arrow[rrddd, bend right, "c_{\llcorner}"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[r, equal] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[d, "\mathrm{bch}"'] & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[d, "i^*"', "\simeq"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^{\times})_{\mathbf{Q}_p} \arrow[dd, "\mathrm{bch}"', "\simeq"] & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r_L^{\mathrm{PD},0})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[d, "p_0^*"', "\simeq"] & & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r,N=0}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L}) \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes_L^{\blacksquare} B^+_{\mathrm{pdR}}
\end{tikzcd}$$

commutes, where we put $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C$. We need to identify $c^{\urcorner}$ and $c_{\llcorner}$ respectively with some more explicit maps, then show the natural equivalence between them.

First, let us begin with identifying $c^{\urcorner}$. Consider the following diagram

(2.3.3.3)

$$\begin{tikzcd}
 & R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[rr, "\mathrm{can}"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \arrow[dd, "\mathrm{bch}"', "\simeq"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[ur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] \arrow[rdd, "c^{\urcorner}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \arrow[urr, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"'] & & \\
 & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^{\times})_{\mathbf{Q}_p} \arrow[d, "\mathrm{bch}"', "\simeq"] & & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B^+_{\mathrm{pdR}} \\
 & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[rr, "\mathrm{can}"] \arrow[urr, "\mathrm{bch}"] & & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}} \arrow[u, "\mathrm{bch}"', "\simeq"]
\end{tikzcd}$$

where all blocks commute, the right diamond-shaped block commutes by crystalline-theoretic base change compatibility.

Next, we consider the diagram

(2.3.3.4)
$$
\begin{tikzcd}
 & R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \arrow[dd, bend left, "\mathrm{bch}", "\simeq"'] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[ur, "\mathrm{bch}"] \arrow[ddr, "c^{\urcorner}"] & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[d, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \\
 & & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B^+_{\mathrm{pdR}} \\
 & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[ur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}} \arrow[u, "\mathrm{bch}", "\simeq"']
\end{tikzcd}
$$

whose upper right block commutes by the definition of $\iota^{\mathrm{geom}}_{\mathrm{HK}}$.

Next, the diagram

(2.3.3.5)
$$
\begin{tikzcd}
 & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[urr, "\mathrm{bch}"] \arrow[ddrr, "c^{\urcorner}"] & & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[d, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] \\
 & & & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B^+_{\mathrm{pdR}} \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[uurr, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] & R\Gamma(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[uurr, "\mathrm{bch}"] \arrow[r, "\iota^{\mathrm{arith}}_{\mathrm{HK}}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \arrow[ur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L) \otimes^{\blacksquare}_L B^+_{\mathrm{pdR}} \arrow[u, "\mathrm{bch}", "\simeq"']
\end{tikzcd}
$$

commutes.

Hence, to identify $c^{\urcorner}$ and $c_{\llcorner}$, we are left to show the commutativity of the diagram

(2.3.3.6)
$$
\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[r, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"', "\simeq"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X})^{\varphi=p^r}_{\mathbf{Q}_p} & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1)^{\varphi=p^r}_{\mathbf{Q}_p} \arrow[d, "i^*"', "\simeq"] & (R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^0_1/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/r^{\mathrm{PD},0}_L)^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[d, "p_0^*"', "\simeq"] & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})^{\varphi=p^r, N=0}_{\mathbf{Q}_p} \arrow[uur, "\mathrm{bch}"] \arrow[r, "\mathrm{can}"] & R\Gamma(\mathfrak{Z}^0_1/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[uur, "\mathrm{bch}"] &
\end{tikzcd}
$$

particularly that of the left block.

For sufficiently large n, we have factorisations of Frobenius $\mathrm{Frob}^n : \mathfrak{Z}_1 \xrightarrow{F_n} \mathfrak{Z}^0_1 \hookrightarrow \mathfrak{Z}_1$ and $\mathrm{Frob}^n : \mathfrak{Z}^0_1 \hookrightarrow$

$\mathfrak{z}_1 \xrightarrow{F_n} \mathfrak{z}_1^0$. Consider the commutative diagram of log-schemes

(2.3.3.7)
$$\begin{tikzcd}
\mathfrak{X}_1^0 \arrow[r, hook] \arrow[d, "\theta_3^0"] & \mathfrak{X}_1 \arrow[r] \arrow[d, "F_n\theta_3"] & \overline{S}_1^{\times} \arrow[d, "F_{L,n}\theta_L"] \\
\mathfrak{z}_1^0 \arrow[r, "\mathrm{Frob}^n"] & \mathfrak{z}_1^0 \arrow[r] & S_{L,1}^0
\end{tikzcd}$$

which we can denote by π. These data π determine a lifting $i_{l_\pi}^* : r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}_{L,1}^{\times}$.

Consider the diagram before taking Frobenius fixed points

(2.3.3.8)
$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{z})_{\mathbf{Q}_p} \arrow[rr, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[dl, "\simeq", "\kappa^*_{l_\pi}"'] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0)_{\mathbf{Q}_p} \arrow[d, "i^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\pi,\mathrm{st}})^{N-\mathrm{nilp}}_{\mathbf{Q}_p} & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}^0_{\breve{F}})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}} \arrow[u, "\varepsilon^{\mathrm{st}}_{\mathrm{HK}}", "\simeq"'] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p} \arrow[d, bend right, "p_0^*"'] & R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0/r_L^{\mathrm{PD},0})^{N-\mathrm{nilp}}_{\mathbf{Q}_p} \arrow[l, "(\mathrm{Frob}^n)^*"'] \arrow[u, "\pi^*"] & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[u, bend right, "\iota_0"'] \arrow[uurr, bend right, "\mathrm{bch}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0/\mathcal{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[l, "(\mathrm{Frob}^n)^*", "\simeq"'] \arrow[u, "\iota_0"] &
\end{tikzcd}$$

where ι_0 is the (unique) natural φ-equivariant F_L-linear Hyodo-Kato section; the lower right block commutes by the definition of $\varepsilon^{\mathrm{st}}_{\mathrm{HK}}$; the lower left circuit commutes (even becomes an isomorphism) after taking $(-)^{N=0}$; the upper left vertical base change morphism becomes an isomorphism after taking $(-)^{\varphi=p^r}$; and i^* becomes isomorphism after taking $(-)^{N=0}$ of the target.

For the commutativity of the $(-)^{\varphi=p^r}$ invariant of the upper left block of (2.3.3.8), we may look at the following commutative diagram

(2.3.3.9)
$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{z}) \arrow[rrr, bend left, "\mathrm{bch}"] \arrow[d, "\mathrm{bch}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{z}) \arrow[l, "(\mathrm{Frob}^n)^*"'] \arrow[d, "\mathrm{bch}"] & & R\Gamma_{\mathrm{cris}}(\mathfrak{X})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B^+_{\mathrm{cris}}} B^+_{\mathrm{st}} \arrow[ddl, bend left, "\kappa^*_{l_\pi}"] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0) \arrow[d, "i^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0) \arrow[d, "i^*"] & & \\
R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0/r_L^{\mathrm{PD},0}) & R\Gamma_{\mathrm{cris}}(\mathfrak{z}_1^0/r_L^{\mathrm{PD},0}) \arrow[l, "(\mathrm{Frob}^n)^*"'] \arrow[r, "\pi^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\pi,\mathrm{st}})_{\mathbf{Q}_p} &
\end{tikzcd}$$

which is clear by base change maps.

Putting these altogether, we obtain the commutativity of (2.3.3.6), which is natural since it is insensitive to the $n \gg 0$ chosen, cf. [19, Proof of Theorem 2.22, Independence of the choice of π and ξ_0]. This completes the proof that $c^{\urcorner} \simeq c_{\llcorner}$ after $-\otimes^{\blacksquare}_L B^+_{\mathrm{pdR}}$, whence $c^{\urcorner} \simeq c_{\llcorner}$ by taking derived $\mathscr{G}_L$-fixed points. □

2.3.4 - Remark. Given a $\mathcal{O}_{F_L}$-class $l \in (\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$ associated with $i_l^* : r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}_{L,1}^{\times}$ lifting $r_L^{\mathrm{PD}} \twoheadrightarrow \mathcal{O}^0_{F_L,1}$

along $\mathcal{O}_{L,1}^{\times} \twoheadrightarrow \mathcal{O}_{F_L,1}^{0}$, and assuming that $v(l) = \frac{1}{e}$, there is another approach providing an equivalence

$$\alpha_l : R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(\mathfrak{Z}, r) \simeq R\Gamma_{\mathrm{syn}}^{\mathrm{BK}}(\mathfrak{Z}_\eta, r),$$

a priori depending on the choice of l, which however will be of other interests. It consists of finding a $\underline{\mathscr{G}_L}$-equivariant morphism $\varepsilon_{\mathrm{HK},l}^{\mathrm{st}} : R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B_{\mathrm{cris}}^+} B_{\mathrm{st}}^+$ equivalent to $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$. For the pair (π, n) associated with the data (2.3.3.7) such that $\frac{p^n}{e} \geq 1$, the $\varepsilon_{\mathrm{HK},l}^{\mathrm{st}}$ is defined as the first row of the following commutative diagram

$$\begin{tikzcd}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[r, "\iota_{0,l}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}^{N-\mathrm{nilp}} \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l,\mathrm{st}})_{\mathbf{Q}_p}^{N-\mathrm{nilp}} & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B_{\mathrm{cris}}^+} B_{\mathrm{st}}^+ \arrow[l, "\simeq"'] \\
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0) \arrow[u, "(\mathrm{Frob}^n)^*"] \arrow[r, "\iota_0"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}^{N-\mathrm{nilp}} \arrow[u, "(\mathrm{Frob}^n)^*"] \arrow[r, "\pi^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\pi,\mathrm{st}})_{\mathbf{Q}_p}^{N-\mathrm{nilp}} \arrow[u, "\mu_n^*"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \otimes^{\blacksquare}_{B_{\mathrm{cris}}^+} B_{\mathrm{st}}^+ \arrow[l, "\simeq"'] \arrow[u, equal]
\end{tikzcd}$$

where the third vertical arrow is induced by

$$\mu_n : \widehat{A}_{l_\pi,\mathrm{st}} \to \widehat{A}_{l,\mathrm{st}}, \quad t_{a^{p^n}} \mapsto (t_a)^{p^n}, \quad \forall a \in \tau_{\frac{1}{e}} := \left\{ a \in (\mathcal{O}_C \backslash \{0\})/(1 + \mathfrak{m}_C) \middle| v(a) = \frac{1}{e} \right\}.$$

All but the left square commute by base change compatibility; that of the left follows from the invertibility of Frob^* on $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ and the uniqueness of φ-equivariant section $\iota_{0,l}$ [19, Theorem 2.12 (2)]. Thus it provides the homotopy between $\varepsilon_{\mathrm{HK},l}^{\mathrm{st}}$ and $\varepsilon_{\mathrm{HK}}^{\mathrm{st}}$.

2.3.5 - Corollary. *For $Z \in \mathcal{R}\mathrm{ig}_K$, one has a natural isomorphism*

$$R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(Z, r) \simeq R\Gamma_{\mathrm{syn}}^{\mathrm{BK}}(Z, r).$$

Proof. The equivalence produced in the proposition being natural and not involving any special choices of elements of L, it glues to the desired global equivalence. □

2.3.6. Therefore, our new arithmetic syntomic cohomology agrees with Colmez-Nizioł's arithmetic syntomic cohomology by [17, Proposition 4.32], although we still do not know whether the Hyodo-Kato morphism agree with theirs.

We will denote $R\Gamma_{\mathrm{syn}}(Z, r) := R\Gamma_{\mathrm{syn}}^{\mathrm{BK}}(Z, r)$ from now on.

3 Syntomic descent spectral sequence

3.1 Syntomic-proétale period map

3.1.1. Fundamental exact sequence in p-adic Hodge-theory. For $r \in \mathbf{N}$ and $i \geq 0$, we have a strict exact sequence of topological abelian groups

$$0 \to \mathbf{Q}_p(r) \to (t^{-i} B_{\log})^{\varphi = p^r, N = 0} \to t^{-i} B_{\mathrm{dR}}^+ / t^r B_{\mathrm{dR}}^+ \to 0$$

For any $X \in \mathcal{R}\mathrm{ig}_C$, the above exact sequence upgrades to an exact sequence of sheaves on $X_{\mathrm{pro\acute{e}t}}$

$$0 \to \mathbf{Q}_p(r) \to (t^{-i} \mathbf{B}_{\log})^{\varphi = p^r, N = 0} \to t^{-i} \mathbf{B}_{\mathrm{dR}}^+ / t^r \mathbf{B}_{\mathrm{dR}}^+ \to 0,$$

where $\mathbf{B}_{\log} = \mathbf{B}[U]$ [9, Proposition 2.25, Definition 2.27] with $\underline{\mathscr{G}_K}$-action $g(U) = U + \log[\frac{g(p^\flat)}{p^\flat}]$, Frobenius action $\varphi(U) = pU$ and monodromy action $N = -\frac{d}{dU}$. This can be also written as a bicartesian square (or a fibre square in the derived category)

$$\begin{tikzcd} \mathbf{Q}_p(r) \arrow[r] \arrow[d] & (t^{-i}\mathbf{B}_{\log})^{\varphi=p^r, N=0} \arrow[d] \\ t^r\mathbf{B}_{\mathrm{dR}}^+ \arrow[r] & t^{-i}\mathbf{B}_{\mathrm{dR}}^+. \end{tikzcd}$$

Taking its cohomology and then $i \to +\infty$, one obtains a fibre square

$$\begin{tikzcd} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\log})[\frac{1}{t}]^{\varphi=p^r, N=0} \arrow[d] \\ R\Gamma_{\mathrm{pro\acute{e}t}}(X, t^r\mathbf{B}_{\mathrm{dR}}^+) \arrow[r] & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{dR}}^+)[\frac{1}{t}]. \end{tikzcd} \tag{3.1.1.1}$$

refining the fibre square

$$\begin{tikzcd} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\log}[\frac{1}{t}])^{\varphi=p^r, N=0} \arrow[d] \\ R\Gamma_{\mathrm{pro\acute{e}t}}(X, t^r\mathbf{B}_{\mathrm{dR}}^+) \arrow[r] & R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{dR}}) \end{tikzcd}$$

which we obtain by first taking $i \to +\infty$ then its cohomology.

3.1.2. Compatibilities with the geometric Hyodo-Kato morphism. Assume that $X = Z_C$ for some $Z \in \mathcal{R}\mathrm{ig}_K$. Then there are a natural isomorphism $R\Gamma_{\mathrm{pro\acute{e}t}}(X, t^\bullet\mathbf{B}_{\mathrm{dR}}^+) \simeq \mathrm{Fil}^\bullet(R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}})$ compatible with filtrations [9, Theorem 5.2] and a natural φ-equivariant morphism $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}) \leftarrow R\Gamma_B(X) \simeq (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log})^{N=0}$ [9, Theorem 4.1]. They are compatible with the geometric Hyodo-Kato morphism [9, Theorem 5.3] for $r \in \mathbf{N}$, i.e. there is a natural commutative diagram

$$\begin{tikzcd} R\Gamma_{\mathrm{pro\acute{e}t}}(X, t^{-r}\mathbf{B}_{\mathrm{dR}}^+) & R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} t^{-r}B_{\mathrm{dR}}^+ \arrow[l] \\ R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}) \arrow[u] & (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log})^{N=0}. \arrow[l] \arrow[u, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \end{tikzcd} \tag{3.1.2.1}$$

If moreover X is qcqs, then $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{dR}}) \simeq R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}}$, and $R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\log}[\frac{1}{t}])^{N=0} \xleftarrow{\simeq} R\Gamma(X, \mathbf{B}[\frac{1}{t}]) \xleftarrow{\simeq} (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log}[\frac{1}{t}])^{N=0}$; in general, by covering X with qcqs opens, we obtain natural equivalences and morphisms

$$\begin{aligned} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\mathrm{dR}}) &\simeq \lim_{U \text{ qcqs}} R\Gamma_{\mathrm{pro\acute{e}t}}(U, \mathbf{B}_{\mathrm{dR}}) \\ &\simeq \lim_{U \text{ qcqs}} (R\Gamma_{\mathrm{dR}}(U/K) \otimes_K^{\blacksquare} B_{\mathrm{dR}}) \quad \leftarrow (\lim_{U \text{ qcqs}} R\Gamma_{\mathrm{dR}}(U/K)) \otimes_K^{\blacksquare} B_{\mathrm{dR}} \\ &\qquad\qquad \simeq R\Gamma_{\mathrm{dR}}(X) \otimes_K^{\blacksquare} B_{\mathrm{dR}} \end{aligned} \tag{3.1.2.2}$$

$$\begin{aligned} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\log}[\frac{1}{t}])^{N=0} &\simeq \lim_{U \text{ qcqs}} R\Gamma_{\mathrm{pro\acute{e}t}}(U, \mathbf{B}_{\log}[\frac{1}{t}])^{N=0} \\ &\simeq \lim_{U \text{ qcqs}} (R\Gamma_{\mathrm{HK}}(U) \otimes_{\breve{F}}^{\blacksquare} B_{\log}[\frac{1}{t}])^{N=0} \quad \leftarrow ((\lim_{U \text{ qcqs}} R\Gamma_{\mathrm{HK}}(U)) \otimes_{\breve{F}}^{\blacksquare} B_{\log}[\frac{1}{t}])^{N=0} \\ &\qquad\qquad \simeq (R\Gamma_{\mathrm{HK}}(X) \otimes_{\breve{F}}^{\blacksquare} B_{\log}[\frac{1}{t}])^{N=0} \end{aligned} \tag{3.1.2.3}$$

compatible with geometric Hyodo-Kato morphism.

3.1.2.4 - Lemma. *There is a natural (φ,N)-equivariant morphism*

$$R\Gamma_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve F} B_{\log}\to R\Gamma(X,\mathbf{B}_{\log})$$

inducing after taking $(-)^{N=0}$ the φ-equivariant morphisms in (3.1.2.3).

Proof. We will go through the proofs of [9, Theorem 4.1, Theorem 4.3]. Working locally, we may assume $X=\mathfrak{X}_\eta$ where $\mathfrak{X}\in\mathcal{M}^{\mathrm{ss},b}_C$. For compact intervals $I\subset(0,+\infty)$, consider the (φ,N)-equivariant isomorphisms

$$\begin{aligned} R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{\breve F} B_{\log,I} &\xrightarrow[\simeq]{\varepsilon^{\mathrm{st}}_{\mathrm{HK}}} R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}})\otimes^{\blacksquare}_{A_{\mathrm{cris}}} B_{\log,I}\\ &\simeq (R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}})\otimes^{\blacksquare}_{A_{\mathrm{cris}}} B_I)\otimes^{\blacksquare}_{B} B_{\log,I}\\ &\to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_I)\otimes^{\blacksquare}_{B_I} B_{\log,I}\\ &\xrightarrow{\simeq} R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\log,I}) \end{aligned} \tag{3.1.2.5}$$

where we have used [9, Proof of Theorem 4.3] for the third morphism. By taking limit over all compact intervals $I\subset(0,+\infty)$, taking into account the fact that $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}$ is represented by a bounded complex of $\breve F$-Banach spaces [9, Proof of Theorem 3.15 (ii)], we see that

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}\otimes^{\blacksquare}_{\breve F}\widehat{B}_{\log}\to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\widehat{\mathbf{B}}_{\log}).$$

We conclude the construction by taking $(-)^{N-\mathrm{nilp}}$, using the N-nilpotency on $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1^0/\mathcal{O}^0_{\breve F})_{\mathbf{Q}_p}$ [9, Proof of Theorem 3.15 (ii)]. Its compatibility with the given isomorphism is checked just as in [9, Proof of Theorem 4.1]. □

3.1.2.6 - Corollary. *For $r\in\mathbf{N}$, there is a natural commutative diagram*

$$\begin{tikzcd} R\Gamma_{\mathrm{pro\acute{e}t}}(X,t^{-r}\mathbf{B}^+_{\mathrm{dR}}) & R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K t^{-r}B^+_{\mathrm{dR}} \arrow[l] \\ R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\log}) \arrow[u] & R\Gamma_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve F} B_{\log}. \arrow[l] \arrow[u, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"'] \end{tikzcd} \tag{3.1.2.7}$$

which is compatible with (3.1.2.1) after taking $(-)^{N=0}$. □

3.1.3. syntomic-proétale period map. By (3.1.2), the fundamental exact sequence (3.1.1) induces natural morphisms of fibre sequences

$$\begin{aligned} R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) &\xrightarrow{\simeq}\left[\begin{array}{ccc} & & R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\log}[\tfrac{1}{t}])^{\varphi=p^r,N=0}\\ & & \big\downarrow{\scriptstyle\mathrm{can}}\\ R\Gamma_{\mathrm{pro\acute{e}t}}(X,t^r\mathbf{B}^+_{\mathrm{dR}}) & \longrightarrow & R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\mathrm{dR}}) \end{array}\right]\\ &\leftarrow\left[\begin{array}{ccc} & & (R\Gamma_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve F} B_{\log}[\tfrac{1}{t}])^{\varphi=p^r,N=0}\\ & & \big\downarrow{\scriptstyle\iota^{\mathrm{geom}}_{\mathrm{HK}}}\\ \mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B_{\mathrm{dR}}) & \longrightarrow & R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B_{\mathrm{dR}} \end{array}\right]\\ &\leftarrow\left[\begin{array}{ccc} & & (R\Gamma_{\mathrm{HK}}(X)\otimes^{\blacksquare}_{\breve F} B^+_{\mathrm{st}})^{\varphi=p^r,N=0}\\ & & \big\downarrow{\scriptstyle\iota^{\mathrm{geom}}_{\mathrm{HK}}}\\ \mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}}) & \longrightarrow & R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \end{array}\right]\\ &= R\Gamma_{\mathrm{syn}}(X,r) \end{aligned} \tag{3.1.3.1}$$

where the second morphism becomes an isomorphism if X is qcqs. The diagram (3.1.3.1) composes to a geometric syntomic-proétale comparison morphism $\rho_{\mathrm{syn}}^{\mathrm{geom}}: R\Gamma_{\mathrm{syn}}(X,r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$ between syntomic cohomology and proétale cohomology, which induces an arithmetic one $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ by taking $\underline{\mathscr{G}_K}$-invariants as follows

$$
\begin{tikzcd}
R\Gamma_{\mathrm{syn}}(Z,r) \arrow[r, dashed, "\rho_{\mathrm{syn}}^{\mathrm{arith}}"] \arrow[d] & R\Gamma_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r)) \arrow[d] \\
R\Gamma_{\mathrm{syn}}(X,r) \arrow[r, "\rho_{\mathrm{syn}}^{\mathrm{geom}}"] & R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)).
\end{tikzcd}
\tag{3.1.3.2}
$$

3.1.4 - Remark. By (3.1.2.7), the maps in the construction of the geometric syntomic-proétale comparison map (3.1.3.1) are actually induced by maps of data (A,B,ι,r), where $A \in \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_{\mathbf{Q}_p}^{\mathrm{cond}})$, $B \in \mathscr{DF}(\mathrm{Mod}_{\mathbf{Q}_p}^{\mathrm{cond}}) := \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathscr{D}(\mathrm{Mod}_{\mathbf{Q}_p}^{\mathrm{cond}}))$, $\iota = \iota_{\mathrm{HK}}^{\mathrm{geom}}$ and $r \in \mathbf{N}$.

3.1.5. Fontaine-Messing period map. We would like to compare the above constructed period maps (3.1.3.2) with another collection of period maps, the *Fontaine-Messing period maps*

$$\alpha_r^{\mathrm{FM}}: R\Gamma_{\mathrm{syn}}(X,r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)),$$

which are defined by globalising the Fontaine-Messing period maps on semistable models $\alpha_r^{\mathrm{FM}}: R\Gamma_{\mathrm{syn}}(\mathfrak{X},r) \to R\Gamma_{\mathrm{\acute{e}t}}(\mathfrak{X},\mathbf{Z}_p(r))_{\mathbf{Q}_p} \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X},\mathbf{Q}_p(r))$, both in the arithmetic and geometric cases [17, §7] (recall the definition of the Fontaine-Messing syntomic cohomology (2.3.1.2), (2.3.2.2)). They are constructed in a natural way so that they satisfy Galois equivariance: if $X = \mathfrak{X}_\eta$ with $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C$ such that $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L, then $\alpha_r^{\mathrm{FM,geom}}$ are $\underline{\mathscr{G}_L}$-equivariant; in particular, if $X = Z \otimes_K C$, then $\alpha_r^{\mathrm{FM,geom}}$ is $\underline{\mathscr{G}_K}$-equivariant.

Whether these seemingly two types of period maps are homotopic, i.e. $\rho_{\mathrm{syn}} = \alpha^{\mathrm{FM}}$ in the underlying homotopy category (both in geometric and arithmetic cases), is closely related to the uniqueness of (geometric) p-adic period morphisms addressed by Nizioł [47, 48] in the algebraic setting and by Sally Gilles [29] in the formally algebraic setting. Although the latter have treated seemingly only the case of proper semistable models, its proofs contain a great amount of local constructions which we will employ to obtain the following proposition.

3.1.6 - Proposition. *The geometric syntomic-proétale period map $\rho_{\mathrm{syn}}^{\mathrm{geom}}$ is naturally homotopic to the geometric Fontaine-Messing period map $\alpha_r^{\mathrm{FM,geom}}$.*

Proof. By éh-hyperdescent, one may reduces to the case of $X = \mathfrak{X}_\eta$, where $\mathfrak{X} \in \mathcal{M}_C^{\mathrm{ss},b}$ is affine and descends to $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L. Essentially by construction, it suffices to proves the natural commutativity of the following diagram

$$
\begin{tikzcd}
R\Gamma_{\mathrm{\acute{e}t}}(\mathfrak{X}_\eta,\mathbf{Z}_p(r))_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_\eta,\mathbf{B}_I) \arrow[r] & R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_\eta,\mathbf{B}_{\mathrm{dR}}) \\
R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(\mathfrak{X},r) \arrow[u, "\alpha_r^{\mathrm{FM,geom}}"] \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \arrow[u, "\mathrm{can}"] \arrow[ur, "\mathrm{can}"] &
\end{tikzcd}
$$

where we have used the canonical isomorphism $R\Gamma(\mathfrak{X}) \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})$ to identify the absolute log-crystalline cohomology defining the Fontaine-Messing syntomic cohomology. Here, the vertical canonical morphism can is defined in [9, Proof of Theorem 4.3] (as was used in (3.1.2.5) for the third morphism), which actually factors as $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A_{\mathrm{cris}}^{\times})_{\mathbf{Q}_p} \xrightarrow{\gamma^{\mathrm{CK}}} R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_\eta,\mathbf{A}_{\mathrm{inf}}) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_\eta,\mathbf{B}_I)$ where the first is induced by Cesnavicius-Koshikawa's φ-equivariant comparison isomorphism [10, Theorem 5.4] and the latter is induced by the canonical map of proétale period sheaves $\mathbf{A}_{\mathrm{inf}} \to B_I$. Moreover, the slanted canonical morphism can is induced by the morphisms of sites $(\mathfrak{X}_\eta/B_{\mathrm{dR},m}^+)_{\mathrm{inf}} \to (\mathfrak{X}/A_{\mathrm{cris}}^{\times})_{\mathrm{cris}}$. The right triangle commutes by [9, Proposition 5.11], so it remains to show that the left square commutes.

The upper left arrow being induced by maps of proétale sheaves $\mathbf{Z}_p(r) \to \mathbf{A}_{\mathrm{inf}} \to B_I$, we are thus reduced to showing the natural commutativity of the following diagram

$$\begin{tikzcd} R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{A}_{\mathrm{inf}}) \\ R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{X}, r) \arrow[u, "\alpha_r^{\mathrm{FM,geom}}"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}})_{\mathbf{Q}_p}. \arrow[u, "\gamma^{\mathrm{CK}}"'] \end{tikzcd}$$

For this, we apply the aforementioned local constructions in Gilles's works [29]: she constructed in *op. cit.*, §8 for such $\mathfrak{X}$ a natural Lazard type period map $\alpha_r^0 : R\Gamma_{\mathrm{syn}}(\mathfrak{X}, r) \to R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p}$ such that $\tau^{\leq r}\alpha_r^0$ is an equivalence[13], then she proved a natural identification $\alpha_r^0 \simeq \alpha_r^{\mathrm{FM,geom}}$ in *op. cit.*, Théorème 9.1[14] on the one hand, and that on the other hand the analogue of the above commutative diagram

$$\begin{tikzcd} R\Gamma_{\text{ét}}(\mathfrak{X}_\eta, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \arrow[r] & R\Gamma_{\text{proét}}(\mathfrak{X}_\eta, \mathbf{A}_{\mathrm{inf}}) \\ R\Gamma^{\mathrm{FM}}_{\mathrm{syn}}(\mathfrak{X}, r) \arrow[u, "\alpha_r^0"] \arrow[r, "\mathrm{can}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/A^{\times}_{\mathrm{cris}})_{\mathbf{Q}_p}. \arrow[u, "\gamma^{\mathrm{CK}}"'] \end{tikzcd}$$

commutes by the step (ii) in the proof of *op. cit.*, Lemme 9.11 (this step was a purely local calculation)[15]; hence we are done. □

3.1.7 - Corollary. *The arithmetic syntomic-proétale period map $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ is homotopic to the arithmetic Fontaine-Messing period map $\alpha_r^{\mathrm{FM,arith}}$.*

Proof. This follows from the Galois descent construction of $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ and (3.1.6), since when composed with the natural map $R\Gamma_{\text{proét}}(Z, \mathbf{Q}_p(r)) \to R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r))$, the period maps $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ and $\alpha_r^{\mathrm{FM,arith}}$ give rise to homotopic morphisms $\rho_{\mathrm{syn}}^{\mathrm{geom}} \simeq \alpha_r^{\mathrm{FM,geom}}$ by (3.1.6). □

3.1.8 - Corollary. *Let $Z \in \mathcal{R}\mathrm{ig}_K$ and $r \in \mathbf{N}$. The the syntomic-proétale comparison maps $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ for Z and $\rho_{\mathrm{syn}}^{\mathrm{geom}}$ for Z_C become isomorphisms after truncation $\tau^{\leq r}$.*

Proof. The geometric case follows from (!) [9, Theorem 7.2]. But alternatively, the geometric and arithmetic cases follow respectively from the above proposition (3.1.6) and corollary ((3.1.6)), since we already know that the truncated isomorphisms holds for the Fontaine-Messing period maps [17, Corollary 7.3] for smooth varieties, which extends to singular cases by éh-hyperdescent. □

[13] In *op. cit.*, §8, the morphism α_r^0, or more precisely its local and integral version $\alpha_{r,\Sigma,\lambda}$, was constructed directly as quasi-isomorphisms of $\tau^{\leq r}$*-truncated complexes* but not for untruncated complexes , see the formula just before *op. cit.*, Proposition 8.8. This formula was deduced from *op. cit.*, Proposition 8.6. But in fact, if we carefully look at the proof of this last proposition, we find that:

- Firsly, multiplication-by-$t^\bullet$ morphism is defined even before the truncation, though it becomes an quasi-isomorphism only after the truncation;
- Secondly, the multiplication-by-t^r morphism and the morphism β there are quasi-isomorphisms even before the truncation, cf. *op. cit.*, Lemma 5.3 and Lemma 5.8 respectively.

Therefore, as it turns out, the *op. cit.*, Proposition 8.6 actually shows the existence of a forward morphism $\beta' : \mathrm{Kosz}(\varphi, \partial, F^r R_{\Sigma,\Lambda}^{[u,v]}) \to \mathrm{Kosz}(\varphi, \Gamma_{\Sigma,\Lambda}, R_{\Sigma,\Lambda}^{[u,v]}(r))$ which becomes a quasi-isomorphism after the truncation $\tau^{\leq r}$. Taking respective Frobenius eigenspaces, one obtains the untruncated map $\alpha^0_{r,\Sigma,\Lambda}$, which is then globalised to be the untruncated morphism α_r^0 in the our text.

[14] Again, her statement is for $\tau^{\leq r}$-truncated complexes, but what she proved was not that stronger. In fact, in her proof:

- Firstly, the properness hypothesis was not used there;
- Secondly, the morphism *op. cit.*, (33) there which rewrites the (truncated) syntomic cohomology as (truncated) Frobenius eigenspace of $\mathbf{A}_{\mathrm{cris}}(-)$ construction holds for untruncated complexes;
- Thirdly, the proof of identification reduces to that of *op. cit.*, Lemma 9.2, which says that $\tau^{\leq r}\alpha^{\mathrm{FM}}_{r,\Sigma,\Lambda} \simeq \tau^{\leq r}\alpha^0_{r,\Sigma,\Lambda}$, actually stays valid for untruncated complexes, see *op. cit.*, Proposition 7.5, or look at the original source of ideas [16, Theorem 4.16].

[15] Let us use the notation of *loc. cit.*. As mentioned in the footnote (13), the Lazard type period map α_r^0 well-defined even before the truncation, especially because the morphism β is a quasi-isomorphism even before the truncation. Therefore, in the diagram *loc. cit.*, (55), it is unnecessary to take the truncation $\tau^{\leq r}$ in order to obtain commutativity, the diagram itself untruncated is already commutative by the same proof, chiefly thanks to the commutativity of the diagram *loc. cit.*, (57), which was no more than a direct computation showing that Bhatt-Morrow-Scholze's $\widetilde{\beta}$ is compatible with the morphism β' of *loc. cit.*.

3.1.9 - Remark. Although one might argue that the corollary (3.1.6) can be proven directly by a geometric truncated quasi-isomorphism statement (say of Bosco [9]) plus Galois descent, without passing through the identification (3.1.7), we should stress that this Galois descent step might not work naively due to one important difference between the rigid-analytic setting and the algebraic setting that, the Galois action on the rigid-analytic geometric syntomic cohomology not being smooth nor having vanishing higher continuous Galois cohomology groups, taking Galois fixed points is not an exact operation on the complex, which is contrary to the case of algebraic varieties.

3.2 Construction of morphisms of spectral sequences

Before constructing morphisms of spectral sequences, we introduce a technical result of Nekovář-Nizioł.

3.2.1. Postnikov system on Bloch-Kato type diagrams. Let R be a condensed ring. Let $A \in \mathscr{D}_{(\varphi,N)}(\mathrm{Mod}_R^{\mathrm{cond}})$ and $B \in \mathscr{D}\mathscr{F}(\mathrm{Mod}_R^{\mathrm{cond}}) = \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathscr{D}(\mathrm{Mod}_R^{\mathrm{cond}}))$ and a morphism $\iota : A \to B$. Also, let $r \in \mathbf{N}$. Given such data (A, B, ι, r), we will consider a specific type of *Postnikov system* of $C_{\mathrm{syn}}(r) := \mathrm{Syn}_{(A,B,\iota,r)} := \mathrm{fib}(A^{\varphi=p^r, N=0} \to B)$ (where the fixed points are derived) on the *Bloch-Kato type limit diagram*

$$\mathrm{Syn}_{(A,B,\iota,r)} \simeq \left[\begin{tikzcd} & \mathrm{Fil}^r B \arrow[d] \\ A \arrow[r, "(\varphi-p^r{,}\iota)"] \arrow[d, "N"] & A \oplus B \arrow[d, "(N{,}0)"] \\ A \arrow[r, "p\varphi-p^r"] & A \end{tikzcd}\right]$$

in the direction of homotopy limit; namely, it consists of a finite collection of adjacent triangles

$$\begin{tikzcd} \mathrm{gr}^0 C_{\mathrm{syn}}(r) \arrow[dr] & & \mathrm{gr}^1 C_{\mathrm{syn}}(r) \arrow[dr] & & \mathrm{gr}^2 C_{\mathrm{syn}}(r) \arrow[dr] & \\ C_{\mathrm{syn}}(r)^0 \arrow[u] & & C_{\mathrm{syn}}(r)^1 \arrow[ll, "{[1]}"] \arrow[u] & & C_{\mathrm{syn}}(r)^2 \arrow[ll, "{[1]}"] \arrow[u] & C_{\mathrm{syn}}(r)^3 \arrow[l, "{[1]}"] \end{tikzcd}$$

where

$$C_{\mathrm{syn}}(r)^0 := \mathrm{Syn}_{(A,B,\iota,r)} := \left[\begin{tikzcd} & \mathrm{Fil}^r B \arrow[d] \\ A \arrow[r, "(\varphi-p^r{,}\iota)"] \arrow[d, "N"] & A \oplus B \arrow[d, "(N{,}0)"] \\ A \arrow[r, "p\varphi-p^r"] & A \end{tikzcd}\right], \quad C_{\mathrm{syn}}(r)^1 := \left[\begin{tikzcd} & A \oplus B \arrow[d, "(N{,}0)"] \\ A \arrow[r, "p\varphi-p^r"] & A \end{tikzcd}\right], \quad C_{\mathrm{syn}}(r)^2 := [A], \quad C_{\mathrm{syn}}(r)^3 := 0.$$

so that

$$\mathrm{gr}^0 C_{\mathrm{syn}}(r) = A \oplus \mathrm{Fil}^r B, \quad \mathrm{gr}^1 C_{\mathrm{syn}}(r) = A \oplus (A \oplus B), \quad \mathrm{gr}^2 C_{\mathrm{syn}}(r) = A.$$ [16]

This gives rise to a *Postnikov exact couple*

$$D_1^{i,j} = H^j(C_{\mathrm{syn}}(r)^i) \xrightarrow{g} H^j(C_{\mathrm{syn}}(r)^{i-1}[1]) = H^{j+1}(C_{\mathrm{syn}}(r)^{i-1}) = D^{i-1,j+1}$$

with associated spectral sequence which we call *Postnikov spectral sequence*

$$E_1^{i,j} = H^j(\mathrm{gr}^i C_{\mathrm{syn}}(r)) \Rightarrow \varinjlim(\cdots \to D_1^{i,j} \xrightarrow{g} D_1^{i-1,j+1} \to \cdots)$$

[16] To extend to $C_{\mathrm{syn}}(r)^\bullet$ for any index $\bullet \in \mathbf{Z}$, one may set $C_{\mathrm{syn}}(r)^{\bullet \geq 3} := 0$ and $C_{\mathrm{syn}}(r)^i := C_{\mathrm{syn}}(r)[i]$ for $i \leq 0$.

(under certain conditions). When the filtration is *finitely exhaustive*, i.e. when $C_{\mathrm{syn}}(r)^i[-i] \xrightarrow{\simeq} C_{\mathrm{syn}}$ for $i \ll 0$, this colimit becomes

$$\varinjlim(\cdots \to H^{i+j}(C_{\mathrm{syn}}(r)^i[-i]) \xrightarrow{g} H^{i+j}(C_{\mathrm{syn}}(r)^{i-1}[-i+1]) \to \cdots) \xrightarrow{\simeq} H^{i+j}(C_{\mathrm{syn}}(r)).$$

3.2.2. Postnikov and hypercohomology spectral sequences. Let R and S be two condensed rings. Consider a left exact functor $F : \mathrm{Mod}_R^{\blacksquare} \to \mathrm{Mod}_S^{\blacksquare}$, e.g. $F = \underline{\mathrm{Hom}}_R(S, -)$ if S is an R-algebra. We denote by RF its right derived functor.

Let $A \in \mathscr{D}^+_{(\varphi,N)}(\mathrm{Mod}_R^{\blacksquare})$ and $B \in \mathscr{D}\mathscr{F}^+(\mathrm{Mod}_R^{\blacksquare})$ and a morphism $\iota : A \to B$, and $r \in \mathbf{N}$. Applying F to the above Posnikov system, one obtains a Postnikov system of $RF(C_{\mathrm{syn}}(r))$ with graded pieces $\mathrm{gr}^i = RF(\mathrm{gr}^i\, C_{\mathrm{syn}}(r))$, and whence the associated Postnikov spectral sequence

$${}^{\lim}E_1^{i,j} = R^jF(\mathrm{gr}^i\, C_{\mathrm{syn}}(r))) \Rightarrow R^{i+j}F(C_{\mathrm{syn}}(r)).$$

On the other hand, without any Postnikov datum, we still have the *hypercohomology exact couple*

$${}^{\mathrm{hyp}}D_2^{i,j} = R^{i+j}F(\tau^{\le j-1}C_{\mathrm{syn}}(r)) \to R^{i+j}F(\tau^{\le j}C_{\mathrm{syn}}(r)) =^{\mathrm{hyp}} D_2^{i-1,j+1}$$

and associated *hypercohomology spectral sequence*

$${}^{\mathrm{hyp}}E_2^{i,j} = R^iF(H^j(C_{\mathrm{syn}}(r))) \Rightarrow R^{i+j}F(C_{\mathrm{syn}}(r)).$$

The main technical result that we need is the following theorem, which relates the two spectral sequences under very restrictive but still reasonable conditions.

3.2.3 - Proposition. *Assume that we are in the setting (3.2.2). If the sequence*

$$0 \to H^j(C_{\mathrm{syn}}(r)) \to H^j(\mathrm{gr}^0\, C_{\mathrm{syn}}(r)) \to H^j(\mathrm{gr}^1\, C_{\mathrm{syn}}(r)) \to H^j(\mathrm{gr}^2\, C_{\mathrm{syn}}(r)) \to 0$$

is exact, or equivalently, the natural morphism in $\mathscr{D}(\mathrm{Mod}_R^{\blacksquare})$

$$H^j(C_{\mathrm{syn}}(r)) \to \left[\begin{tikzcd} & H^j(\mathrm{Fil}^r B) \arrow[d] \\ H^j(A) \arrow[r, "(\varphi - p^r, \iota)"] \arrow[d, "N"] & H^j(A) \oplus H^j(B) \arrow[d, "(N,0)"] \\ H^j(A) \arrow[r, "p\varphi - p^r"] & H^j(A) \end{tikzcd}\right] \tag{3.2.3.1}$$

is an isomorphism for any $j \in \mathbf{Z}$, then there is a natural morphism of exact couples $({}^{\lim}D_2^{i,j}, {}^{\lim}E_2^{i,j}) \to ({}^{\mathrm{hyp}}D_2^{i,j}, {}^{\mathrm{hyp}}E_2^{i,j})$. Consequently, there is a natural morphism of spectral sequences ${}^{\lim}E_t^{i,j} \to {}^{\mathrm{hyp}}E_t^{i,j}$ which starts from the E_2-page with common abutment $R^{i+j}F(C_{\mathrm{syn}}(r))$, and for spectral filtraions we have ${}^{\lim}F \subset {}^{\mathrm{hyp}}F$.

Proof. This is a special case of Nekovář-Nizioł's result [46, Theorem 2.18]. □

To apply this result, we consider the fundamental diagram for $X \in \mathcal{R}\mathrm{ig}_C$.

3.2.4. C_{st}-conjecture and fundamental diagrams. For $X \in \mathcal{R}\mathrm{ig}_C$ partially proper or affinoid or $X \in \mathcal{R}\mathrm{ig}_C^{\dagger}$

quasi-compact, we say that *the C_{st}-conjecture holds* for X if the following commutative diagram

$$(\mathrm{FD}^+_{i,r}) \qquad \begin{tikzcd} H^i_{\mathrm{syn}}(X,r) \arrow[r] \arrow[d] & H^i(\mathrm{Fil}^r R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}})) \arrow[d] \\ (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0} \arrow[r, "\iota^{\mathrm{geom}}_{\mathrm{HK}}"] & H^i_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}), \end{tikzcd}$$

which we call the *fundamental diagram*, is bicartesian for any $i = r \geq 0$, or equivalently (since by assumption on X and using $\otimes^{\blacksquare}$-flatness of $B_{\log}$ we have $H^i((R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0}) \simeq (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0}$ in the derived sense) that

$$(\mathrm{FD}^+_{i,r}{}') \qquad H^i_{\mathrm{syn}}(X,r) \xrightarrow{\simeq} \left[\begin{tikzcd} & H^i(\mathrm{Fil}^r R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}})) \arrow[d] \\ H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log} \arrow[r, "(\varphi-p^r,\iota)"] \arrow[d, "N"] & H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log} \oplus H^i_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \arrow[d, "(N,0)"] \\ H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log} \arrow[r, "p\varphi-p^r"] & H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log} \end{tikzcd} \right]$$

is an equivalence for $i = r \geq 0$.

3.2.5 - Remark. Though the original conjecture [20] is stated with B^+_{st} in place of $B_{\log}$, we prefer using the latter as it is more geometric from the point of view of the Fargues-Fontaine curve [26]. We will refer to the original conjecture as the *B^+_{st}-coefficient C_{st}-conjecture.*

3.2.6 - Remark. Assume that $X = Z_C$ with $Z \in \mathcal{R}\mathrm{ig}^{\dagger}_K$ or $Z \in \mathcal{R}\mathrm{ig}_K$.

(i) Assume that Z is smooth quasi-Stein (e.g. Stein or affinoid). Then *for $i > r$, the diagram ($\mathrm{FD}^+_{i,r}$) is automatically bicartesian.* Indeed, when $i > r$, the right vertical arrow in ($\mathrm{FD}^+_{i,r}$) is an isomorphism, so we are left to show that $H^i_{\mathrm{syn}}(X,r) \xrightarrow{\simeq} (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0}$. For this, recall that from definition (2.3.1.1) and similarly argument concerning classicality of cohomology of the Hyodo-Kato part, we get an exact sequence

$$H^{i-1}((R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})/F^r) \to H^i_{\mathrm{syn}}(X,r) \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B^+_{\mathrm{st}})^{\varphi=p^r,N=0} \to H^i((R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})/F^r).$$

But

$$\begin{aligned} (R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})/F^r &\simeq (\Omega^{\bullet}_{Z/K}(Z) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})/F^r \\ &\simeq \Omega^{\bullet}_{Z/K}(Z/K) \otimes^{\blacksquare}_K (B^+_{\mathrm{dR}}/t^{\max\{r-\bullet,0\}}) \end{aligned}$$

which belongs to $\mathscr{D}^{[0,r-1]}(\mathrm{Mod}^{\blacksquare}_K)$, whence the left and right most terms of the above exact sequence vanish for $i > r$.

(ii) Assume that Z is dagger qcqs or partially proper. *If the C_{st}-conjecture holds for Z_C, then ($\mathrm{FD}^+_{i,r}$) is bicartesian for* $0 \leq i \leq r$. Indeed, fix $i \geq 0$ and run induction on $r \geq i$. When $r = i$, this is the C_{st}-conjecture. Now for the induction step, consider the following commutative diagram

(3.2.6.1)

$$\begin{tikzcd} 0 \arrow[r] & H^i_{\mathrm{syn}}(X,r) \arrow[r] \arrow[d, "t", "\simeq"'] & (H^r_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^r,N=0} \arrow[d, hook, "t"] & \oplus & H^i(F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})) \arrow[r] \arrow[d, "t", "\simeq"'] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \arrow[r] \arrow[d, hook, "t"] & 0 \\ 0 \arrow[r] & H^i_{\mathrm{syn}}(X,r+1) \arrow[r] \arrow[d] & (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^{r+1},N=0} \arrow[d] & \oplus & H^i(F^{r+1}(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})) \arrow[r] \arrow[d] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \arrow[r] \arrow[d, two heads] & 0 \\ & 0 \arrow[r] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C & \oplus & 0 \arrow[r] & H^i_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C \arrow[r] & 0 \end{tikzcd}$$

whose top row is exact by induction hypothesis, whose second vertical arrow is exact by [13, Lemma

3.39, and proof of Proposition 3.36] (cf. [9, Formula (7.15) and its following paragraph] for the case of $B_{\log}$ in place of B_{st}^+), and whose third vertical arrow is an isomorphism since $i \le r$. Therefore, the bottom row is isomorphism, from which we deduce by diagram chasing that the middle row is also exact, thus showing the bicartesianness for $(i, r+1)$.

3.2.7 - Example. Many cases of the C_{st}-conjecture have been established.

(i) For X proper, this is done by Colmez-Nizioł in [20, Theorem 6.2, Corollary 6.15] and by Bosco in [9, Theorem 7.4] (plus the argument of [20, Corollary 6.15]).

(ii) For X smooth Stein, this is done by Colmez-Nizioł in [20, Theorem 6.14, Corollary 6.19] and by Bosco in [9, Theorem 7.7] (plus the argument of [20, Corollary 6.19]).

(iii) For X smooth dagger affinoid, this is done by Colmez-Nizioł in [20, Theorem 6.14, Corollary 6.19] and by Bosco in [9, Proof of Theorem 7.7] (plus the argument of [20, Corollary 6.19]).

(iv) For X smooth affinoid curve, this is done by Bosco in [9] (plus the argument of [20, Corollary 6.19]).

3.2.8. Now, we invert t in the fundamental diagram ($\mathrm{FD}^+_{i,r}$), and define the complex in $\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p})$

$$C_{\mathrm{syn}}(r) := \left[(R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}])^{\varphi = p^r, N=0} \overset{\iota_{\mathrm{HK}}^{\mathrm{geom}}}{\longrightarrow} (R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})/F^r \right].$$

Consider the commutative diagram

$$(\mathrm{FD}_{i,r}) \qquad \begin{array}{ccc} H^i(C_{\mathrm{syn}}(r)) & \longrightarrow & H^i(F^r(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})) \\ \downarrow & & \downarrow \\ (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}])^{\varphi=p^r, N=0} & \xrightarrow{\iota_{\mathrm{HK}}^{\mathrm{geom}}} & H^i_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}} \end{array}$$

When X is qcqs, the natural map $C_{\mathrm{syn}}(r) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ is qcqs, see (3.1.3.1).

3.2.9 - Remark. (i) For $Z \in \mathcal{R}\mathrm{ig}_K$, the natural morphism

$$\mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}) \to \mathrm{Fil}^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})$$

becomes an isomorphism after taking the canonical truncation $\tau^{\le r}$. We may replace B_{dR} by $t^{-j}B^+_{\mathrm{dR}}$ for $j \in \mathbf{N}$ and prove the same statement, hence $t^{-j}B_{\mathrm{dR}}$ is a K-Banach space so that [9, Corollary A.67] is applicable. By éh-hyperdescent, we reduce to the case where Z is smooth affinoid, this is because the cofibre of $\mathrm{Fil}^r(\Omega^\bullet_{Z/K} \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}) \to \mathrm{Fil}^r(\Omega^\bullet_{Z/K} \otimes^{\blacksquare}_K B_{\mathrm{dR}})$ is concentrated in degrees $> r$.

(ii) For $X \in \mathcal{R}\mathrm{ig}_C$, the natural morphism

$$\mathrm{Fil}^r R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \to \mathrm{Fil}^r(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})$$

becomes an isomorphism after taking the canonical truncation $\tau^{\le r}$. Again, we may replace B_{dR} by $t^{-j}B^+_{\mathrm{dR}}$ for $j \in \mathbf{N}$. By éh-hyperdescent, we may assume X to be smooth affinoid, then using Elkik's algebraisation technique [25, Theorem 7, Remark 2], we may assume X descends to a smooth affinoid Z over some finite extension L/K. Then the statement follows from (i).

3.2.10 - Lemma. *For partially proper $X \in \mathcal{R}\mathrm{ig}_C$, we there is a natural exact sequence*

$$0 \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^i, N=0} \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}])^{\varphi=p^i, N=0} \to H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Proof. For X smooth affinoid, this is [9, Formula (7.15)]. In general, we would like to take inverse limit; but to avoid the problem of exchanging countable inverse limit and tensor product $- \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]$, we start by showing that [9, Formula (7.15)] remains true for partially proper spaces. Indeed, for qcqs $U \in \mathcal{R}\mathrm{ig}_C^\dagger$, we have an exact sequence

$$0 \to \mathcal{E}(H^i_{\mathrm{HK}}(U)) \otimes_{\mathcal{O}} \mathcal{O}(i) \to \mathcal{E}(H^i_{\mathrm{HK}}(U)) \otimes_{\mathcal{O}} \mathcal{O}(i+j) \to \iota_*(H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}}) \to 0$$

of coherent sheaves on the Fargues-Fontaine curve, where, $\mathcal{E}(H^i_{\mathrm{HK}}(U))$ is the vector bundle over the Fargues-Fontaine curve attached to the finite (φ, N)-module $H^i_{\mathrm{HK}}(U)$ over $\breve{F}$. Taking global sections, we obtain an exact sequence of $\breve{F}$-vector spaces

$$0 \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} B_{\log})^{\varphi=p^i, N=0} \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B_{\log})^{\varphi=p^i, N=0} \to H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0. \tag{3.2.10.1}$$

because the first cohomology of $\mathcal{E}(H^i_{\mathrm{HK}}(U)) \otimes_{\mathcal{O}} \mathcal{O}(i)$ vanishes by non-negative Harder-Narasimhan slopes considerations [9, Theorem 3.30 (ii)]. As N is nilpotent on $H^i_{\mathrm{HK}}(U)$ and one can write $B_{\log} = \widehat{B_{\log}}^{N=\mathrm{nilp}}$ where $\widehat{B_{\log}} := \varprojlim_I B_I\langle U\rangle$ is a $\breve{F}$-Fréchet space, the above exact sequence can be written as

$$0 \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} \widehat{B_{\log}})^{\varphi=p^i, N=0} \to (H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}\widehat{B_{\log}})^{\varphi=p^i, N=0} \to H^i_{\mathrm{HK}}(U) \otimes_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Now $\widehat{B_{\log}}$ being a $\breve{F}$-Fréchet space, passing to limit with respect to an strictly increasing covering of any partially proper X by qcqs smooth dagger affinoid U, and using [9, Corollary A.67 (i)] and the vanishing $R^1 \varprojlim_U (H^i_{\mathrm{HK}}(U) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B_{\log})^{\varphi=p^i, N=0} = 0$ for $i, j \in \mathbf{N}$ (cf. proof of [9, Formula (7.17)]), one obtains the exact sequence

$$0 \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} \widehat{B_{\log}})^{\varphi=p^i, N=0} \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}\widehat{B_{\log}})^{\varphi=p^i, N=0} \to H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Again by N-nilpotency on $R\Gamma_{\mathrm{HK}}(X)$, this can be rewritten as the exact sequence

$$0 \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p^i, N=0} \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B_{\log})^{\varphi=p^i, N=0} \to H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} t^{-j}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0. \tag{3.2.10.2}$$

Finally, letting $j \to +\infty$, one obtains the desired exact sequence. □

3.2.11 - Lemma. *For partially proper $X \in \mathcal{R}\mathrm{ig}_C$, the natural morphism*

$$R\Gamma_{\mathrm{syn}}(X, r) \to C_{\mathrm{syn}}(r)$$

becomes an isomorphism after taking the canonical truncation $\tau^{\leq r}$.

Proof. This follows from the observation that $\tau^{\leq r} C_{\mathrm{syn}}(r)$ is naturally isomorphic to the filtered colimit over $i \geq r$ of $\tau^{\leq r} R\Gamma_{\mathrm{syn}}(X, i)$ by (3.2.9, ii), the exact sequence (3.2.10.2) and the diagram (3.2.6.1) but assuming in the last diagram the exactness of the first two rows only at the middle terms. □

3.2.12 - Proposition (Colmez-Nizioł, Bosco). *The square ($\mathrm{FD}_{i,r}$) is bicartesian for all $i, r \geq 0$ such that $i \leq r$ in the following cases:*

(i) X is a proper rigid space over C.

(ii) X is a smooth dagger affinoid rigid space over C.

(iii) X is a smooth Stein (dagger[17]) rigid space over C.

Proof. (i) For X proper, the semistable comparison (cf. [20, Theorem 6.2], [9, Theorem 7.4]) and the degeneration at the E_1-page of Hodge-to-de Rham spectral sequence imply that the diagram ($\mathrm{FD}_{i,r}$) is bicartesian for any

[17] For partially proper dagger rigid space $X \in \mathcal{R}\mathrm{ig}_C^\dagger$, there is a natural isomorphism $R\Gamma(X, \mathcal{F}^\dagger) \to R\Gamma(\widehat{X}, \mathcal{F})$ for éh-sheaves $\mathcal{F} \in \{R\Gamma_{\mathrm{HK}}(-), F^\bullet R\Gamma_{\mathrm{inf}}(-/B^+_{\mathrm{dR}})\}$ on $\mathcal{R}\mathrm{ig}_C^\dagger$ (2.2.1).Therefore, there is no need here to distinguish between dagger and genuine rigid spaces in the partially proper case.

$i, r \geq 0$. Indeed, by *loc. cit.*, for any $i \geq 0$, we have $H^i_{\text{proét}}(X, \mathbf{Q}_p) \simeq H^i_{\text{ét}}(X, \mathbf{Q}_p)$ which is a finite-dimensional $\mathbf{Q}_p$-vector space, and we have a natural isomorphism

$$H^i_{\text{ét}}(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \simeq H^i_{\text{HK}}(X) \otimes_{\breve{F}} B_{\log}[\frac{1}{t}]$$

compatible with (φ, N)-action, which induces a natural isomorphism

$$H^i_{\text{ét}}(X, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\text{dR}} \simeq H^i_{\text{inf}}(X/B^+_{\text{dR}}) \otimes^{\blacksquare}_{B^+_{\text{dR}}} B_{\text{dR}}$$

compatible with filtrations. In fact, the first natural isomorphism is induced by $R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \simeq \tau^{\leq r} R\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \simeq R\Gamma_{\text{syn}}(X, r) \to R\Gamma_{\text{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]$ for $r \gg 0$.

(ii) For X smooth dagger affinoid, the diagram ($\text{FD}_{i,r}$) is bicartesian for any $i, r \geq 0$. Indeed, we may freely use Tate twist to reduce to the case $i = r$. The bicartesiannes then follows from the bicartesianess of the ($\text{FD}^+_{i,r}$) and [9, Formula (7.15)]. More precisely, we may assume by smoothness and [25, Theorem 7, Remark 2] that $X = Z_C$ for some smooth dagger affinoid rigid space over some finite extension L/K. We have

$$\begin{tikzcd}
0 \arrow[r] & H^r_{\text{syn}}(X, r) \arrow[r] \arrow[d, "\simeq"', "(3.2.11)"] & (H^r_{\text{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi = p^r, N=0} \arrow[d, hook] & \oplus & H^r(\text{Fil}^r(R\Gamma_{\text{dR}}(Z/L) \otimes_L B^+_{\text{dR}})) \arrow[r] \arrow[d, "\simeq"', "(3.2.9)"] & H^r_{\text{dR}}(Z/L) \otimes_L B^+_{\text{dR}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r, tail] & H^r(C_{\text{syn}}(r)) \arrow[r, tail] & (H^r_{\text{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}])^{\varphi = p^r, N=0} \arrow[d, two heads] & \oplus & H^r(\text{Fil}^r(R\Gamma_{\text{dR}}(Z/L) \otimes_L B_{\text{dR}})) \arrow[r] \arrow[d] & H^r_{\text{dR}}(Z/L) \otimes_L B_{\text{dR}} \arrow[r] \arrow[d, two heads] & 0 \\
 & & H^r_{\text{dR}}(Z/L) \otimes_L B_{\text{dR}}/B^+_{\text{dR}} & \oplus & 0 \arrow[r, "(\text{id},0)"] & H^r_{\text{dR}}(Z/L) \otimes_L B_{\text{dR}}/B^+_{\text{dR}} \arrow[r, tail] & 0,
\end{tikzcd}$$

whose first row and all columns are exact. Here, the first row is exact by the bicartesianess of the ($\text{FD}^+_{i,r}$) in the case of X smooth dagger affinoid, and the second column is exact by (3.2.10). We deduce from it the exactness of the second row.

(iii) For X smooth Stein, the proof is the same as in (ii). □

3.2.13 - Remark. The proof of (ii) actually shows that for partially proper $X \in \mathcal{R}ig_C$, by using suitable Tate twists, the bicartesianness of ($\text{FD}^+_{i,r}$) implies formally that of ($\text{FD}_{i,r}$) thanks to (3.2.9), (3.2.10) and (3.2.11). Therefore, any partially proper $X \in \mathcal{R}ig_C$ satisfying C_{st}-conjecture, such as small varieties[18] listed in [20, Theorem 8.1] (including those in (??), e.g. proper, smooth Stein, smooth dagger affinoid, smooth affinoid curve) and the varieties that are products of proper varieties and Stein varieties [20, Proposition 8.17][19], are immediate instances at hand.

We are now ready to construct morphisms of spectral sequences.

3.2.14. Assume ($\text{FD}_{i,r}$) is bicartesian for any $i, r \geq 0$. Then there exists by (3.2.3) a natural morphism of spectral sequences

$$\begin{tikzcd}
{}^{\lim}E^{i,j}_t \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\text{syn}}(r)^{\underline{\mathcal{G}_K}}) \arrow[d, equal] \\
{}^{\text{hyp}}E^{i,j}_t \arrow[r, Rightarrow] & H^{i+j}(C_{\text{syn}}(r)^{\underline{\mathcal{G}_K}})
\end{tikzcd}$$

starting from the E_2-page with

$${}^{\lim}E^{i,j}_1 := H^j(\text{gr}^i C_{\text{syn}}(r)^{\underline{\mathcal{G}_K}})$$

[18]Recall that small varieties include proper varieties, qcqs dagger affinoids, analytification of algebraic varieties, certain tubular neighbourhoods of subvarieties of proper varieties or complements of such tubular neighbourhoods. More concretely speaking, a (smooth) dagger variety is said to be *small* if its de Rham cohomology is finite-dimensional.

[19]In *loc. cit.*, the proper factor was assumed to be smooth. But in fact, the argument there goes through even if we drop out the smoothness of the proper factor, cf. [9, Proof of Theorem 7.4].

$$ {}^{\mathrm{hyp}}E_2^{i,j} := H^i(\underline{\mathscr{G}_K}, H^j(C_{\mathrm{syn}}(r))). $$

3.2.15. Before Postnikov limit spectral sequence, we have another natural morphism of spectral sequences mapping to it

$$
\begin{tikzcd}
{}^{\mathrm{syn}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(R\Gamma_{\mathrm{syn}}(Z,r)) \arrow[d] \arrow[r, phantom, "\simeq"] & H^{i+j}_{\mathrm{syn}}(Z,r) \\
{}^{\mathrm{lim}}E_t^{i,j} \arrow[r, Rightarrow] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) &
\end{tikzcd}
$$

starting from the E_1-page from the Postnikov limit type *syntomic descent spectral sequence* associated to the similar Postnikov system on $R\Gamma_{\mathrm{syn}}(Z,r)$

$$ {}^{\mathrm{syn}}E_1^{i,j} := H^j(\mathrm{gr}^i\, R\Gamma_{\mathrm{syn}}(Z,r)), $$

whose E_2-page yields

$$ {}^{\mathrm{syn}}E_2^{i,j} := H^i(\mathrm{Syn}_{(H^j_{\mathrm{HK}}(Z), H^j_{\mathrm{dR}}(Z/K), \iota^{\mathrm{arith}}_{\mathrm{HK}}, H^j(F^r R\Gamma_{\mathrm{dR}}(Z/K)))}). $$

3.2.16. The hypercohomology spectral sequence maps naturally to the Hochschild-Serre spectral sequence for the proétale cohomology

$$
\begin{tikzcd}
{}^{\mathrm{hyp}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d] & \\
{}^{\mathrm{HS}}E_t^{i,j} \arrow[r, Rightarrow] & H^{i+j}(R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))^{\underline{\mathscr{G}_K}}) \arrow[r, phantom, "\simeq"] & H^{i+j}_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r))
\end{tikzcd}
$$

starting from the E_2-page by (3.1.3.1), with

$$ {}^{\mathrm{HS}}E_2^{i,j} := H^i(\underline{\mathscr{G}_K}, H^j_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))). $$

3.2.17. Morphisms of spectral sequences. Combining the constructions (3.2.15), (3.2.14) and (3.2.16) altogether, one obtains a sequence of natural morphisms of spectral sequences

$$
\begin{tikzcd}
{}^{\mathrm{syn}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}_{\mathrm{syn}}(Z,r) \arrow[d] \\
{}^{\mathrm{lim}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d, equal] \\
{}^{\mathrm{hyp}}E_t^{i,j} \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(C_{\mathrm{syn}}(r)^{\underline{\mathscr{G}_K}}) \arrow[d] \\
{}^{\mathrm{HS}}E_t^{i,j} \arrow[r, Rightarrow] & H^{i+j}_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r))
\end{tikzcd}
$$

starting at the E_2-page whenever the fundamental diagram ($\mathrm{FD}_{i,r}$) is bicartesian for Z_C and for any $i,r \geq 0$; this is satisfied for example if Z is partially proper and if the C_{st}-conjecture (3.2.4) holds for Z_C, by (3.2.13) and (3.2.6, ii).

3.3 Syntomic descent spectral sequence

Let us first make a digression to certain Galois cohomology groups, following [28, Chapitre I, §3].

3.3.1 - Definition (Fontaine-Perrin-Riou). Let V be a solid $\mathbf{Q}_p$-representation of $\underline{\mathscr{G}_K}$.

(i) We define $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, V)$ as the i-th cohomology group of the homotopy limit

$$\left[\begin{tikzcd} & H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \arrow[d] \\ H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \arrow[r, "(\varphi - p^r, \iota)"] \arrow[d, "N"] & H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \oplus V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}) \arrow[d, "(N,0)"] \\ H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \arrow[r, "p\varphi - p^r"] & H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \end{tikzcd}\right]$$

and call it *geometric continous Galois cohomology of V*.

(ii) An extension $0 \to V \to W \to W' \to 0$ of solid $\mathbf{Q}_p$-representations of $\underline{\mathscr{G}_K}$ is called an *st-extension* if the sequence

$$0 \to H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to H^0(\underline{\mathscr{G}_K}, W' \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to 0$$

is exact in $\mathrm{Mod}^{\blacksquare}_F$.

(iii) An extension $0 \to V \to W \to W' \to 0$ of solid $\mathbf{Q}_p$-representations of $\underline{\mathscr{G}_K}$ is called an *B^+_{dR}-exact extension* if the sequence

$$0 \to H^0(\underline{\mathscr{G}_K}, V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, W' \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to 0$$

is exact in $\mathrm{Mod}^{\blacksquare}_K$.

3.3.2 - Remark. If V is a continuous *finite-dimensional* $\mathbf{Q}_p$-representation of $\mathscr{G}_K$, then $H^0(\underline{\mathscr{G}_K}, V \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}])$ is always finite-dimensional with F-dimension at most equal to $\dim_{\mathbf{Q}_p} V$ by Fontaine's period ring formalism.

3.3.3 - Remark. Let S be any profinite set and $W' = \mathbf{Q}_p[S]^{\blacksquare}$ with trivial $\underline{\mathscr{G}_K}$-action. Then an extension $0 \to V \to W \to \mathbf{Q}_p \to 0$ is an st-extension (*resp.* a B^+_{dR}-exact extension) if and only if there is a Galois equivariant $\mathbf{Q}_p$-linear section of $X \to \mathbf{Q}_p[S]^{\blacksquare}$, where X is defined as the pushout

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \arrow[r] & X \end{tikzcd} \quad \left(\textit{resp.} \quad \begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ V \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}} \arrow[r] & X \end{tikzcd} \right)$$

in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p}$. Indeed, the "exact" extension property amounts to a F-linear (*resp.* K-linear) section of $H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \to F[S]^{\blacksquare}$ (*resp.* of $H^0(\underline{\mathscr{G}_K}, W \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) \to K[S]^{\blacksquare}$) by (the proof of) [28, I, Proposition 3.3.7]. Using the fact that $H^0(\underline{\mathscr{G}_K}, \mathbf{Q}_p[S]^{\blacksquare} \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) = F[S]^{\blacksquare}$ (*resp.* $H^0(\underline{\mathscr{G}_K}, \mathbf{Q}_p[S]^{\blacksquare} \otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}}) = K[S]^{\blacksquare}$) which is a compact projective object in $\mathrm{Mod}^{\blacksquare}_F$ (*resp.* in $\mathrm{Mod}^{\blacksquare}_K$), this is equivalent to the desired existence of section.

3.3.4 - Remark. Let V be a flat solid $\mathbf{Q}_p$-representation of $\mathscr{G}_K$. We compare $H^i_{\mathrm{st}}(\mathscr{G}_K, V)$ with $H^i_g(\mathscr{G}_K, V)$ of Fontaine-Riou [28, I, 3.3.3].

First, recall the definition of *loc. cit.*: consider the sequence in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p[\mathscr{G}_K]}$

$$(S_g) \qquad 0 \to \mathbf{Q}_p \to B_{\log}[\frac{1}{t}] \to B_{\log}[\frac{1}{t}] \oplus B_{\log}[\frac{1}{t}] \oplus B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to B_{\log}[\frac{1}{t}] \to 0$$

with $a \mapsto a$, $b \mapsto ((\varphi - 1)b, Nb, \iota_p(b))$, $(b_1, b_2, c) \mapsto (Nb_1 - (p\varphi - 1)b_2)$, which is exact; tensoring this with V, we still get an exact sequence in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p[\mathscr{G}_K]}$

$$(S_g(V)) \qquad 0 \to V \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\log}[\frac{1}{t}] \oplus B_{\log}[\frac{1}{t}] \oplus B_{\mathrm{dR}}/B^+_{\mathrm{dR}}) \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \to 0.$$

The termwise Galois invariant $H^0(\underline{\mathscr{G}_K},-)$ of this resolution complex calculates the cohomology $H^i_g(\underline{\mathscr{G}_K},V)$.

Mimicking this procedure, we have exact sequences in $\mathrm{Mod}^{\blacksquare}_{\mathbf{Q}_p[\mathscr{G}_K]}$

$$(S_{\mathrm{st}})\qquad 0 \to \mathbf{Q}_p \to B_{\log}[\frac{1}{t}] \oplus B^+_{\mathrm{dR}} \to B_{\log}[\frac{1}{t}] \oplus B_{\log}[\frac{1}{t}] \oplus B_{\mathrm{dR}} \to B_{\log}[\frac{1}{t}] \to 0$$

with $a \mapsto (a,a)$, $(b,c) \mapsto ((\varphi-1)b, Nb, \iota_p(b)-c)$, $(b_1,b_2,c) \mapsto (Nb_1-(p\varphi-1)b_2)$, and

$$(S_{\mathrm{st}}(V))\qquad 0 \to V \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\log}[\frac{1}{t}] \oplus B^+_{\mathrm{dR}}) \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} (B_{\log}[\frac{1}{t}] \oplus B_{\log}[\frac{1}{t}] \oplus B_{\mathrm{dR}}) \to V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \to 0.$$

By our definition, the termwise Galois invariant $H^0(\underline{\mathscr{G}_K},-)$ of this resolution complex calculates the cohomology $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K},V)$.

3.3.5 - Proposition. *Let V be a flat solid $\mathbf{Q}_p$-representation of $\underline{\mathscr{G}_K}$. Let $\nu \in \{g,\mathrm{st}\}$.*

(i) The natural map $H^0_\nu(\underline{\mathscr{G}_K},V) \to H^0(\underline{\mathscr{G}_K},V)$ is an isomorphism.

*(ii) The natural map $H^1_\nu(\underline{\mathscr{G}_K},V) \to H^1(\underline{\mathscr{G}_K},V)$ is injective, and identifies $H^1_\nu(\mathscr{G}_K,V)(S)$ with the sub-$\mathbf{Q}_p$-vector space of $H^1(\mathscr{G}_K,V)(S)$ classifying the st-extensions if $\nu = g$ (*resp. *st- and B^+_{dR}-exact extension if $\nu = \mathrm{st}$) W of $\mathbf{Q}_p[S]^{\blacksquare}$ by V.*

In particular, $H^1_{\mathrm{st}}(\underline{\mathscr{G}_K},V) \hookrightarrow H^1_g(\underline{\mathscr{G}_K},V) \hookrightarrow H^1(\underline{\mathscr{G}_K},V)$.

We ignore whether the map $H^1_{\mathrm{st}}(\underline{\mathscr{G}_K},V) \hookrightarrow H^1_g(\underline{\mathscr{G}_K},V)$ is in practice an isomorphism or not.

Proof. For $\nu = g$, it is essentially contained in [28, I, Proposition 3.3.7]. It remains to establish the identification statement. Notice that $H^1(\underline{\mathscr{G}_K},V)(S) = \mathrm{Ext}^1_{\mathbf{Q}_p[\mathscr{G}_K]}(\mathbf{Q}_p[S]^{\blacksquare},V)$ and $H^1(\underline{\mathscr{G}_K},V\otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}])(S) = \mathrm{Ext}^1_{\mathbf{Q}_p[\mathscr{G}_K]}(\mathbf{Q}_p[S]^{\blacksquare},V\otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}])$. Then we may argue exactly as in the proof of *loc. cit.*, simply by replacing $\mathbf{Q}_p$ by $\mathbf{Q}_p[S]^{\blacksquare}$, using the remark (3.3.3).

For $\nu = \mathrm{st}$, the proof is the similar, noticing that $[W] \in \ker(H^1(\mathscr{G}_K,V) \to H^1(\mathscr{G}_K,V\otimes^{\blacksquare}_{\mathbf{Q}_p}(B_{\log}[\frac{1}{t}]\oplus B^+_{\mathrm{dR}})))$ if and only if $[W] \in \ker(H^1(\mathscr{G}_K,V) \to H^1(\mathscr{G}_K,V\otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}])$ as well as $[W] \in \ker(H^1(\mathscr{G}_K,V) \to H^1(\mathscr{G}_K,V\otimes^{\blacksquare}_{\mathbf{Q}_p} B^+_{\mathrm{dR}})$, so if and only if $0 \to V \to W \to \mathbf{Q}_p \to 0$ is an st- and B^+_{dR}-exact extension by (3.3.3). Similarly for S-valued points. $\square$

Now let us come back to our syntomic descent spectral sequence. We are going to interpretate the E_2-page of the syntomic descent spectral sequence (3.2.15) as Galois cohomology groups.

3.3.6. First, we interpretate each term of ${}^{\mathrm{syn}}E_1^{i,j}$, which are on the arithmetic level, as the Galois invariants of geometric objects.

(i) For de Rham cohomology, we have $H^j_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \simeq H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})$ for any $Z \in \mathcal{R}\mathrm{ig}^{(\dagger)}_K$ by flatness of B^+_{dR} in $\mathrm{Mod}^{\blacksquare}_K$. Using the nuclearity of $H^j_{\mathrm{dR}}(Z/K)$ over K, we obtain by (1.3.5) that

$$H^j_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K H^0(\underline{\mathscr{G}_K},B^{(+)}_{\mathrm{dR}}) \simeq H^0(\underline{\mathscr{G}_K},H^j_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B^{(+)}_{\mathrm{dR}}),$$

whence

$$H^j_{\mathrm{dR}}(Z/K) \simeq H^0(\underline{\mathscr{G}_K},H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})) \simeq H^0(\underline{\mathscr{G}_K},H^j_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})\otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}).$$

(ii) As for the cohomology of the filtration part, we have that

$$H^j(F^rR\Gamma_{\mathrm{dR}}(Z/K)) \simeq H^0(\underline{\mathscr{G}_K},H^j(F^rR\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}))) \simeq H^0(\underline{\mathscr{G}_K},H^j(F^r(R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}})\otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})))$$

for any $Z \in \mathcal{R}\mathrm{ig}^{(\dagger)}_K$. Indeed, the filtration $F^\bullet R\Gamma_{\mathrm{dR}}(Z/K)$ is finite (and separated), because Z is éh-locally smooth of the same dimension d, so the naive truncation filtration on $F^\bullet\Omega^\bullet_{Z/K,\text{éh}}$ becomes zero at F^{d+1}. We can write $\mathrm{Fil}^r_{\mathrm{Hdg}}R\Gamma_{\mathrm{inf}}(Z_C/B^+_{\mathrm{dR}}) \simeq F^r(R\Gamma_{\mathrm{dR}}(Z/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}})$ as an iterated extension of

$\mathrm{gr}^0 R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K t^r B^+_{\mathrm{dR}}$ by $\mathrm{gr}^1 R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K t^{r-1} B^+_{\mathrm{dR}}$, ..., $\mathrm{gr}^{r-1} R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K tB^+_{\mathrm{dR}}$ and then by $F^r R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}$; then applying the Galois cohomology computation (1.3.11, i), we get the first isomorphism. For the second isomorphism, since the filtration $F^\bullet R\Gamma_{\mathrm{dR}}(Z/K)$ is bounded from below, say stablising from $\bullet \geq N+1$ on for some $N \geq r$; so $F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})$ is an iterated extension of $F^N R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}}/t^{r-N} B^+_{\mathrm{dR}}$ by $F^{N-1} R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C(r-N+1)$, ..., $F^{r+1} R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K C(-1)$ then by $F^r(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}})$; then applying the Galois cohomology computation (1.3.11, i), we get the second isomorphism.

(iii) For the Hyodo-Kato part, we have

$$H^j_{\mathrm{HK}}(Z) \simeq H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes_{\breve{F}} B_{\log}[\frac{1}{t}])$$

for $Z \in \mathcal{R}\mathrm{ig}_K^{(\dagger)}$ which are qcqs (2.1.22) (2.2.6) or partially proper (2.2.7).

As a result, for any $Z \in \mathcal{R}\mathrm{ig}_K^{(\dagger)}$ that is qcqs or partially proper, the E_2-term ${}^{\mathrm{syn}}E_2^{i,j}$ is the i-th cohomology group of the homotopy limit

$$\left[\begin{tikzcd} & H^0(\underline{\mathscr{G}_K}, H^j(\mathrm{Fil}^r(R\Gamma_{\inf}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}))) \arrow[d] \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) \arrow[r, "(\varphi - p^r, \iota)"] \arrow[d, "N"] & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}] \oplus H^j_{\inf}(Z_C/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}) \arrow[d, "{(N,0)}"] \\ H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) \arrow[r, "p\varphi - p^r"] & H^0(\underline{\mathscr{G}_K}, H^j_{\mathrm{HK}}(Z_C) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}]) \end{tikzcd}\right],$$

thus finishing our interpretation.

Finally, we focus on the proper case.

3.3.7. Proper case. Let $Z \in \mathcal{R}\mathrm{ig}_K$ be proper. By the semistable comparison theorem (3.2.12, i), the above diagram is identified with

$$\left[\begin{tikzcd} & H^0(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} t^r B^+_{\mathrm{dR}})) \arrow[d] \\ H^0(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \arrow[r, "(\varphi - p^r, \iota)"] \arrow[d, "N"] & H^0(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}] \oplus H^j_{\text{ét}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}) \arrow[d, "{(N,0)}"] \\ H^0(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]) \arrow[r, "p\varphi - p^r"] & H^0(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\log}[\frac{1}{t}]). \end{tikzcd}\right]$$

Hence, we can identify

$$ {}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p(r))). \tag{3.3.7.1}$$

A sanity check through the constructions implies that the morphism of spectral sequences on the E_2-page ${}^{\mathrm{syn}}E_2^{i,j} \to {}^{\mathrm{HS}}E_2^{i,j}$ is identified with the natural map $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p(r))) \to H^i(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(Z_C, \mathbf{Q}_p(r)))$.

3.3.8 - Remark. (i) As we can see by degeneration of Hodge-to-de Rham spectral sequence at the E_1-page for proper $Z \in \mathcal{R}\mathrm{ig}_K$, the map $H^j(F^r R\Gamma_{\mathrm{dR}}(Z/K)) \to H^j_{\mathrm{dR}}(Z)$ is injective onto $F^r H^j_{\mathrm{dR}}(Z)$, with cokernel identified with $H^j_{\mathrm{dR}}(Z/K)/F^r \simeq H^0(\underline{\mathscr{G}_K}, (H^j_{\mathrm{dR}}(Z/K) \otimes_K B_{\mathrm{dR}})/F^r) \simeq H^0(\underline{\mathscr{G}_K}, (H^j_{\inf}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})/F^r)$. The same argument as above using the semistable comparison theorem (3.2.12, i) shows that we can also identify

$${}^{\mathrm{syn}}E_2^{i,j} = H^i_g(\underline{\mathscr{G}_K}, H^j_{\text{ét}}(X, \mathbf{Q}_p(r))).$$

(ii) Alternatively, one can show directly that $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, V) \xrightarrow{\simeq} H^i_g(\underline{\mathscr{G}_K}, V)$ under the condition that $V \otimes^{\blacksquare}_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq (M, \mathrm{Fil}^\bullet) \otimes^{\blacksquare}_K B_{\mathrm{dR}}$ is an isomorphism compatible with filtrations for certain filtered K-module $(M, \mathrm{Fil}^\bullet)$. This applies for example to $V = H^j_{\mathrm{ét}}(Z_C, \mathbf{Q}_p(r))$ for proper $Z \in \mathcal{R}\mathrm{ig}_K$ by semi-stable comparison theorem (3.2.12, i). Indeed, the exact sequence $0 \to \mathrm{Fil}^r M \to M \to M/\mathrm{Fil}^r \to 0$ is identified, by computation of Galois cohomology of $C(i)$ (1.3.11, i), with the exact sequence

$$0 \to H^0(\underline{\mathscr{G}_K}, \mathrm{Fil}^r(M \otimes_K B_{\mathrm{dR}})) \to H^0(\underline{\mathscr{G}_K}, M \otimes_K B_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, (M \otimes_K B_{\mathrm{dR}})/F^r) \to 0.$$

Under the assumption, this is again identified with

$$0 \to H^0(\underline{\mathscr{G}_K}, V \otimes_{\mathbf{Q}_p} t^r B^+_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}) \to H^0(\underline{\mathscr{G}_K}, V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}/t^r B^+_{\mathrm{dR}}) \to 0. \tag{3.3.8.1}$$

This implies immediately that $H^i_{\mathrm{st}}(\underline{\mathscr{G}_K}, V) \xrightarrow{\simeq} H^i_g(\underline{\mathscr{G}_K}, V)$ by comparing the termwise Galois invariants $H^0(\underline{\mathscr{G}_K}, -)$ of the complexes ($S_g(V)$) and ($S_{\mathrm{st}}(V)$).

4 Chern classes of vector bundles and regulators

4.1 First Chern class maps

4.1.1. Crystalline construction. Let us first recall Tsuji's construction of the log-crystalline first Chern class (following Kato) [55, (2.2.3)]. Let $\mathscr{X}$ be an integral and quasi-coherent log-scheme over a quasi-coherent log pd-scheme $S^\sharp = (S, \mathcal{L}, I, \gamma)$ with $p \in \mathcal{O}_S$ nilpotent.

Assume first that there is a log pd-$S^\sharp$-smooth thickening $\mathscr{X} \hookrightarrow P$ with log pd-envelope $\mathscr{X} \hookrightarrow D$ (if $\mathscr{X}$ is moreoever log affine, then there is a universal coordinate such thickenings P^{univ} [4, 1.4, Remark (iii)], which however is not necessarily of finite type). Our goal is to construct a map

$$M^{\mathrm{gp}}_{\mathscr{X}} \to \mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^\bullet_{P/S^\sharp}[1]$$

in $\mathscr{D}(\mathscr{X}_{\mathrm{ét}}, \mathbf{Z})$. This map will be constructed only in the derived category, as the composition

$$M^{\mathrm{gp}}_{\mathscr{X}} \xleftarrow{\simeq} (1 + J_D \to M^{\mathrm{gp}}_D) \xrightarrow{(\log, \mathrm{dlog})} (\mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^1_{P/S^\sharp} \to \cdots) = \mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^\bullet_{P/S^\sharp}[1] \tag{4.1.1.1}$$

in $\mathscr{D}(\mathscr{X}_{\mathrm{ét}}, \mathbf{Z})$, where $1 + J_D$ and M^{gp} sit respectively at cohomological degrees -1 and 0.

In general, we follow the procedure of [4, 1.6, Remark] using an embedding system $E = \{\mathscr{X}_\bullet \hookrightarrow P_\bullet\}$; such liftings form a cofiltered system [35, (2.21)]. Let $\mathscr{X}_\bullet \to D_\bullet$ be the log pd-envelopes of $\mathscr{X}_\bullet \to P_\bullet$, defined by the pd-ideals $J_{D_\bullet} := \ker(\mathcal{O}_{D_\bullet} \to \mathcal{O}_{\mathscr{X}_\bullet})$. We have an adjoint pair of topos

$$\theta^* : \mathscr{X}^\sim_{\mathrm{ét}} \to \mathscr{X}^\sim_{\bullet,\mathrm{ét}} : \theta_*$$

and their derived functors $\theta^* \dashv R\theta_*$. We have a map by (4.1.1.1)

$$\theta^* M^{\mathrm{gp}}_{\mathscr{X}} = M^{\mathrm{gp}}_{\mathscr{X}_\bullet} \to \mathcal{O}_{D_\bullet} \otimes_{\mathcal{O}_{P_\bullet}} \omega^\bullet_{P_\bullet/S^\sharp}[1] \tag{4.1.1.2}$$

in $\mathscr{D}(\mathscr{X}_{\mathrm{ét}}, \mathbf{Z})$, which induces by adjunction the *log-crystalline first Chern class map*

$$c^{\mathrm{cris}}_{1,\mathscr{X}/S^\sharp} : M^{\mathrm{gp}}_{\mathscr{X}} \to R\theta_*(\mathcal{O}_{D_\bullet} \otimes_{\mathcal{O}_{P_\bullet}} \omega^\bullet_{P_\bullet/S^\sharp})[1] \simeq Ru^{\log}_{\mathscr{X}/S^\sharp *} \mathcal{O}_{\mathscr{X}/S^\sharp}[1] \tag{4.1.1.3}$$

where the last canonical isomorphism is given by [35, Proposition 2.20].

4.1.2 - Lemma (Compatibility of log-crystalline first Chern class maps). *For any commutative diagram*

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{g} & \mathscr{X}' \\ \downarrow & & \downarrow \\ S^\sharp & \xrightarrow{f} & S'^\sharp \end{array}$$

with $\mathscr{X}'$ an integral and quasi-coherent log-scheme over another quaso-coherent log pd-scheme $S'^\sharp$ with $p \in \mathcal{O}_{S'}$ nilpotent such that the underlying morphism of schemes is locally of finite type, the following diagram

$$\begin{array}{ccc} M^{\mathrm{gp}}_{\mathscr{X}} & \longleftarrow & g^* M^{\mathrm{gp}}_{\mathscr{X}'} \\ \Big\downarrow c^{\mathrm{cris}}_{1,\mathscr{X}/S^\sharp} & & \Big\downarrow c^{\mathrm{cris}}_{1,\mathscr{X}'/S'^\sharp} \\ Ru^{\log}_{\mathscr{X}/S^\sharp *}\mathcal{O}_{\mathscr{X}/S^\sharp}[1] & \longleftarrow & g^* Ru^{\log}_{\mathscr{X}'/S'^\sharp *}\mathcal{O}_{\mathscr{X}'/S'^\sharp}[1] \end{array}$$

in $\mathscr{D}(\mathscr{X}_{\text{ét}}, \mathbf{Z})$ commutes.

Proof. It is easily checked by choosing compatible embedding systems, which is possible by existence of universal coordinate thickenings. □

4.1.3. Characteristic p setting. By this we mean that $S^\sharp$ is a fine log pd-scheme endowed with a Frobenius action F lifting that of $S^\sharp/p$ and $\mathscr{X}$ is a fine log-scheme in characteristic p over $S^\sharp$.

4.1.4 - Lemma. *Let $\mathscr{X} \to S^\sharp$ be a characteristic p setting. Consider the natural Frobenius action φ on $Ru_*\mathcal{O}_{\mathscr{X}/S^\sharp}$. Then $c^{\mathrm{cris}}_{1,\mathscr{X}/S^\sharp}$ factors through*

$$M^{\mathrm{gp}}_{\mathscr{X}} \to (Ru_*\mathcal{O}_{\mathscr{X}/S^\sharp})^{\varphi=p}.$$

Proof. The embedding systems considered in (4.1.1) can be taken to be equipped with compatible Frobenius liftings $E = \{\mathscr{X}_\bullet \hookrightarrow P_\bullet\}$, whose log pd-envelopes $\mathscr{X}_\bullet \hookrightarrow D_\bullet$ are equipped with induced compatible Frobenius liftings $F_{D_\bullet}$. The Frobnius action on $Ru_*\mathcal{O}_{\mathscr{X}/S^\sharp} \simeq R\theta_*(\mathcal{O}_{D_\bullet} \otimes_{\mathcal{O}_{P_\bullet}} \omega^\bullet_{P_\bullet/S^\sharp})$ is induced by $F_{D_\bullet}$ and $F_{P_\bullet}$. The map (4.1.1.2) whence (4.1.1.3) is Frobenius equivariant. Then we are done since $\varphi = p$ on $M^{\mathrm{gp}}_{\mathscr{X}}$ for $\mathscr{X}$ in characteristic p. □

4.1.5 - Example. Let L be a finite extension of $\mathbf{Q}_p$ with residue field k_L. Let $\mathfrak{Z}$ be a semistable formal scheme over $\mathcal{O}_L$ with log-structure $M_{\mathfrak{Z}} \to \mathcal{O}_{\mathfrak{Z}}$ induced by its special fibre. Let $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L^\times} \mathcal{O}_C^\times$.

Consider the following p-adic formal log pd-schemes:

- $\operatorname{Spf} \mathbf{Z}_p$ with trivial log-structure,
- $\operatorname{Spf} \mathcal{O}^{\mathrm{triv}}_{F_L}$,
- $\operatorname{Spf} r^{\mathrm{PD},0}_L$ with log-structure induced by the t_a's for $a \in (\mathfrak{m}_L/\mathfrak{m}_L^2)\backslash\{0\}$, together with log pd-thickening $p_0 : \operatorname{Spec} \mathcal{O}^0_{F_L,1} \hookrightarrow \operatorname{Spf} r^{\mathrm{PD},0}_L$,
- $\operatorname{Spf} r^{\mathrm{PD}}_L$ with log-structure induced by the t_a's for $a \in (\mathfrak{m}_L/\mathfrak{m}_L^2)\backslash\{0\}$, where l is a $\mathcal{O}_{F_L}$-class in $(\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$ determining a log pd-thickening $p_l : \operatorname{Spec} \mathcal{O}^\times_{L,1} \hookrightarrow \operatorname{Spf} r^{\mathrm{PD}}_L$ lifting p_0,
- $\operatorname{Spf} \mathcal{O}^0_{F_L} := W_n(k_L)^0$,
- $\operatorname{Spf} \mathcal{O}^\times_L$,
- $\operatorname{Spf} A^\times_{\mathrm{cris}}$,
- $\operatorname{Spf} \widehat{A}_{l,\mathrm{st}}$, where l is a $\mathcal{O}_{\breve{F}}$-class in $(\mathfrak{m}_C/p\mathfrak{m}_C)\backslash\{0\}$.

They have respectively $\operatorname{Spec}\mathbf{F}_p$, $\operatorname{Spec}k_L^{\mathrm{triv}}$, $\operatorname{Spec}k_L\{t_a\}$ and $\operatorname{Spec}k_L^0$ as reduction modulo p which all receive maps from $\mathfrak{Z}_1^0$, and are related by morphisms of p-adic formal log pd-rings

$$\mathbf{Z}_p \to \mathcal{O}_{F_L}^{\mathrm{triv}} \to r_L^{\mathrm{PD},0} \xrightarrow{p_0} \mathcal{O}_{F_L}^0.$$

Applying the construction (4.1.1) to these settings, one obtains respective first Chern class maps

- $c_{1,\mathfrak{Z}_n}^{\mathrm{cris}}$ for $\mathfrak{Z}_n$ over $S^\sharp = \mathbf{Z}_p$ or $\mathcal{O}_{F_L}^{\mathrm{triv}}$,
- $c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}$ for $S^\sharp = r_L^{\mathrm{PD},0}$, and $c_{1,\mathfrak{Z}_1,l}^{\mathrm{PD}}$ for $S^\sharp = r_L^{\mathrm{PD}}$,
- $c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}$ for $S^\sharp = \mathcal{O}_{F_L}^0$,
- $c_{1,\mathfrak{Z}_n}^{\mathrm{crdR}}$ for $\mathfrak{Z}_n$ over $S^\sharp = \mathcal{O}_L^\times$,
- $c_{1,\mathfrak{X}_n}^{A_{\mathrm{cris}}}$ for $\mathfrak{X}_n$ over $S^\sharp = A_{\mathrm{cris}}^\times$,
- $c_{1,\mathfrak{X}_1,l}^{\mathrm{st}}$ for $S^\sharp = \widehat{A}_{l,\mathrm{st}}$.

4.1.6 - Lemma. *The first Chern class maps $c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}$ and $c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}$ factor respectively through*

$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}} : M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}})^{\varphi=p,N=0}[1]$$
$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}} : M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{HK},0}})^{\varphi=p,N=0}[1].$$

Proof. The compatibility (4.1.2) affirms that the diagram

$$\begin{tikzcd}
M_{\mathfrak{Z}_1}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1}^{\mathrm{cris}}"] \arrow[d, equal] & Ru_*\mathcal{O}_{\mathfrak{Z}_1/\mathbf{Z}_p}[1] \arrow[d, "\simeq"] \\
M_{\mathfrak{Z}_1}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1}^{\mathrm{cris}}"] \arrow[d] & Ru_*\mathcal{O}_{\mathfrak{Z}_1/\mathcal{O}_{F_L}^{\mathrm{triv}}}[1] \arrow[d] \\
M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1^0}^{\mathrm{cris}}"] \arrow[d, equal] & Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^{\mathrm{triv}}}[1] \arrow[d] \\
M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}"] \arrow[d, equal] & Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}}[1] \arrow[d] \\
M_{\mathfrak{Z}_1}^{\mathrm{gp}} \arrow[r, "c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}"] & Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0}[1]
\end{tikzcd}$$

commutes. The fourth right vertical map is N-equivariant. Moreover, by [36, Lemma 4.2], the third right vertical map factors through $Ru_*\mathcal{O}_{\mathfrak{Z}_1/\mathcal{O}_{F_L}^{\mathrm{triv}}} \xrightarrow{\simeq} (Ru_*\mathcal{O}_{\mathfrak{Z}_1/r_L^{\mathrm{PD},0}})^{N=0}$. Hence $c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}}$ and $c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}}$ factor respectively through

$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{PD}} : M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0}})^{N=0}[1]$$
$$c_{1,\mathfrak{Z}_1^0}^{\mathrm{HK}} : M_{\mathfrak{Z}_1^0}^{\mathrm{gp}} \to (Ru_*\mathcal{O}_{\mathfrak{Z}_1^0/r_L^{\mathrm{HK},0}})^{N=0}[1].$$

Then (4.1.4) helps conclude. □

4.1.7. De Rham construction. Next, let us recall the construction of the *rigid-analytic de Rham first Chern class map*

(4.1.7.1) $$c_1^{\mathrm{dR}} : \mathcal{O}_Z^\times \to \mathrm{Fil}^1\Omega^\bullet_{Z/K}[1]$$

of éh-sheaves over Z for an rigid-analytic variety Z over K, and that of the *infinitesimal B_{dR}^+ first Chern class* *map*

$$c_1^{\inf} : \mathcal{O}_X^\times \to Ru_{X/B_{\mathrm{dR}}^+ *}\mathcal{O}_{\inf, X/B_{\mathrm{dR}}^+}[1] \tag{4.1.7.2}$$

of éh-sheaves over X for an rigid-analytic variety X over C. Here, $u_{X/B_{\mathrm{dR}}^+ *}$ denotes the canonical morphism of sites from $(X/B_{\mathrm{dR}}^+)_{\inf}$ to $X_{\text{éh}}$.

Assume first that everything is smooth. In the geometric case, one can proceeds as in the crystalline case using embedding systems, so as to reduce to the case where there exists $X \hookrightarrow P$ with latter smooth over B_{dR}^+. In this case, letting D be the envelope of X in P [31, Definition 2.2.1, and §4.1], our map is then constructed as

$$\mathcal{O}_X^\times \xleftarrow{\simeq} (1 + \mathcal{J}_{\inf,D} \to \mathcal{O}_{\inf,D}^\times) \xrightarrow{(\log,\mathrm{dlog})} (\mathcal{O}_{\inf,D} \to \mathcal{O}_{\inf,D} \otimes_{\mathcal{O}_{\inf,P}} \Omega^1_{P/B_{\mathrm{dR}}^+} \to \cdots) = \mathcal{O}_{\inf,D} \otimes_{\mathcal{O}_{\inf,P}} \Omega^\bullet_{P/B_{\mathrm{dR}}^+}[1]. \tag{4.1.7.3}$$

in $\mathscr{D}(X_{\text{ét}}, \mathbf{Z})$, where the last complex is identified with $R\Gamma_{\inf}(X/B_{\mathrm{dR}}^+)$ [31, Theorem 4.1.1]. The logarithm is well-defined. Indeed, by definition, if we let I be defining ideal of X in P, which is a coherent ideal, then $D := \varinjlim_{n\in\mathbf{N}} h_{P_n}$ where P_n is the n-th infinitesimal neighbourhood of X in P, defined by the ideal I^{n+1}; and $\mathcal{O}_D := \varprojlim_{n\in\mathbf{N}} \mathcal{O}_{P_n}$, $\mathcal{J}_{\inf,D} := \ker(\mathcal{O}_D \to \mathcal{O}_X) = \varprojlim_{n\in\mathbf{N}} \ker(P_n \to \mathcal{O}_X)$. The logarithm is well-defined on $1 + I^{n+1} \to \mathcal{O}_{P_n}$ by I-nilpotency and since we are over $\mathbf{Q}_p$; then it suffices to pass to the limit.

For the arithmetic case, one could do the same using the infinitesimal site $(Z/K)_{\inf}$ which calculates de Rham cohomology [31, Theorem 1.2.1 (iii)], but it would be far more elementary if we define c_1^{dR} directly as the map of genuine complexes

$$\mathcal{O}_Z^\times \xrightarrow{\mathrm{dlog}} (\Omega^1_{Z/K} \to \Omega^2_{Z/L} \to \cdots) = \mathrm{Fil}^1\Omega^\bullet_{Z/K}[1].$$

In general for singular X, éh-descent suffices to conclude.

Using infinitesimal interpretation, we see that c_1^{dR} and $c_1^{\inf}$ are compatible.

4.1.8 - Lemma. *The map c_1^{dR} factors through*

$$c_1^{\inf} : \mathcal{O}_X^\times \to \mathrm{Fil}^1_{\mathrm{Hdg}}\Omega^\bullet_{X/B_{\mathrm{dR}}^+}.$$

Proof. In the case where there exists $X \hookrightarrow P$ with latter smooth over B_{dR}^+ and D is the envelope of X in P, it is easily from the expression that the second map in (4.1.7.3) factors through $(\mathcal{J}_{\inf,D} \to \mathcal{O}_{\inf,D} \otimes_{\mathcal{O}_{\inf,P}} \Omega^1_{P/B_{\mathrm{dR}}^+} \to \cdots) = \mathrm{Fil}^1(\mathcal{O}_{\inf,D} \otimes_{\mathcal{O}_{\inf,P}} \Omega^\bullet_{P/B_{\mathrm{dR}}^+})[1]$. In general, it follows from simplicial construction then éh-descent. □

4.1.9. Syntomic construction. For $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$, we want to define its *syntomic first Chern class map*

$$c_{1,\mathfrak{Z}}^{\mathrm{syn}} : M_{\mathfrak{Z}}^{\mathrm{gp}} \to R\Gamma_{\mathrm{syn}}(\mathfrak{Z}_\eta, 1)[1] \tag{4.1.9.1}$$

in $\mathscr{D}((\mathfrak{Z}_\eta)_{\text{ét}}, \mathbf{Z})$.

By (2.3.3), we may use the natural identification $R\Gamma_{\mathrm{syn}}(\mathfrak{Z}_\eta, 1) \simeq [R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p}_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)/\mathrm{Fil}^1]$. Consider the following diagram

$$\begin{tikzcd}
\Gamma(\mathfrak{Z}, M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[r, "c_1^{\mathrm{cris}}"] \arrow[d, "\simeq"] \arrow[rr, bend right, "c_1^{\mathrm{crdR}}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z})^{\varphi=p}_{\mathbf{Q}_p}[1] \arrow[r] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^\times)_{\mathbf{Q}_p}[1] \arrow[d, "\simeq"] \\
\Gamma(\mathfrak{Z}_\eta, \mathcal{O}^\times_{\mathfrak{Z}_\eta}) \arrow[r, "c_1^{\mathrm{dR}}"] & \mathrm{Fil}^1 R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1] \arrow[r] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1]
\end{tikzcd} \tag{4.1.9.2}$$

where the left vertical isomorphism results from $M_{\mathfrak{Z}}^{\mathrm{gp}} \simeq j_*\mathcal{O}^\times_{\mathfrak{Z}_\eta}$, and the right vertical isomorphism will be recalled below. Now, for commutativity, the maps c_1^{cris} and c_1^{crdR} are naturally compatible by (4.1.2), and we will now show that c_1^{crdR} and c_1^{dR} are compatible by similarity between constructions of crystalline and infinitesimal cohomologies. Let $\alpha : (\mathfrak{Z}_\eta/L)_{\inf} \to (\mathfrak{Z}/\mathcal{O}_L^\times)_{\mathrm{cris}}$ be the morphism of sites defined via the generic fibre functor sending $(\mathfrak{U} \hookrightarrow \mathfrak{T})$ over $\mathcal{O}_L^\times$ to $\mathfrak{U}_\eta \hookrightarrow \mathfrak{T}_\eta$ over L. Then α induces a morphism $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^\times) \xrightarrow{\simeq} R\alpha_* R\Gamma_{\mathrm{inf}}(\mathfrak{Z}_\eta/L)$. Since $\mathfrak{Z}$ is semistable, $\mathfrak{Z}$ is log-smooth over $\mathcal{O}_L^\times$ and $\mathfrak{Z}_\eta$ is smooth over L, hence in our definition of first Chern class maps, one can choose $\mathfrak{Z} \to \mathfrak{P}$ and $\mathfrak{Z}_\eta \hookrightarrow P$ to be identities. This gives the following commutative diagram

$$\begin{tikzcd}
M_{\mathfrak{Z}}^{\mathrm{gp}} \arrow[d, "\simeq"] & (1 \to M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[l, "\simeq"'] \arrow[r, "(\log,\mathrm{dlog})"] \arrow[d, "\simeq"] & (\mathcal{O}_{\mathfrak{Z}} \to \omega^1_{\mathfrak{Z}/\mathcal{O}_L^\times} \to \cdots) \arrow[r, equal] \arrow[d, "\simeq"] & \omega^\bullet_{\mathfrak{Z}/\mathcal{O}_L^\times}[1] \arrow[d, "\simeq \text{ after } (-)_{\mathbf{Q}_p}"] \\
\alpha_*\mathcal{O}_{\mathfrak{Z}_\eta}^\times & (1 \to \alpha_*\mathcal{O}_{\mathrm{inf},\mathfrak{Z}_\eta}^\times) \arrow[l, "\simeq"'] \arrow[r, "(\log,\mathrm{dlog})"] & (\alpha_*\mathcal{O}_{\mathrm{inf},\mathfrak{Z}_\eta} \to \alpha_*\Omega^1_{\mathfrak{Z}_\eta/L} \to \cdots) \arrow[r, "\simeq"] & R\alpha_*\Omega^\bullet_{\mathfrak{Z}_\eta/L}[1]
\end{tikzcd}$$

which calculates the diagram at the beginning of the proof involving c_1^{crdR} and c_1^{dR}; for the right vertical isomorphism, we notice that $R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathfrak{Z}/\mathcal{O}_L^\times})_{\mathbf{Q}_p} \xrightarrow{\simeq} R\Gamma(\mathfrak{Z}_\eta, \Omega^\bullet_{\mathfrak{Z}_\eta/K})$ for affine $\mathfrak{Z}$ semistable over $\mathcal{O}_L$).

By commutativity of (4.1.9.2), the morphism crystalline Chern class map c_1^{cris} factors through the fibre $[R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}^{\varphi=p} \to R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)/\mathrm{Fil}^1] = R\Gamma_{\mathrm{syn}}(\mathfrak{Z}_\eta, 1)$, thus defining (4.1.9.3).

Finally, for $Z \in \mathcal{R}ig_K$, by éh-descent, one defines its *syntomic first Chern class map*

$$c_1^{\mathrm{syn}} : \mathcal{O}_Z^\times \to R\Gamma_{\mathrm{syn}}(Z,1)[1] \tag{4.1.9.3}$$

in $\mathscr{D}(Z_{\text{éh}}, \mathbf{Z})$. Taking first cohomology, we obtain (by abuse of notation) a morphism

$$c_1^{\mathrm{syn}} : \mathrm{Pic}(Z) \simeq H^1_{\text{ét}}(Z, \mathcal{O}_Z^\times) \to H^2_{\mathrm{syn}}(Z,1). \tag{4.1.9.4}$$

In other words, we may associate with any line bundle $\mathcal{L}$ on $Z_{\text{ét}}$ a class $c_1^{\mathrm{syn}}(\mathcal{L}) \in H^2_{\mathrm{syn}}(Z,1)$.

4.1.10 - Remark. We could also have defined syntomic first Chern class using c_1^{HK} and c_1^{dR} if we have proven the compatibility between them, however, it seems that this might exist locally and there depend on the choice of a uniformizer of varying base fields L. The construction is as follows.

4.1.11 - Proposition. *Let $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L, and choose a uniformizer $\varpi \in L$. Then the following diagram commutes*

$$\begin{tikzcd}
R\Gamma(\mathfrak{Z}, M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[r] \arrow[d] & R\Gamma(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \arrow[r, "c_1^{\mathrm{HK}}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}[1] \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{arith}}"] \\
R\Gamma(\mathfrak{Z}_\eta, \mathcal{O}_{\mathfrak{Z}_\eta}^\times) \arrow[r, "c_1^{\mathrm{dR}}"] & \mathrm{Fil}^1 R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K) \arrow[r] & R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)[1]
\end{tikzcd}$$

whose homotopy depends on the class $l_\varpi = [\varpi]$ in $\mathfrak{m}_L/p\mathfrak{m}_L$ and commutes with N. Moreover, the syntomic first Chern class map defined by this is equivalent via certain homotopy depending on π to that of (4.1.9.3).

Lacking independency nor naturality on L of this homotopy, we are not sure how to deal with its globalisation. But the lemma remains useful, because in applications, we will often reduce the statement to certain local statement by other naturality results, where we then use this lemma to check isomorphisms.

Proof. The last statement follows from the commutativity, the equivalence (2.3.4), the compatibility between c_1^{HK} and c_1^{cris} (4.1.2), and that between c_1^{cris} and c_1^{dR} (4.1.9.2). We only need to treat the commutativity diagram.

We may assume $\mathfrak{Z}$ to be qcqs by Zariski descent; then $R\Gamma_{\mathrm{dR}}(\mathfrak{Z}_\eta/K)$ is represented by a bounded complex of K-Banach spaces, so that (1.3.11, ii) applies. Let $\mathfrak{X} = \mathfrak{Z} \otimes_{\mathcal{O}_L} \mathcal{O}_C \in \mathcal{M}_C^{\mathrm{ss},b}$. By definition of Hyodo-Kato morphisms (2.1.4) and compatibility between c_1^{dR} and c_1^{inf} (4.1.7), it suffices to prove that the following diagram

$$\begin{tikzcd}
R\Gamma(\mathfrak{Z}, M_{\mathfrak{Z}}^{\mathrm{gp}}) \arrow[r] \arrow[d] & R\Gamma(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \arrow[r, "c_1^{\mathrm{HK}}"] & R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}[1] \arrow[d, "\iota_{\mathrm{HK}}^{\mathrm{geom}}"] \\
R\Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta}^\times) \arrow[rr, "c_1^{\mathrm{inf}}"] & & R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+)[1]
\end{tikzcd}$$

commutes $\mathscr{G}_K$-equivariantly. The choice of uniformizer $\varpi \in \mathcal{O}_L$ determines a morphism $r_L^{\mathrm{PD}} \to \widehat{A}_{l_\varpi,\mathrm{st}}$, whence identifying $\iota_{\mathrm{HK}}^{\mathrm{geom}}$ through a homotopy α_π with

$$\begin{aligned}
R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \xrightarrow{\iota_{0,l}} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{\mathbf{Q}_p}^{N-\mathrm{nilp}} &\to R\Gamma_{\mathrm{cris}}(\mathfrak{X}_1/\widehat{A}_{l_\varpi,\mathrm{st}})_{\mathbf{Q}_p}^{N-\mathrm{nilp}} \\
&\xleftarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A_{\mathrm{cris}}^\times)_{\mathbf{Q}_p} \otimes_{B_{\mathrm{cris}}^+}^{\blacksquare} B_{\mathrm{st}}^+ \\
&\to R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+).
\end{aligned}$$

Here, $\iota_{0,l}$ is the Hyodo-Kato section, which is (φ, N)-equivariant [19, Theorem 2.12, Proposition 2.14].

Hence c_1^{PD} is compatible with $c_{1,l_\varpi}^{\mathrm{st}}$ by naturality (4.1.2). On the other hand, still by naturality, $c_{1,l_\varpi}^{\mathrm{st}}$ is compatible with $c_1^{A_{\mathrm{cris}}}$, which is in turn compatible with c_1^{inf} by similarity between crystalline and infinitesimal cohomologies via the morphism of sites $(\mathfrak{X}_\eta/B_{\mathrm{dR},m}^+)_{\mathrm{inf}} \to (\mathfrak{X}/A_{\mathrm{cris}}^\times)_{\mathrm{cris}}$ defined by the functor sending a pair $(\mathfrak{U} \hookrightarrow \mathfrak{T} = \operatorname{Spf} P)$ over $A_{\mathrm{cris}}^\times$ to $(\mathfrak{U}_\eta \hookrightarrow \mathfrak{T}_{B_{\mathrm{dR},m}^+} := \mathfrak{T}_\eta \times_{\operatorname{Spa} B_{\mathrm{cris}}^+} \operatorname{Spa} B_{\mathrm{dR},m}^+ = \operatorname{Spa}(B \hat{\otimes}_{A_{\mathrm{cris}}} B_{\mathrm{dR},m}^+))$ over $B_{\mathrm{dR},m}^+$.

It remains to show that there is a homotopy $\iota_{0,l} \circ c_1^{\mathrm{HK}} \simeq c_1^{\mathrm{PD}}$ commuting with N. It is enough to show that their difference $\delta := \iota_{0,l} \circ c_1^{\mathrm{HK}} - c_1^{\mathrm{PD}}$ is homotopic to zero. By compatibility of c_1^{PD} and c_1^{HK} via the natural map $p_0 : r_L^{\mathrm{PD}} \to \mathcal{O}_{F_L}^0$, and by the (φ, N)-equivariance of the isomorphism $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} r_L^{\mathrm{PD}}[\frac{1}{p}] \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{\mathbf{Q}_p}$, the difference δ factors φ-equivariantly through

$$\delta' : R\Gamma(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_1[1]$$

where $I_1 = \ker(r_L^{\mathrm{PD}} \to \mathcal{O}_{F_L})[\frac{1}{p}]$, which is a F_L-Banach space. Now, we need to show that the map (by abuse of notation) on cohomology groups

$$\delta' : H_{\text{ét}}^i(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \to H_{\mathrm{cris}}^{i+1}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_1 \tag{4.1.11.1}$$

is equal to zero.

For this, we need more notation. For $n \in \mathbf{N}$, denote

$$I_n := \left\{ \sum_{i=n}^{+\infty} a_i \frac{t_a^i}{\lfloor \frac{i}{e} \rfloor !} \mid a_i \in F_L, \lim_{i \to +\infty} a_i = 0 \right\} \subset r_L^{\mathrm{PD}}[\frac{1}{p}]. \tag{4.1.11.2}$$

This is a F-linear closed (whence Banach) subspace (even an ideal) with finite-dimensional complement $\bigoplus_{i=0}^{n-1} F_L \frac{t_a^i}{\lfloor \frac{i}{e} \rfloor !}$, and does not depend on the choice of $a \in (\mathfrak{m}_L/\mathfrak{m}_L^2) \backslash \{0\}$. The n-th power of frobenius φ^n on r_L^{PD} when restricted to I_1 factors through I_n, i.e. $\varphi^n(I_1) \subset I_n$.

Now we treat the vanishing of (4.1.11.1). Knowing that $\varphi(c_1^{\mathrm{HK}}) = p c_1^{\mathrm{HK}}$ and $\varphi(c_1^{\mathrm{PD}}) = p c_1^{\mathrm{PD}}$, we have $\varphi(\delta') = p\delta'$. By φ-equivariance of δ' and invertibility of $\varphi = p$ on $R\Gamma_{\text{ét}}(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}})_{\mathbf{Q}_p}$, we have $\delta' = \frac{1}{p^n}\varphi^n\delta'$ factoring as

$$\delta' : H_{\text{ét}}^i(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \to H_{\mathrm{cris}}^{i+1}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_n \to H_{\mathrm{cris}}^{i+1}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_1.$$

Hence δ' factors through

$$\delta'' : H_{\text{ét}}^i(\mathfrak{Z}_1^0, M_{\mathfrak{Z}_1^0}^{\mathrm{gp}}) \to \varprojlim_n (H_{\mathrm{cris}}^{i+1}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} t_a^{p^n-1} r_L^{\mathrm{PD}}[\frac{1}{p}])$$

where the transition maps are induced by inclusions of ideals $I_n = t_a^{p^n-1} r_L^{\mathrm{PD}} \subset r_L^{\mathrm{PD}}$. Since $\mathfrak{Z}$ is affine,

$R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ is represented by a bounded complex of F_L-Banach spaces (2.1.8). Hence $H^{i+1}_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ is the cokernel of a morphism between F_L-Banach spaces. The lemma (1.2.9.1) allows us to conclude. □

4.1.12 - Remark. We could also have resorted to the derived limit approach, knowing that $R\Gamma_{\mathrm{HK}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$ is a bounded complex of F_L-Banach spaces, for which tensor product commutes with countable products on each factor. Apparently however, this will not work, because $R\varprojlim_n t_a^{p^n-1} r_L^{\mathrm{PD}}[\frac{1}{p}] \simeq R^1\varprojlim_n t_a^{p^n-1} r_L^{\mathrm{PD}}[\frac{1}{p}][-1]$ which does not vanish by easy computation, e.g. $F\langle T\rangle \to \prod_{\mathbf{N}} F$ is not surjective. However, the last map is injective, which was our principal motivation of vanishing results in (1.2.9.2).

4.1.13. Étale construction. For Z either schemes or analytic adic spaces (so that $\mathcal{O}_Z^\times$ is an étale sheaf [34, 2.2.6]) over K (which is a nonarchimedean field of characteristic 0), there is a short exact sequence of étale sheaves named *Kummer sequence* on $Z_{\text{ét}}$

$$0 \to \mu_{p^n} \to \mathcal{O}_Z^\times \to \mathcal{O}_Z^\times \to 0$$

for any $n \in \mathbf{N}$, from which we obtain a mod p^n *étale Chern class map* $\mathcal{O}_Z^\times \to \mu_{p^n}[1]$ in $\mathscr{D}(Z_{\text{ét}}, \mathbf{Z}/p^n) \hookrightarrow \mathscr{D}(Z_{\text{ét}}, \mathbf{Z}_p)$, whence an *étale Chern class map*

$$c_1^{\text{ét}} : \mathcal{O}_Z^\times \to \mathbf{Z}_p(1)[1].$$

The algebraic and analytic étale Chern class maps are compatible through the analytification functor.

4.2 Projective bundle formula and $\mathbf{A}^1$-homotopy invariance

4.2.1. The (relative) cup product is defined in general for morphisms of ringed spaces [52, Remark 0B68]. In many cases, it can be quite explicit and such formula could be useful for proving compatibilities.

The relative cup product is compatible with respect to commutative squares, namely, for any commutative diagram of ringed spaces

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

and $F, G \in \mathrm{Mod}_{\mathcal{O}_X}$, the following diagram commutes naturally

$$\begin{tikzcd} g^*(Rf_*F \otimes^L Rf_*G) \arrow[r, "g^*(-\cup-)"] \arrow[d] & g^*Rf_*(F \otimes G) \arrow[d] \\ (Rf'_*g'^*F) \otimes^L (Rf'_*g'^*G) \arrow[r, "-\cup-"] & Rf'_*(g'^*F \otimes^L g'^*G). \end{tikzcd}$$

This follows formally from *loc. cit.*. Indeed, the vertical maps are induced by $g^*Rf_* \to Rf'_*g'^*$, adjoint to $f'^*g^*Rf_* \simeq g'^*f^*Rf_* \to g'^*$ the $g'^* \circ \mathrm{counit}$. The adjoint diagram (associated to $f'^* \dashv Rf'_*$) of the above is then

$$\begin{tikzcd} g'^*f^*(Rf_*F \otimes^L Rf_*G) \arrow[r, "g'^*f^*(-\cup-)"] \arrow[d, "\mathrm{unit}\circ g'^*\circ\mathrm{counit}"] & g'^*f^*Rf_*(F \otimes G) \arrow[d, "g'^*\circ\mathrm{counit}"] \\ f'^*(Rf'_*g'^*F) \otimes^L f'^*(Rf'_*g'^*G) \arrow[r, "\mathrm{counit}\circ g'^*"] & g'^*F \otimes^L g'^*G. \end{tikzcd}$$

The lower left two morphisms composed to a diagonal morphism $g'^* \circ \mathrm{counit}$, hence its natural commutativity is reduced to the commutativity of the upper triangle, and *a fortiori* that of the triangle

$$\begin{tikzcd} f^*(Rf_*F \otimes^L Rf_*G) \arrow[r, "f^*(-\cup-)"] \arrow[dr, "\text{counit}"'] & f^*Rf_*(F \otimes^L G) \arrow[d, "\text{counit}"] \\ & F \otimes^L G. \end{tikzcd}$$

But the diagonal morphism here is by definition adjoint to the cup product $-\cup-$, whence the commutativity.

4.2.2. Product structure on log-crystalline cohomology. Let $\mathscr{X}$ be an integral and quasi-coherent log-scheme over a quasi-coherent log pd-scheme $S^\sharp$, as in the setting of (4.1.1. The product structure on log-crystalline cohomology $R\Gamma_{\text{cris}}(\mathscr{X}/S^\sharp)$ is described as induced from the product structure of $\mathcal{O}_D \otimes_{\mathcal{O}_P} \omega^\bullet_{P/S^\sharp}$ when $\mathscr{X} \hookrightarrow P$ is a log pd-$S^\sharp$-smooth thickening with log pd-envelope $\mathscr{X} \hookrightarrow D$, then by étale hyperdescent in general cases. It satisfies higher associativity relations. The product structure thus defined is natural with respect to morphism of log-crystalline sites, and is compatible with Frobenius action in the characteristic p setting (4.1.3).

4.2.3 - Example. Let us continue the example (4.1.5): let $\mathfrak{Z} \in \mathcal{M}_K^{\text{ss}}$ with splitting field L; let l be a $\mathcal{O}_{F_L}$-class in $(\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$. The natural morphisms

$$R\Gamma_{\text{cris}}(\mathfrak{Z}) \to R\Gamma_{\text{cris}}(\mathfrak{Z}_1/r_L^{\text{PD}})_l \to R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/r_L^{\text{PD},0}) \to R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)$$

are compatible with φ-equivariant product structures.

4.2.4 - Lemma. *Consider a $\mathcal{O}_{F_L}$-class $l \in (\mathfrak{m}_L/p\mathfrak{m}_L)\backslash\{0\}$ associated with $i_l^* : r_L^{\text{PD}} \twoheadrightarrow \mathcal{O}_{L,1}^\times$ lifting $r_L^{\text{PD}} \twoheadrightarrow \mathcal{O}_{F_L,1}^0$ along $\mathcal{O}_{L,1}^\times \twoheadrightarrow \mathcal{O}_{F_L,1}^0$. Then the Hyodo-Kato section $\iota_{0,l} : R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\text{cris}}(\mathfrak{Z}_1/r_L^{\text{PD}})_{l,\mathbf{Q}_p}$ is compatible with product structures.*

Proof. The same strategy of the last part of the proof of (4.1.11) applies: one looks at the difference of two maps δ, which factors through $R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_1[1]$. By φ-equivariance of δ' and invertibility of φ on $R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}$, we have $\delta' = \varphi^n\delta'\varphi^{-n}$ factoring as

$$\delta' : R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{L\blacksquare} R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_n[1] \to R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \otimes_{F_L}^{\blacksquare} I_1[1].$$

Then we apply (1.2.9.1) to conclude. □

4.2.5 - Example. Let us continue the example (4.2.3), focusing now on the monodromy operators. We claim that for $M \in \{R\Gamma_{\text{cris}}(\mathfrak{Z}_1/r_L^{\text{PD}})_{l,\mathbf{Q}_p}, R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/r_L^{\text{PD},0})_{\mathbf{Q}_p}, R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}\}$, the product structure is compatible with the monodromy operator in the sense that the following diagram commutes

$$\begin{tikzcd} M \otimes_{F_L}^{L\blacksquare} M \arrow[r] \arrow[d, "N\otimes \mathrm{id}_M + \mathrm{id}_M \otimes N"] & M \arrow[d, "N"] \\ M \otimes_{F_L}^{L\blacksquare} M \arrow[r] & M \end{tikzcd}$$

and satisfies higher associativity relations. Indeed, in these cases, the monodromy operators are induced (in a canonical way[20]) from the Lie algebra action of $\mathbf{G}_m^\natural$-action on such M (cf. [19, Paragraphs around (2.15)]), and the concerned product structures are all $\mathbf{G}_m^\natural$-equivariant. The equivariance of the Lie algebra action is then illustrated exactly by the diagram above.

4.2.6 - Lemma. *Let (A,ϕ), (B,ψ) and (C,χ) be three objects with endormorphism in a $\otimes$-stable ∞-category (hence have fibre sequences and additive structure on* Hom*'s), together with a pairing $\mu : A \otimes B \to C$ such that*

[20] We took the rational coefficients to make the monodromy action canonical, i.e. $N = \frac{1}{e_L} t\partial_t$ [19, The paragraph after (2.15)].

$\psi_A \otimes \chi_B + \chi_A \otimes \psi_B$ is compatible with ψ_C, where $\chi_A \in \mathrm{End}(A)$ and $\chi_B \in \mathrm{End}(B)$. Then μ induces a unique pairing $\mathrm{fib}(\psi_A) \otimes \mathrm{fib}(\psi_B) \to \mathrm{fib}(\chi)$.

Proof. This is formal, and can be read off from the following commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(\psi_A) \otimes \mathrm{fib}(\psi_B) \arrow[r, dashed] \arrow[d] & \mathrm{fib}(\psi_C) \arrow[d] \\
A \otimes B \arrow[r, "\mu"] \arrow[d, "\psi_A\otimes\chi_B+\chi_A\otimes\psi_B"] & C \arrow[d, "\psi_C"] \\
A \otimes B \arrow[r, "\mu"] & C
\end{tikzcd}$$

whose left column composes naturally to zero. □

4.2.7 - Example. The natural morphisms

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}^{N=0} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}^{N=0} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}^{N=0}$$

are compatible with φ-equivariant product structures, where the last three objects are endowed with the canonical product structure by (4.2.6).

4.2.8 - Example. We want to study product structures on Frobenius fixed points $(-)^{\varphi=p^r}$ for $r \in \mathbf{Z}$. Intuitively speaking, the product of a $\varphi = p^r$ eigenvector and a $\varphi = p^s$ eigenvector should be a $\varphi = p^{r+s}$ eigenvector, so we should expect a product structure on Frobenius fixed points of the form

$$A^{\varphi=p^r} \otimes B^{\varphi=p^s} \to C^{\varphi=p^{r+s}}.$$

We claim that it is indeed the case. For this, in order to apply (4.2.6), consider $\psi_A = \varphi_A - p^r$, $\chi_B = p^s$, $\chi_A = p^r\varphi_A$ and $\psi_B = \varphi_B - p^s$. Then $\psi_A \otimes \chi_B + \chi_A \otimes \psi_B = \varphi_A \otimes \varphi_B - p^{r+s}$ is compatible with $\varphi_C - p^{r+s}$. It is direct but probably tedious to check that the product structure thus defined satisfies higher associativity relations.

As a result, the natural morphisms

$$R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}^{\varphi=p^r} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p}^{\varphi=p^r,N=0} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}^{\varphi=p^r,N=0} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}^{\varphi=p^r,N=0}$$

for $r \in \mathbf{Z}$ are compatible with product structures.

4.2.9. Product structure on de Rham cohomology. For $Z \in \mathcal{R}\mathrm{ig}_K$, we have a product structure

$$\mathrm{Fil}^r R\Gamma_{\mathrm{dR}}(Z/K) \otimes_K^{L\blacksquare} \mathrm{Fil}^s R\Gamma_{\mathrm{dR}}(Z/K) \to \mathrm{Fil}^{r+s} R\Gamma_{\mathrm{dR}}(Z/K)$$

for $r, s \in \mathbf{Z}$ induced from the one on the level of complexes of sheaves. For $X \in \mathcal{R}\mathrm{ig}_C$, similarly (combining the the log-crystalline setup (4.2.2) and the de Rham setup above), we obtain a product structure

$$\mathrm{Fil}^r R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \otimes_K^{L\blacksquare} \mathrm{Fil}^s R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \to \mathrm{Fil}^{r+s} R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+)$$

for $r, s \in \mathbf{Z}$. They all satisfy higher associativity relations.

For $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L and $\mathfrak{X} \in \mathcal{M}_C^{\mathrm{ss}}$, the natural "base change" morphisms $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}/\mathcal{O}_L^\times) \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_\eta/K)$ and $R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A_{\mathrm{cris}}^\times) \to R\Gamma_{\mathrm{inf}}(\mathfrak{X}_\eta/B_{\mathrm{dR}}^+)$ are compatible with product structures, again by similarity between crystalline and infinitesimal constructions.

To induce the product structure on syntomic cohomologies from the above, one needs the following lemma.

4.2.10 - Lemma. *The Hyodo-Kato section* $\iota_0 : R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}$ *is compatible with product structures.*

Proof. The same proof as (4.2.4) (or by composing $\iota_{0,l}$ with the natural base change map $R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1/r_L^{\mathrm{PD}})_{l,\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})\mathbf{Q}_p$). □

4.2.11. Product structure on syntomic cohomology. Let $Z \in \mathcal{R}ig_K$ and $r, r' \in \mathbf{N}$, there is a natural product structure [55, §2.2]

$$R\Gamma_{\mathrm{syn}}(Z,r) \otimes^{L\blacksquare}_{\mathbf{Q}_p} R\Gamma_{\mathrm{syn}}(Z,r') \to R\Gamma_{\mathrm{syn}}(Z,r+r') \tag{4.2.11.1}$$

given by the formula $(x,y)\otimes(x',y') \mapsto (xx', (-1)^q xy' + y\varphi_{r'}(x'))$, but we explain it now using previous constructions. For this, taking the Bloch-Kato point of view, we need to show that $\iota_{\mathrm{HK}}^{\mathrm{arith}} : R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p}^{\varphi=p^r, N=0} \to R\Gamma_{\mathrm{dR}}(Z/K)/\mathrm{Fil}^r$ is compatible with product structures hence induces a canonical product structure on its fibre $R\Gamma_{\mathrm{syn}}^{\mathrm{BK}}(Z,r)$. Again, we may work locally and assume that $Z = \mathfrak{Z}_\eta$ with $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ with splitting field L. Going through the construction of (2.1.4), using (4.2.3), (4.2.7), (4.2.8) and (4.2.9), we only need to prove that the Hyodo-Kato section $\iota_0 : R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)_{\mathbf{Q}_p} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z}_1^0/r_L^{\mathrm{PD},0})_{\mathbf{Q}_p}$ is compatible with product structures; but this has just been done in (4.2.10).

Moreoever, the maps of the diagrams in the proof of (2.3.3) are compatible with product structures again by (4.2.3), (4.2.7), (4.2.8), (4.2.9) and (4.2.10), whence it is the same product structure as defined by $R\Gamma_{\mathrm{syn}}^{\mathrm{FM}}(\mathfrak{Z},r) = [R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}^{\varphi=p^r} \to R\Gamma_{\mathrm{cris}}(\mathfrak{Z})_{\mathbf{Q}_p}/\mathrm{Fil}^r]$.

4.2.12 - Lemma. *For* $Z \in \mathcal{R}ig_K$, *the natural syntomic-proétale period map* $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ *is compatible with the product structure on syntomic cohomology.*

Proof. In light of the construction of the syntomic-proétale period map (3.1.3) by taking Galois invariants of the diagram (3.1.3.1), it suffices to show that each morphism in this last diagram is compatible with product structures, and that the maps $R\Gamma_{\mathrm{syn}}(Z,r) \to R\Gamma_{\mathrm{syn}}(Z_C,r)$ is compatible with product structures. The second compatibility is clear by construction. The first compatibility follows from the (φ,N)-equivariance of the comparison map $R\Gamma_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{F}} B_{\log}[\frac{1}{t}] \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{B}_{\log}[\frac{1}{t}])$ as can be deduced from (3.1.2.4). □

4.2.13. Products with first Chern classes. Let $\mathcal{E}$ be an (analytic/étale) vector bundle of rank $d+1$, $d \geq 0$, on Z. Let $\pi : \mathbf{P}_Z(\mathcal{E}) \to Z$ be its associated projective bundle and $\mathcal{O}(1)$ be its canonical bundle. The syntomic first Chern class (4.1.9.4) defines maps

$$c_1^{\mathrm{syn}}(\mathcal{O}(1))^i \cup \pi^* : R\Gamma_{\mathrm{syn}}(Z, r-i) \xrightarrow{\pi^*} R\Gamma_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}), r-i) \xrightarrow{c_1^{\mathrm{syn}}(\mathcal{O}(1))^i \cup -} R\Gamma_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}), r)[2i]$$

for $0 \leq i \leq r$. Indeed, (4.2.11.1) induces a map

$$R\Gamma_{\mathrm{syn}}(Z,1) \to R\,\mathrm{Hom}(R\Gamma_{\mathrm{syn}}(Z,r-1), R\Gamma_{\mathrm{syn}}(Z,r))$$

hence taking H^2, one gets

$$H^2_{\mathrm{syn}}(Z,1) \to \mathrm{Hom}(R\Gamma_{\mathrm{syn}}(Z,r-1), R\Gamma_{\mathrm{syn}}(Z,r)[2]).$$

The image of a class c is the cup product denoted by $c \cup -$. By itertation on i, one obtains

$$\prod_i H^2_{\mathrm{syn}}(Z,1) \to \mathrm{Hom}(R\Gamma_{\mathrm{syn}}(Z,r-i), R\Gamma_{\mathrm{syn}}(Z,r)[2i])$$

for $0 \leq i \leq r$. The image of the constant tuple $c_1^{\mathrm{syn}}(\mathcal{O}(1))^{[[1,i]]}$ is denoted by $c_1^{\mathrm{syn}}(\mathcal{O}(1))^i \cup$.

Similarly, for Z a scheme or analytic adic space over K, the étale first Chern class (4.1.13) defines maps

$$c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^* : R\Gamma_{\text{ét}}(Z, \mu_{p^n}^{\otimes(r-i)}) \to R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mu_{p^n}^{\otimes r})[2i]$$

whence maps

$$c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^* : R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r-i)) \to R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Z}_p(r))[2i].$$

These commute with the analytification functor since so do $c_1^{\text{ét}}$ and $-\cup-$ by (4.2.1).

4.2.14 - Lemma. *Let $\mathfrak{Z} \in \mathcal{M}_K^{\text{ss}}$ with splitting field L, and choose a uniformizer $\varpi \in L$. Then the product structures on $R\Gamma_{\text{cris}}(\mathfrak{Z}_1^0/\mathcal{O}_{F_L}^0)$ and $R\Gamma_{\text{dR}}(\mathfrak{Z}_\eta/K)$ are compatible across the arithmetic Hyodo-Kato morphism $\iota_{\text{HK}}^{\text{arith}}$, whose homotopy depends on the class $l_\varpi = [\varpi]$ in $\mathfrak{m}_L/p\mathfrak{m}_L$.*

Proof. The same strategy as in the proof of (4.1.11), whose most essential part is dealt with in (4.2.4). □

4.2.15 - Proposition (Projective bundle formula). *For $r \geq d$, there is a natural isomorphism*

$$\bigoplus_{i=0}^{d}(c_1^{\text{syn}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} R\Gamma_{\text{syn}}(Z, r-i)[-2i] \xrightarrow{\simeq} R\Gamma_{\text{syn}}(\mathbf{P}_Z(\mathcal{E}), r).$$

Proof. By admissible descent, we may assume that $\mathcal{E} = \mathcal{O}_Z^{\oplus(d+1)}$, then $\mathbf{P}_Z(\mathcal{E}) = \mathbf{P}_Z^d$. By éh-descent, one may assume that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z} \in \mathcal{M}_K^{\text{ss}}$ is affine with splitting field L, which we may furthermore assume to be algebraizable to an affine scheme $\mathscr{Z}$ flat and η-smooth over $\mathcal{O}_L$ whose p-adic formal completion is $\mathfrak{Z}$ (by [25, Theorem 7], cf. [53, Corollary 3.3.2]). By compatiblity of first Chern class maps (4.1.11) and product structures (4.2.14), we are reduced to showing that

$$(\mathrm{Proj}_{\mathrm{HK}}) \qquad \bigoplus_{i=0}^{d}(c_1^{\text{HK}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} R\Gamma_{\text{HK}}(Z)[-2i] \xrightarrow{\simeq} R\Gamma_{\text{HK}}(\mathbf{P}_Z^d)$$

$$(\mathrm{Proj}_{\mathrm{dR},r}) \qquad \bigoplus_{i=0}^{d}(c_1^{\text{dR}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} \mathrm{Fil}^{r-i} R\Gamma_{\text{dR}}(Z/K)[-2i] \xrightarrow{\simeq} \mathrm{Fil}^r R\Gamma_{\text{dR}}(\mathbf{P}_Z^d/K)$$

are isomorphisms. Since Z has a semistable model $\mathfrak{Z}$, ($\mathrm{Proj}_{\mathrm{HK}}$) reduces to ($\mathrm{Proj}_{\mathrm{dR},r}$) with $r = 0$ by the Hyodo-Kato isomorphism (2.1.9). So it suffices to prove ($\mathrm{Proj}_{\mathrm{dR},r}$). Recall that the algebraic analogue of it holds for $\mathscr{Z}_\eta$, cf. [46, Proof of Proposition 5.2]; indeed, the statement, being analytically local, is refined to a sheaf theoretic isomorphism

$$(\mathrm{Proj}_{\mathrm{dR},r}^{\mathrm{alg}}) \qquad \bigoplus_{i=0}^{d}(c_1^{\text{dR}}(\mathcal{O}(1))^i \cup \pi^{\text{alg}*}) : \bigoplus_{i=0}^{d} \Omega^{\bullet \geq r-i}_{\mathscr{Z}_\eta/K}[-2i] \xrightarrow{\simeq} R\pi_*^{\text{alg}} \Omega^{\bullet \geq r}_{\mathbf{P}^d_{\mathscr{Z}_\eta}/K},$$

which can be easily checked on stalks [52, Proposition 0FMR, Proposition 0FMT]. By relative GAGA [21, Example 3.2.6, Appendix (A.1.1)] [40, §p. 43-53] and compatibility between algebraic and rigid-analytic de Rham first Chern class maps, one obtains

$$(\mathrm{Proj}'_{\mathrm{dR},r}) \qquad \bigoplus_{i=0}^{d}(c_1^{\text{dR}}(\mathcal{O}(1))^i \cup \pi^{\text{alg}*}) : \bigoplus_{i=0}^{d} \Omega^{\bullet \geq r-i}_{(\mathscr{Z}_\eta)^{\text{an}}/K}[-2i] \xrightarrow{\simeq} R\pi_* \Omega^{\bullet \geq r}_{\mathbf{P}^d_{(\mathscr{Z}_\eta)^{\text{an}}}/K}.$$

Since $Z = \mathfrak{Z}_\eta = \mathscr{Z}_\eta^{\text{rig}}$ is an affinoid open subspace of $(\mathscr{Z}_\eta)^{\text{an}}$, one obtains ($\mathrm{Proj}_{\mathrm{dR},r}$) by taking $R\Gamma(Z, -)$ of this isomorphism. □

4.2.16 - Proposition ($\mathbf{A}^1$-homotopy invariance). *Let $\pi : \mathbf{A}_Z^1 \to Z$ be the relative analytic affine line over Z. Then for any $r \in \mathbf{N}$, the pullback induces a natural isomorphism*

$$\pi^* : R\Gamma_{\text{syn}}(Z, r) \to R\Gamma_{\text{syn}}(\mathbf{A}_Z^1, r).$$

Proof. We are reduced to showing the $\mathbf{A}^1$-homotopy invariance of $R\Gamma_{\mathrm{HK}}(-,r)$ and $\mathrm{Fil}^\bullet R\Gamma_{\mathrm{dR}}(-/K)$ respectively for $Z \in \mathcal{R}\mathrm{ig}_K$ and $r \in \mathbf{N}$. By éh-descent, one may assume that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z} \in \mathcal{M}_K^{\mathrm{ss}}$ is affine with splitting field L, which we may furthermore assume to be algebraizable to an affine scheme $\mathscr{Z}$ flat, of finite type and η-smooth over $\mathcal{O}_L$ whose p-adic formal completion is $\mathfrak{Z}$, as precedingly. The Hyodo-Kato statement reduces to the Fil^0 de Rham statement by the Hyodo-Kato isomorphism (2.1.9). One can check the latter directly, or by Gysin sequence using the previous proposition.

Let us explain the second approach. Let $\bar{\pi} : \mathbf{P}^1_Z \to Z$ be the relative projective line, $j : \mathbf{A}^1_Z \to \mathbf{P}^1_Z$ and $i_\infty : Z \hookrightarrow \mathbf{P}^1_Z$ be the usual excision couple. For any coherent sheaves $\mathcal{F}$ on $\mathbf{A}^1_Z$, the higher direct images $R^i j_* \mathcal{F} = 0$ for $i \geq 1$; indeed, $H^i(U \cap \mathbf{A}^1_Z, \mathcal{F})$ vanishes for affinoid open $U \subset \mathbf{P}^1_Z$, since $U \cap \mathbf{A}^1_Z$ is then quasi-Stein and Kiehl's Cartan's Theorem B [39, Satz 2.4] (cf. [45, Remark 1.4]). As a result, $R j_* \Omega^{\bullet \geq r}_{\mathbf{A}^1_Z/K}$ is represented by the genuine complex $j_* \Omega^{\bullet\geq r}_{\mathbf{A}^1_Z/K}$. There is an exact sequence of complexes of étale sheaves

$$0 \to \Omega^{\bullet\geq r}_{\mathbf{P}^1_Z/K} \to j_*\Omega^{\bullet\geq r}_{\mathbf{A}^1_Z/K} \xrightarrow{\mathrm{Res}} i_{\infty *}\Omega^{\bullet\geq r-1}_{Z/K} \to 0$$

where Res is taking the residue along this closed subspace. The de Rham first Chern class of the tautological bundle is $c_1^{\mathrm{dR}}(\mathcal{O}(1)) = [\mathrm{dlog}\, t]$, whose representative differential form $\mathrm{dlog}\, t$ induces by wedge product the connecting morphism in derived category. So $\delta : i_{\infty *}\Omega^{\bullet\geq r-1}_{Z/K}[-1] \to \Omega^{\bullet\geq r}_{\mathbf{P}^1_Z/K}$ is identified with $c_1^{\mathrm{dR}}(\mathcal{O}(1)) \cup \bar{\pi}^*$. Now we obtain the following commutative diagram

$$\begin{tikzcd}
\Omega^{\bullet\geq r-1}_{Z/K}[-1] \arrow[r] \arrow[d, "\simeq"] & \Omega^{\bullet\geq r-1}_{Z/K}[-1] \oplus \Omega^{\bullet\geq r}_{Z/K} \arrow[r] \arrow[d, "\simeq", "(c_1^{\mathrm{dR}}(\mathcal{O}(1))\cup\pi^*)\oplus\pi^*"'] & \Omega^{\bullet\geq r}_{Z/K} \arrow[d, "\pi^*"] \\
R\bar{\pi}_* i_{\infty *}\Omega^{\bullet\geq r-1}_{Z/K}[-1] \arrow[r, "R\bar{\pi}(\delta)"] & R\bar{\pi}_*\Omega^{\bullet\geq r}_{\mathbf{P}^1_Z/K} \arrow[r] & R\bar{\pi}_* j_*\Omega^{\bullet\geq r}_{\mathbf{A}^1_Z/K}
\end{tikzcd}$$

where rows are exact triangles, the first two vertical arrows are isomorphisms since $\bar{\pi} i_\infty = \mathrm{id}_Z$ and ($\mathrm{Proj}'_{\mathrm{dR},r}$), whence so is the third. □

4.2.17 - Proposition (Projective bundle formula for (pro)étale cohomology). *For $r \geq d$, there is a natural isomorphism*

$$\bigoplus_{i=0}^{d} (c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r-i))[-2i] \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Z}_p(r)).$$

In particular, there are natural isomorphisms

$$\bigoplus_{i=0}^{d} (c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} R\Gamma_{\text{ét}}(Z, \mathbf{Q}_p(r-i))[-2i] \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Q}_p(r))$$

$$\bigoplus_{i=0}^{d} (c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} R\Gamma_{\text{proét}}(Z, \mathbf{Q}_p(r-i))[-2i] \xrightarrow{\simeq} R\Gamma_{\text{proét}}(\mathbf{P}_Z(\mathcal{E}), \mathbf{Q}_p(r)).$$

Proof. The second isomorphism is the first with p inverted. The third one reduces to the case where Z is affine by analytic descent, which agrees with the second by quasi-compactness of Z.

For the first isomorphism, it suffices to prove that

$$\bigoplus_{i=0}^{d} (c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^*) : \bigoplus_{i=0}^{d} \mu_{p^n}^{\otimes(r-i)}[-2i] \xrightarrow{\simeq} R\pi_*\mu_{p^n}^{\otimes r}. \tag{4.2.17.1}$$

As these are discrete sheaves, once proven, it can be upgraded to an isomorphism in $\mathscr{D}(Z_{\text{ét}}, \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})$.

The question being éh-local, we may assume that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z}$ is a p-adic formal scheme algebraizable to an affine scheme $\mathscr{Z}$ flat, of finite type and η-smooth over $\mathcal{O}_L$ for some finite extension L/K (cf. beginning of the proof of (4.2.15)). There is a projective bundle formula for mod p^n étale cohomology of schemes over K

(cf. [57, Theorem 6.1.7]), so there is a natural isomorphism

$$\bigoplus_{i=0}^{d}(c_1^{\text{ét}}(\mathcal{O}(1))^i \cup \pi^{\text{alg}*}) : \bigoplus_{i=0}^{d} \mu_{p^n}^{\otimes(r-i)}[-2i] \xrightarrow{\simeq} R\pi_* \mu_{p^n}^{\otimes r}$$

in $\mathcal{D}((\mathcal{X}_\eta)_{\text{ét}}, Z/p^n)$. By Huber's comparison [34, Theorem 3.8.1] and its compatibility between products with first Chern classes (4.2.13), this implies the desired (4.2.17.1). □

4.2.18 - Proposition ($\mathbf{A}^1$-homotopy invariance of integral étale cohomology). *Let $\pi : \mathbf{A}^1_Z \to Z$ be the relative analytic affine line over Z. Then for any $r \in \mathbf{N}$, the pullback induces a natural isomorphism*

$$\pi^* : R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r)) \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{A}^1_Z, \mathbf{Z}_p(r)).$$

In particular, the pullback induces a natural isomorphism

$$\pi^* : R\Gamma_{\text{ét}}(Z, \mathbf{Q}_p(r)) \xrightarrow{\simeq} R\Gamma_{\text{ét}}(\mathbf{A}^1_Z, \mathbf{Q}_p(r)).$$

Proof. The second isomorphism reduces to the first, since by definition $R\Gamma_{\text{ét}}(Z, \mathbf{Q}_p(r)) := R\Gamma_{\text{ét}}(Z, \mathbf{Z}_p(r))_{\mathbf{Q}_p}$.

To prove the first, we may assume by éh-descent that $Z = \mathfrak{Z}_\eta$ where $\mathfrak{Z}$ is a p-adic formal scheme algebraizable to an affine scheme $\mathcal{X}$ flat, of finite type and η-smooth over $\mathcal{O}_L$ for some finite extension L/K (cf. beginning of the proof of (4.2.15)). By Huber's comparison theorem [34, Theorem 3.8.1], we have the natural base change isomorphism of étale sheaves

$$\varphi^*_{\mathcal{X}_\eta} R\pi^{\text{alg}}_* \mu_{p^n}^{\otimes r} \xrightarrow{\simeq} R\pi_* \mu_{p^n}^{\otimes r},$$

where $\varphi_{\mathcal{X}_\eta} : (\mathcal{X}_\eta)^{\text{an}}_{\text{ét}} \to (\mathcal{X}_\eta)_{\text{ét}}$ is the natural morphism of sites defined by the analytification functor. But we have $F \xrightarrow{\simeq} R\pi^{\text{alg}}_* \pi^{\text{alg}*} F$ by the $\mathbf{A}^1$-homotopy invariance of torsion sheaves F on $(\mathcal{X}_\eta)_{\text{ét}}$ (since its base field L is of characteristic 0)[21]. Therefore, we obtain isomorphisms of étale sheaves

$$\mu_{p^n}^{\otimes r} = \varphi^*_{\mathcal{X}_\eta} \mu_{p^n}^{\otimes r} \xrightarrow{\simeq} \varphi^*_{\mathcal{X}_\eta} R\pi^{\text{alg}}_* \mu_{p^n}^{\otimes r} \xrightarrow{\simeq} R\pi_* \mu_{p^n}^{\otimes r}.$$

Since they are discrete objects on quasi-compact schemes, these isomorphisms upgrade to isomorphisms in $\mathcal{D}(Z_{\text{ét}}, \mathcal{C}\text{ond}\mathcal{A}\text{b})$. Evaluating them on $Z = \mathfrak{Z}_\eta = \mathcal{X}^{\text{rig}}_\eta$, which is an affinoid open subspace of $(\mathcal{X}_\eta)^{\text{an}}$, one gets the $\mathbf{A}^1$-homotopy invariance for the coefficients $\mu_{p^n}^{\otimes r}$, from which follows the desired isomorphism by taking limits over $n \in \mathbf{N}$. □

4.2.19 - Remark. Unfortunately, there is no natural isomorphism

$$\pi^* : R\Gamma_{\text{proét}}(Z, \mathbf{Q}_p(r)) \xrightarrow{\simeq} R\Gamma_{\text{proét}}(\mathbf{A}^1_Z, \mathbf{Q}_p(r))$$

since $R\Gamma_{\text{proét}}(\mathbf{A}^1_Z, \mathbf{Q}_p(r)) \not\simeq R\Gamma_{\text{ét}}(\mathbf{A}^1_Z, \mathbf{Z}_p(r))_{\mathbf{Q}_p}$. It fails even for $Z = \text{Spa}(K, \mathcal{O}_K)$; indeed, $H^1_{\text{proét}}(\mathbf{A}^1_K, \mathbf{Q}_p(1)) = 0$ and $H^1_{\text{proét}}(\mathbf{A}^1_K, \mathbf{Q}_p(1)) \xrightarrow{\simeq} \Omega^1(\mathbf{A}^1_K)$ by Colmez-Nizioł's computation [18] and the Hochschild-Serre spectral sequence, which is huge.

Nevertheless, we have a so-called *fundamental motivic spectrum* $(R\Gamma_{\text{proét}}(-, \mathbf{Q}_p(r)))_r$ in the sense of [2, Definition 2.3.2]. Recall that the ∞-category of "motivic spectra" is defined as

$$\mathcal{S}\text{p}_{\mathbf{P}^1} := \mathcal{S}\text{p}_{\mathbf{P}^1}(\mathcal{S}\text{hv}_{\text{Zar}}(\mathcal{S}\text{m}_{\mathbf{Z}}, \mathcal{S}\text{pc}_*)) := \mathcal{S}\text{hv}_{\text{Zar}}(\mathcal{S}\text{m}_{\mathbf{Z}}, \mathcal{S}\text{pc}_*)[(\mathbf{P}^1)^{-1}]$$

for the $\mathbf{P}^1$-action by the pointed projective line $(\mathbf{P}^1, \infty)$ on $\mathcal{S}\text{hv}_{\text{Zar}}(\mathcal{S}\text{m}_{\mathbf{Z}}, \mathcal{S}\text{pc})$[22]. A motivic spectra be-

[21]Or by a seemingly stronger result: locally acyclicity of smooth morphisms of schemes.

[22]Recall that the usual spectra construction inverts the usual action of the unit sphere $\mathbf{S}^1$ on $\mathcal{S}\text{pc}_*$, which is the same as telescoping the action $\mathbf{S}^1 \otimes -$ since the cyclic action $\tau = (123) : (\mathbf{S}^1)^{\otimes 3} \to (\mathbf{S}^1)^{\otimes 3}$ is homotopic to id, as a result $(-)[\mathbf{S}^{-1}] \simeq \text{Tel}_{\mathbf{S}^1}(-) := \text{colim}_{\mathbf{S}^1 \otimes -}(-)$ is equivalent to the telescope construction. However, by contrast, $\mathcal{S}\text{hv}_{\text{Zar}}(\mathcal{S}\text{m}_{\mathbf{Z}}, \mathcal{S}\text{pc}_*)[(\mathbf{P}^1)^{-1}]$ is rather the *symmetric* telescoping

ing "fundamental" means roughly that it satifies Bass fundamental exact sequence. Of course, for any $\mathcal{V} \in \mathrm{Mod}_{\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc})}(\mathrm{Pr}^{L,\otimes})$, i.e. any ∞-category presentably tensored over $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc})$ (i.e. presentable ∞-category tensored together with a tensor action of $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc})$ that preserves colimits in each variable), we can define $\mathcal{S}\mathrm{p}_{\mathbf{P}^1}(\mathcal{V}_*)$ by formally inverting the $\mathbf{P}^1$-action on $\mathcal{V}_*$. For example, consider the presentably symmetric monoidal ∞-category $\mathcal{V} := \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K,\mathcal{S}\mathrm{p})$ which admits a symmetric monoidal functor from $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc})$ via $\mathcal{S}\mathrm{pc} \to \mathcal{S}\mathrm{p}$ and base change (left Kan extension) along $\mathcal{S}\mathrm{m}_{\mathbf{Z}} \to \mathcal{S}\mathrm{m}_{\mathcal{O}_K} \to \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ where the last arrow is given by taking the rigid generic fibre $(-)^{\mathrm{rig}}_{\eta}$. For $E = (E_r)_{r\in\mathbf{N}} := (R\Gamma_{\mathrm{pro\acute{e}t}}(-,\mathbf{Q}_p(r)))_{r\in\mathbf{N}}$, we have the following:

(i) The proétale cohomology satisfies Nisnevich descent, even éh-descent, on $\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$.

(ii) The system E forms naturally a $\mathbf{P}^1$-spectrum in $\mathcal{V}$, i.e. one can naturally make $E \in \mathcal{S}\mathrm{p}_{\mathbf{P}^1}(\mathcal{V})$. Indeed, there is an equivalence

$$\pi^* \oplus c_1^{\mathrm{\acute{e}t}}(\mathcal{O}(1)) \cup \pi^* : R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \oplus R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r-1))[-2] \xrightarrow{\simeq} R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{P}^1_X,\mathbf{Q}_p(r))$$

provided by the projective bundle formula (4.2.17). So we have $c_1^{\mathrm{\acute{e}t}}(\mathcal{O}(1)) : E_{r-1} \xrightarrow{\simeq} \underline{\mathrm{Hom}}((\mathbf{P}^1_K,\infty),E)$.

(iii) It is a fundamental $\mathbf{P}^1$-spectrum, i.e. the Bass boundary map

$$\partial^* : E^{\mathbf{S}^1\otimes\mathbf{G}_m} \to E^{\mathbf{P}^1},$$

identified as

$$\partial^*_X : R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,X},\mathbf{Q}_p(r))[-1] \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{P}^1_X,\mathbf{Q}_p(r)),$$

admits a natural right inverse, namely there exists a natural section $s_X : R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{P}^1_X,\mathbf{Q}_p(r)) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,X},\mathbf{Q}_p(r))[-1]$ such that naturally $\partial^*_X s_X \simeq \mathrm{id}$. Indeed, there is a chain of morphisms

$$\begin{aligned} s_X : R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{P}^1_X,\mathbf{Q}_p(r)) &\simeq \lim_{U\in\mathcal{A}\mathrm{ffd}} R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{P}^1_U,\mathbf{Q}_p(r)) \\ &\xleftarrow{\simeq} \lim_{U\in\mathcal{A}\mathrm{ffd}} R\Gamma_{\mathrm{\acute{e}t}}(\mathbf{P}^1_U,\mathbf{Q}_p(r)) \\ &\xrightarrow{(s_U^{\mathrm{\acute{e}t}})_U} \lim_{U\in\mathcal{A}\mathrm{ffd}} R\Gamma_{\mathrm{\acute{e}t}}(\mathbf{G}_{m,U},\mathbf{Q}_p(r))[-1] \\ &\to \lim_{U\in\mathcal{A}\mathrm{ffd}} R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,U},\mathbf{Q}_p(r))[-1] \\ &\simeq R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,U},\mathbf{Q}_p(r))[-1] \end{aligned}$$

where $s_U^{\mathrm{\acute{e}t}}$ is obtained by comparison with algebraic p-adic étale cohomology over the characteristic 0 field K, and satisfies $\partial_U^{*,\mathrm{\acute{e}t}} s_U^{\mathrm{\acute{e}t}} = \mathrm{id}$, whence $\partial^*_U s_U = \mathrm{id}$ thus $\partial^*_X s_X = \mathrm{id}$. Therefore, E is a fundamental motivic spectrum.

4.3 Chern classes for vector bundles

4.3.1. Syntomic Chern classes. Let $Z \in \mathcal{R}\mathrm{ig}_K$. Using the projective bundle formula (4.2.15) and Chern class maps

(4.3.1.1) $$c_0^{\mathrm{syn}} : \mathbf{Q}_p \xrightarrow{\mathrm{can}} R\Gamma_{\mathrm{syn}}(Z,0), \quad c_1^{\mathrm{syn}} : \mathcal{O}_Z^{\times} \to R\Gamma_{\mathrm{syn}}(Z,1)[1],$$

we obtain syntomic Chern classes $c_i^{\mathrm{syn}}(\mathcal{E})$ for any locally free sheaf $\mathcal{E}$ on Z. More precisely, there are unique classes $c_i^{\mathrm{syn}}(\mathcal{E}) \in H^{2i}_{\mathrm{syn}}(Z,i)$ for $i = 1,\dots,d+1$ such that

$$\sum_{i=0}^{d} c_1^{\mathrm{syn}}(\mathcal{O}(1))^i \cup \pi^* c_{d+1-i}^{\mathrm{syn}}(\mathcal{E}) = c_1^{\mathrm{syn}}(\mathcal{O}(1))^{d+1} c_0^{\mathrm{syn}}(1) \in H^{2(d+1)}_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}),d+1).$$

procedure (or the so-called "*symmetric* spectra" $\mathrm{Sp}^{\Sigma}(-)$ construction) and admits a forgetful map $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc}_*)[(\mathbf{P}^1)^{-1}] \to \mathrm{Tel}_{\mathbf{P}^1}(\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc}_*)) := \mathrm{colim}_{\mathbf{P}^1\otimes -}(\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_{\mathbf{Z}},\mathcal{S}\mathrm{pc}_*))$ forgetting the symmetric group action, which is conservative but not an equivalence.

And we put $c_i^{\mathrm{syn}}(\mathcal{E}) = 0$ for $i > d+1$. In other words, $c_i^{\mathrm{syn}}(\mathcal{E})$ can be read off from the minimal polynomial of $c_1^{\mathrm{syn}}(\mathcal{O}(1))$ in the graded algebra $H^{2\bullet}_{\mathrm{syn}}(\mathbf{P}_Z(\mathcal{E}),\bullet)$.

It is easily verified that $c_i^{\mathrm{syn}}(\mathcal{E})$ depends only on the class of $\mathcal{E}$ in the naive[23] zeroth K-group of Z

$$K_0^{\mathrm{naive}}(Z) := \frac{\{\text{Étale/Analytic vector bundles over } Z\}^{\mathrm{gp}}}{\{[\mathcal{E}_1]+[\mathcal{E}_3]-[\mathcal{E}_2] \mid \mathcal{E}_2 \text{ is an } \mathcal{O}_Z\text{-extension of } \mathcal{E}_3 \text{ by } \mathcal{E}_1\}}.$$

Hence we obtain the i-th syntomic Chern class maps

$$c_i^{\mathrm{syn}} : K_0^{\mathrm{naive}}(Z) \to H^{2i}_{\mathrm{syn}}(Z,i), \quad i \in \mathbf{N}$$

extending those in (4.3.1.1).

4.3.2. Étale Chern classes. Similarly as above, using the projective bundle formula (4.2.17) and Chern class maps

$$c_0^{\text{ét}} : \mathbf{Z}_p \xrightarrow{\mathrm{can}} R\Gamma_{\text{ét}}(Z,\mathbf{Z}_p), \quad c_1^{\text{ét}} : \mathcal{O}_Z^\times \to R\Gamma_{\mathrm{syn}}(Z,\mathbf{Z}_p(1))[1], \tag{4.3.2.1}$$

one can define the i-th integral étale Chern class maps

$$c_i^{\text{ét}} : K_0^{\mathrm{naive}}(Z) \to H^{2i}_{\text{ét}}(Z,\mathbf{Z}_p(i))(*), \quad i \in \mathbf{N}$$

extending those in (4.3.2.1).

4.3.3 - Remark. Alternatively, we might also define higher Chern classes using the following universal computation: let GL_n be the rigid-analytic general linear group of rank n over $\mathbf{Q}_p$ and $B_\bullet(-)$ be the simplicial classifying space construction for monoids; then

$$H^\bullet_{\mathrm{syn}}(B_\bullet\mathrm{GL}_{n,L},\bullet) \simeq H^\bullet_{\mathrm{syn}}(L,\bullet)[c_1,\ldots,c_n]$$

for $L = K$ or $L = C$ and $n \geq 1$, where $c_i \in H^{2i}_{\mathrm{syn}}(B_\bullet\mathrm{GL}_{n,L},i)$ is the i-th Chern class of the universal vector bundle of rank n, and similarly

$$H^\bullet_{\text{ét}}(B_\bullet\mathrm{GL}_{n,L},\mathbf{Z}_p(\bullet)) \simeq H^\bullet_{\text{ét}}(L,\mathbf{Z}_p(\bullet))[c_1,\ldots,c_n].$$

Indeed, this follows by standard computation using the respective projective bundle formula and $\mathbf{A}^1$-homotopy invariance, cf. [49, §2.A. BGL(n)].

It would require some work to pull these universal classes back along maps $X \to B_\bullet\mathrm{GL}_n$ in order to define Chern classes. By naturality, these would coincide with our original definition of Chern class maps $c_i(\mathcal{E})$ for vector bundles, cf. (4.5.9) for details of this construction.

Before proving compatibility of syntomic and étale Chern classes, we review some compatibility results.

4.3.4. Some compatibility results with comparison maps. Let us be more precise about the (iso)morphisms (3.1.3.1) used to construct $\rho_{\mathrm{syn}}^{\mathrm{arith}}$ and $\rho_{\mathrm{syn}}^{\mathrm{geom}}$.

(i) *The isomrophism* $\mathrm{Fil}^\bullet(R\Gamma_{\mathrm{dR}}(Z/K) \otimes^\blacksquare_K B_{\mathrm{dR}}) \simeq \mathrm{Fil}^\bullet_{\mathrm{Hdg}}(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^\blacksquare_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}})$ *commutes with product structures and Chern class maps* c_0 *and* c_1. Indeed, arguing éh-locally, we may assume Z smooth affinoid. In this case, the isomorphism is constructed via [9, Lemma 5.16, Lemma 5.17, especially formula (5.14)], where all maps are compatible with product structures and respective Chern class maps c_0 and c_1.

(ii) *The isomorphism* $R\Gamma_{\mathrm{cris}}(\mathfrak{X}) \otimes^\blacksquare_{A_{\mathrm{cris}}} B_I \simeq R\Gamma_{\text{proét}}(X,\mathbf{B}_I)$ *is compatible with product structures.* Indeed, the map is constructed locally for $\mathfrak{X} = \mathrm{Spf}(R)$ framed by (Σ,Λ) [9, Notation 4.12] via the Čech-Alexander

[23] We keep the naive superscript as opposed to Andreychev's analytic K-groups for adic spaces.

computation

$$
\begin{aligned}
R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A^{\times}_{\mathrm{cris}}) &\simeq (D_{\Sigma,\Lambda}(R)(0) \to D_{\Sigma,\Lambda}(R)(1) \to D_{\Sigma,\Lambda}(R)(2) \to \cdots) \\
&\to (D_{\Sigma,\Lambda}(R) \to \underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma_{\Sigma,\Lambda}], D_{\Sigma,\Lambda}(R)) \to \underline{\mathrm{Hom}}(\mathbf{Z}[\Gamma^2_{\Sigma,\Lambda}], D_{\Sigma,\Lambda}(R)) \to \cdots) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, D_{\Sigma,\Lambda}(R)) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, \mathbf{B}_I(R_{\Sigma,\Lambda,\infty})) \\
&\simeq R\Gamma((\mathfrak{X}_\eta)_{\mathrm{\acute{e}t}}, \mathbf{B}_I)
\end{aligned}
$$

where we followed the proof of [9, Corollary 4.16, Theorem 4.3] and used the map $D_{\Sigma,\Lambda}(R) \to \mathbf{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \to \mathbf{B}_I(R_{\Sigma,\Lambda,\infty})$. Or samely, we may use Koszul complex computation

$$
\begin{aligned}
R\Gamma_{\mathrm{cris}}(\mathfrak{X}/A^{\times}_{\mathrm{cris}}) &\simeq \mathrm{Kosz}_{D_{\Sigma,\Lambda}(R)}((\partial_\sigma)_{\sigma\in\Sigma}, (\partial_{\lambda,i})_{\lambda\in\Lambda, 1\le i\le d}) \\
&\to \mathrm{Kosz}_{D_{\Sigma,\Lambda}(R)}((\gamma_\sigma - 1)_{\sigma\in\Sigma}, (\gamma_{\lambda,i} - 1)_{\lambda\in\Lambda, 1\le i\le d}) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, D_{\Sigma,\Lambda}(R)) \\
&\simeq R\Gamma(\underline{\Gamma_{\Sigma,\Lambda}}, \mathbf{B}_I(R_{\Sigma,\Lambda,\infty})) \\
&\simeq R\Gamma((\mathfrak{X}_\eta)_{\mathrm{\acute{e}t}}, \mathbf{B}_I)
\end{aligned}
$$

using [9, Lemma 4.14, Lemma 4.15]. All maps here are compatible with product structures.

(iii) *The isomorphism* $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}(R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) \otimes^{\blacksquare}_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}) \simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathrm{Fil}^\bullet \mathbf{B}_{\mathrm{dR}})$ *is compatible with product structures.* Indeed, the maps is constructed locally via

$$
\begin{aligned}
R\Gamma_{\mathrm{inf}}(X/B^+_{\mathrm{dR}}) &\simeq \mathrm{Kosz}_{D_{\Psi,\Xi,m}(A)}((\partial_u)_{u\in\Psi\amalg\Xi}) \\
&\to \mathrm{Kosz}_{D_{\Psi,\Xi,m}(A)}((\gamma_u - 1)_{u\in\Psi\amalg\Xi} \\
&\to \mathrm{Kosz}_{(\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty})}((\gamma_u - 1)_{u\in\Psi\amalg\Xi}) \\
&\simeq R\Gamma(\underline{\Gamma_{\Psi,\Xi}}, (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty})) \\
&\simeq R\Gamma_{\mathrm{pro\acute{e}t}}(X, (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m))
\end{aligned}
$$

where we used the map $D_{\Psi,\Xi,m}(A) \to (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty})$ [9, Formula (5.19)].

(iv) *The isomorphisms in (ii) and (iii) are compatible.* This follows from the commutativity of [9, Proof Proposition 5.11]

$$
\begin{tikzcd}
D_{\Sigma,\Lambda}(R) \arrow[r] \arrow[d] & \mathbf{B}_I(R_{\Sigma,\Lambda,\infty}) \arrow[d] \\
D_{\Psi,\Xi,m}(A) \arrow[r] & (\mathbf{B}^+_{\mathrm{dR}}/\mathrm{Fil}^m)(A^+_{\Psi,\Xi,\infty}).
\end{tikzcd}
$$

4.3.5 - Theorem. *We have* $\rho^{\mathrm{arith}}_{\mathrm{syn}} \circ c_1^{\mathrm{syn}} = c_1^{\mathrm{\acute{e}t}}$ *as morphisms* $H^1_{\mathrm{\acute{e}t}}(Z, \mathcal{O}^\times_Z) \to H^2_{\mathrm{pro\acute{e}t}}(Z, \mathbf{Q}_p(1))$.

Proof. Let us elaborate the proof of the statement evaluated at a point, i.e. we will prove that for any line bundle $\mathcal{L}$ on $Z_{\mathrm{\acute{e}t}}$, we have the identity

$$
\rho^{\mathrm{arith}}_{\mathrm{syn}}(c_1^{\mathrm{syn}}(\mathcal{L})) = c_1^{\mathrm{\acute{e}t}}(\mathcal{L}) \in H^2_{\mathrm{pro\acute{e}t}}(Z, \mathbf{Q}_p(1))(*).
$$

The proof consists of fastidious reductions to the annulus case where an elementary computation is then done (4.3.5.17), and can be skipped on first reading. For the condensed statement, the arguments are the same.

4.3.5.1. Let us first reduced to its geometric counterpart, and meanwhile giving another characterisation of the first Chern class $c_1(\mathcal{L})$ by using the simplicial classifying stack $B_\bullet\mathbf{G}_m$. For this, let us denote by $\mathbf{G}_m = \mathbf{G}_{m,K}$ the *rigid-analytic torus* over K (i.e. analytification of the algebraic torus over K, to be distinguished from the unit circle torus $\mathbf{T}^1_K := \mathrm{Spa}(K\langle T^{\pm 1}\rangle, \mathcal{O}_K\langle T^{\pm 1}\rangle)$). The étale line bundle $\mathcal{L}$ determines a morphism $f_{\mathcal{L}} : X \to B_\bullet\mathbf{G}_m$

(up to translation by $\mathbf{G}_m$). Let $\mathcal{F}$ be a big étale sheaf over $\operatorname{Spec} K$, and denote

$$R\Gamma_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{F}) := \lim_{\Delta} R\Gamma_{\text{ét}}(\mathbf{G}_m^{\times \bullet}, \mathcal{F}).$$

By the spectral sequence $E_1^{i,j} := H^j_{\text{ét}}(\mathbf{G}_m^{\times i}, \mathcal{F}) \mapsto H^{i+j}_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{F})$, we see that

(i) We have $H^1_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{O}^\times) \simeq H^0(\mathbf{G}_m, \mathcal{O}^\times)$ with a canonical element the coordinate function $T \in H^0(\mathbf{G}_m, \mathcal{O}^\times)$; similarly in the geometric setup, we have $H^1_{\text{ét}}(B_\bullet \mathbf{G}_{m,C}, \mathcal{O}^\times) \simeq H^0(\mathbf{G}_{m,C}, \mathcal{O}^\times)$ with a canonical element the coordinate function $T \in H^0(\mathbf{G}_{m,C}, \mathcal{O}^\times)$.

(ii) We have

$$H^2_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathbf{Z}_p(1)) \simeq H^1_{\text{ét}}(\mathbf{G}_m, \mathbf{Z}_p(1)) \overset{(e^*,\nu^*)}{\xrightarrow{\simeq}} H^1_{\text{ét}}(K, \mathbf{Z}_p(1)) \oplus H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1)) \tag{4.3.5.2}$$

where two component maps are induced by base change respectively along the unit section $e : \operatorname{Spa}(K, \mathcal{O}_K) \to \mathbf{G}_m$ and the projection from the *geometric rigid-analytic torus* $\nu : \mathbf{G}_{m,C} \to \mathbf{G}_m$. Similarly, we have

$$H^2_{\text{ét}}(B_\bullet \mathbf{G}_{m,C}, \mathbf{Z}_p(1)) \simeq H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1)).$$

Let us denote by κ be the canonical pro-(finite étale) $\mathbf{Z}_p(1)$-torsor

$$\widetilde{\mathbf{G}}_{m,C} := \lim_{T \mapsto T^p} \mathbf{G}_{m,C} \to \mathbf{G}_{m,C};$$

then $[\kappa]$ is the canonical topological generator of $H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1)) \simeq \mathbf{Z}_p$. By an explicit calculation, we have

$$c_1^{\text{ét}}(T) = [\kappa] \in H^1_{\text{ét}}(\mathbf{G}_{m,C}, \mathbf{Z}_p(1))$$

for $T \in H^0(\mathbf{G}_{m,C}, \mathcal{O}^\times)$.

(iii) $H^2_{\text{syn}}(B_\bullet \mathbf{G}_m, \mathbf{Z}_p(1)) \simeq H^1_{\text{syn}}(\mathbf{G}_m, \mathbf{Z}_p(1)) \to H^1_{\text{HK}}(\mathbf{G}_m)$ with a canonical element $\operatorname{dlog} t$.

The morphism $f_{\mathcal{L}}$ is naturally chosen (up to translation by $\mathbf{G}_m$) so that $f_{\mathcal{L}}^* : H^0(\mathbf{G}_m, \mathcal{O}^\times) \simeq H^1_{\text{ét}}(B_\bullet \mathbf{G}_m, \mathcal{O}^\times) \to H^1_{\text{ét}}(X, \mathcal{O}^\times)$ is such that $[T] \mapsto [\mathcal{L}]$. By naturality of $c_1^?$, we have $c_1^?(\mathcal{L}) = c_1^?(f_{\mathcal{L}}^*(T)) = f_{\mathcal{L}}^*(c_1^?(T))$ for $? \in \{\text{syn}, \text{ét}\}$. Also, $\rho_{\text{syn}}^{\text{arith}}$ commutes with $f_{\mathcal{L}}^*$, so that $\rho_{\text{syn}}^{\text{arith}}(c_1^{\text{syn}}(\mathcal{L})) = \rho_{\text{syn}}^{\text{arith}} f_{\mathcal{L}}^*(c_1^{\text{syn}}(T)) = f_{\mathcal{L}}^* \rho_{\text{syn}}^{\text{arith}}(c_1^{\text{syn}}(T))$; therefore, we are reduced to showing that

$$\rho_{\text{syn}}^{\text{arith}}(c_1^{\text{syn}}(T)) = c_1^{\text{ét}}(T) \in H^1_{\text{ét}}(\mathbf{G}_m, \mathbf{Q}_p(1)). \tag{4.3.5.3}$$

Similarly as the direct sum decomposition (4.3.5.2), we have

$$H^2_{\text{proét}}(B_\bullet \mathbf{G}_m, \mathbf{Q}_p(1)) \simeq H^1_{\text{proét}}(\mathbf{G}_m, \mathbf{Q}_p(1)) \overset{(e^*,\nu^*)}{\xrightarrow{\simeq}} H^1_{\text{proét}}(K, \mathbf{Q}_p(1)) \oplus H^0(\underline{\mathscr{G}_K}, H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1))). \tag{4.3.5.4}$$

Therefore, (4.3.5.3) is further reduced to the identities

$$e^* \rho_{\text{syn}}^{\text{arith}}(c_1^{\text{syn}}(T)) = e^* c_1^{\text{ét}}(T) \in H^1_{\text{proét}}(K, \mathbf{Q}_p(1)) \tag{4.3.5.5}$$

$$\nu^* \rho_{\text{syn}}^{\text{arith}}(c_1^{\text{syn}}(T)) = \nu^* c_1^{\text{ét}}(T) \in H^0(\underline{\mathscr{G}_K}, H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1))). \tag{4.3.5.6}$$

4.3.5.7. For (4.3.5.5), we notice that both sides are zero: on the one hand, on the other hand, by naturality of $\rho_{\text{syn}}^{\text{arith}}$ and c_1^{syn}, they commutes with e^*, so we have $e^* \rho_{\text{syn}}^{\text{arith}}(c_1^{\text{syn}}(T)) = \rho_{\text{syn}}^{\text{arith}}(c_1^{\text{ét}}(e^* T))$ and $e^* T = 1 \in H^0(K, \mathcal{O}^\times)$; similarly, by naturality of $c_1^{\text{ét}}$, we obtain $e^* c_1^{\text{ét}}(T) = c_1^{\text{ét}}(e^* T) = 0$.

4.3.5.8. Now, we are left with (4.3.5.6), which is the geometric counterpart of our theorem. Again by naturality, since $\nu^* T = T$, (4.3.5.6) amounts to the equality

$$\rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T)) = c_1^{\mathrm{\acute{e}t}}(T) \in H^0(\underline{\mathscr{G}_K}, H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1))). \tag{4.3.5.9}$$

We may discard taking Galois invariants here as it will not help prove the identity.

The rigid-analytic torus $\mathbf{G}_{m,C}$ has a non quasi-compact semistable formal model over $\mathcal{O}_K$, and it also has a projective semistable model by gluing two copies of $\operatorname{Spf}\mathcal{O}_K\langle T\rangle$ (allowing this time horizontal divisors $\{T=0\}$) along $\operatorname{Spf}\mathcal{O}_K\langle T^{\pm 1}\rangle$. According to [16, Corollary 1.10] and [13, 4.3.2], we have computations

(i) $H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) = \breve{F} \cdot c_1^{\mathrm{HK}}(T)$,
(ii) $H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}_{\log})^{N=0} \leftarrow (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{N=0} = B \cdot c_1^{\mathrm{HK}}(T)$,
(iii) $H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}_{\log})^{\varphi=p,N=0} \leftarrow (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=0,N=0} = \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T)$,
(iv) $H^1_{\mathrm{inf}}(\mathbf{G}_{m,C}/B^+_{\mathrm{dR}}) \simeq H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}} \cdot c_1^{\mathrm{dR}}(T)$, where

$$c_1^{\mathrm{dR}}(T) = [\operatorname{dlog} T]$$

by construction of (compatible) de Rham Chern class maps c_1^{inf} and c_1^{dR} (4.1.7).

(v) There is a short exact sequence [8, Formula (6.7)]

$$0 \to H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \to H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \to \Omega^1_{\mathbf{G}_m/K}(\mathbf{G}_m)^{d=0} \otimes^{\blacksquare}_K C(-1) \to 0$$

by taking the first cohomology group of $R\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) = \mathrm{Fil}^0(R\Gamma_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}})$ [8, Theorem 6.5].

(vi) There is a natural map of short exact sequences

$$\begin{tikzcd}
0 \arrow[rr] \arrow[d] & & H^1_{\mathrm{\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \arrow[r, equal] \arrow[d, "\mathrm{can}"] & \mathbf{Q}_p \cdot [\kappa] \arrow[r] \arrow[d, dashed, "\rho'"] & 0 \\
0 \arrow[r] & \mathcal{O}(\mathbf{G}_{m,C})/\mathbf{Q}_p \arrow[r] & H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \arrow[r, "\beta"] & \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T) \arrow[r] & 0.
\end{tikzcd}$$

Here ρ' is supposed to behave like $\rho_{\mathrm{syn}}^{\mathrm{geom},-1}$.

(vii) Consider the composition of natural maps

$$\begin{aligned}
\iota'_{\mathrm{HK}} : \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T) &= (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=p,N=0} \\
&\to H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}_{\log}) \\
&\to H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \simeq H^1\mathrm{Fil}^1(R\Gamma_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B_{\mathrm{dR}}) \\
&\supset H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \\
&= B^+_{\mathrm{dR}} \cdot c_1^{\mathrm{dR}}(T).
\end{aligned} \tag{4.3.5.10}$$

It matches $c_1^{\mathrm{HK}}(T)$ with $c_1^{\mathrm{dR}}(T) = [\operatorname{dlog} T]$ by its compatibility with the geometric Hyodo-Kato morphism [9, Theorem 5.3 (ii)], hence it is injective. Moreover, the composition $\iota'_{\mathrm{HK}}\rho'$ coincides with the composition map

$$\rho'' : H^1_{\mathrm{\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \to H^1_{\mathrm{pro\acute{e}t}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \tag{4.3.5.11}$$

induced by $\mathbf{Q}_p(1) \to B^+_{\mathrm{dR}}$ sending $\varepsilon = (\zeta_{p^n})_n \mapsto \log[\varepsilon]$.

4.3.5.12. We claim that

$$\rho'(\kappa) = c_1^{\mathrm{HK}}(T) \in (H^1_{\mathrm{HK}}(\mathbf{G}_{m,C}) \otimes^{\blacksquare}_{\breve{F}} B_{\log})^{\varphi=0,N=0} = \mathbf{Q}_p \cdot c_1^{\mathrm{HK}}(T). \tag{4.3.5.13}$$

Admitting this, by construction of $\rho_{\mathrm{syn}}^{\mathrm{geom}}$ (3.1.3) and compatibility between differntial symbols, we have $\beta\rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}T) = c_1^{\mathrm{HK}}(T)$, so $\beta\left(c_1^{\text{ét}}(T) - \rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T))\right) = 0$, or equivalently

$$\delta(T) := c_1^{\text{ét}}(T) - \rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T)) \in \ker(H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{Q}_p(1)) \to \mathbf{Q}_p) = \mathcal{O}(\mathbf{G}_{m,C})/\mathbf{Q}_p.$$

we need to show that the difference function $\delta(T) = 0$. For this, consider the squaring morphism $\sigma := (-)^2 : \mathbf{G}_{m,C} \to \mathbf{G}_{m,C}$ over C given by $T^2 \mapsfrom T$. On the one hand, we have

$$\sigma^*(c_1^{\text{ét}}(T) - \rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T))) = c_1^{\text{ét}}(T^2) - \rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T^2)) = 2(c_1^{\text{ét}}(T) - \rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T))) = 2\delta(T)$$

by naturality of $\rho_{\mathrm{syn}}^{\mathrm{geom}}$ and $c_1^?$; on the other hand, by naturality of $\mathcal{O}(\mathbf{G}_{m,C})/\mathbf{Q}_p$,

$$\sigma^*(c_1^{\text{ét}}(T) - \rho_{\mathrm{syn}}^{\mathrm{geom}}(c_1^{\mathrm{syn}}(T))) = \sigma^*\delta(T) = \delta(T^2).$$

Therefore, $2\delta(T) = \delta(T^2)$ whence $\delta(T) = 0$.

4.3.5.14. It now remains to show (4.3.5.13). The injectivity of ι'_{HK} (4.3.5.10) reduces it to

$$\rho''(\kappa) = [\mathrm{dlog}\, T] \in H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) \hookleftarrow H^1_{\mathrm{dR}}(\mathbf{G}_m/K) \otimes^{\blacksquare}_K B^+_{\mathrm{dR}}, \tag{4.3.5.15}$$

where we used $\iota'_{\mathrm{HK}}\rho' = \rho''$ (4.3.5.11) and $\iota'_{\mathrm{HK}}(c_1^{\mathrm{HK}}(T)) = c_1^{\mathrm{dR}}(T) = [\mathrm{dlog}\, T]$ (see the point (x) above).

Before the proof, we introduce certain perfectoid covering of $\mathbf{G}_m$ for our computation. We denote by $X_n := \mathrm{Spa}(K\langle p^n T, p^n T^{-1}\rangle, \mathcal{O}_K\langle p^n T, p^n T^{-1}\rangle) = \{|p^n| \le |T| \le |p^{-n}|\} \subset \mathbf{G}_m$ the arithmetic annuli over K and by $X_{n,C}$ their base change to C. We do not use the pro-(finite étale) $\mathbf{Z}_p(1)$-torsor $\kappa : \widetilde{\mathbf{G}}_{m,C} \to \mathbf{G}_{m,C}$. Instead, consider the closed embeddings

$$X_n \hookrightarrow Y_n := \mathrm{Spa}(K\langle U_n, V_n\rangle, \mathcal{O}_K\langle U_n, V_n\rangle)$$

given by $U_n \mapsto p^n T, V_n \mapsto p^n T^{-1}$. Consider $\widetilde{X}_{n,C}$ the pullback along this closed embedding of the canonical $\mathbf{Z}_p(1)^2$-torsor affinoid perfectoid cover

$$\widetilde{Y}_{n,C} := \mathrm{Spa}(C\left\langle U_n^{1/p^\infty}, V_n^{1/p^\infty}\right\rangle, \mathcal{O}_C\left\langle U_n^{1/p^\infty}, V_n^{1/p^\infty}\right\rangle) \to Y_n.$$

Then we have compatible strict inclusions of perfectoid space $\widetilde{X}_{n,C} \subset^\dagger \widetilde{X}_{n+1,C}$ induced by $pU_n \mapsfrom U_{n+1}, pV_n \mapsfrom V_{n+1}$. Their union $\widetilde{X}_C$ is the canonical $\mathbf{Z}_p(1)^2$-torsor perfectoid cover of $\mathbf{G}_{m,C}$, with a *Stein affinoid perfectoid covering* by $\widetilde{X}_{n,C}$. The system $\{H^i_{\text{proét}}(\widetilde{X}_{n,C}, \mathbf{B}^+_{\mathrm{dR}})\}_{n\in\mathbf{N}}$ is Mittag-Leffer [14, Lemma 3.10], and vanishes if $i > 0$ [50, Theorem 6.5], thus $H^i_{\text{proét}}(\widetilde{\mathbf{G}}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) = B^+_{\mathrm{dR}}(\widetilde{\mathbf{G}}_{m,C})$ if $i = 0$ and vanishes if $i > 0$. Then the Hochschild-Serre spectral sequence and the $\lim^1$-sequence give

$$H^1_{\text{proét}}(\mathbf{G}_{m,C}, \mathbf{B}^+_{\mathrm{dR}}) = H^1(\underline{\mathbf{Z}_p(1)}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_C)) \xrightarrow{\simeq} \lim_n H^1(\underline{\mathbf{Z}_p(1)}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{m,C})).$$

Now the proof reduces to identifying

$$\rho''(\kappa)|_{X_{n,C}} = [\mathrm{dlog}\, T] \in H^1(\underline{\mathbf{Z}_p(1)}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{m,C})). \tag{4.3.5.16}$$

4.3.5.17. Let us prove (4.3.5.16). It is direct to check that $\rho''(\kappa)|_{X_{n,C}} \in H^1_{\text{proét}}(X_{n,C}, \mathbf{B}^+_{\mathrm{dR}}) \simeq H^1(\underline{\mathbf{Z}_p}(1)^2, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{n,C})) \simeq H^1\,\underline{\mathrm{Hom}}((\underline{\mathbf{Z}_p}(1)^2)^{\times\bullet}, \mathbf{B}^+_{\mathrm{dR}}(\widetilde{\mathbf{G}}_{m,C}))$ (by Hochschild-Serre spectral sequence for the first isomorphism and (1.3.4) for the second) is represented by the continuous cocycle

$$(\gamma_u, \gamma_v) \mapsto \log[\gamma_u] - \log[\gamma_v]. \tag{4.3.5.18}$$

Now we will go through the construction of the comparison map $R\Gamma_{\mathrm{dR}}(X/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \to R\Gamma_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{B}^+_{\mathrm{dR}})$ [9, Proof of Theorem 5.9] (see the point (iii) above) in order to identify the image of $[\mathrm{dlog}\, T]$ in $H^1_{\mathrm{pro\acute{e}t}}(X_{n,C},\mathbf{B}^+_{\mathrm{dR}}) \simeq H^1(\underline{\mathbf{Z}_p(1)^2},\mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{n,C}))$. Let $D_n := D_{X_{n,C}}(Y_{n,C})$ be the ring of functions of the B^+_{dR}-envelope of $X_{n,C}$ in $Y_{n,B^+_{\mathrm{dR}}}$ as in [9, Lemma 5.16] (cf. [31, §2.2]). Then $R\Gamma_{\mathrm{dR}}(X_n/K)\otimes^{\blacksquare}_K B^+_{\mathrm{dR}} \simeq \mathrm{Kosz}_{D_n}(\partial_u,\partial_v)$ where $\partial_u = u\frac{\partial}{\partial u}, \partial_v = v\frac{\partial}{\partial v}$ are log derivations, and the class $[\mathrm{dlog}\, T] = [\mathrm{dlog}(p^nT)] = -[\mathrm{dlog}(p^nT^{-1})]$ corresponds to

$$[(1,-1)] \in H^1\mathrm{Kosz}_{D_n}(\partial_u,\partial_v).$$

The map $\mathrm{Kosz}_{D_n}(\partial_u,\partial_v) \to \mathrm{Kosz}_{D_n}(\gamma_u-1,\gamma_v-1)$ of the point (iii) is induced by ([10, Proposition 5.34])

$$\begin{tikzcd}
(D_n \arrow[r, "\partial_u"] \arrow[d, "\mathrm{id}"] & D_n) \arrow[d, "h_{u,\varepsilon}"] \\
(D_n \arrow[r, "\gamma_{u,\varepsilon}"] & D_n)
\end{tikzcd}$$

where $\varepsilon \in \mathbf{Z}_p(1)$ is a chosen generator, $h_\varepsilon := \sum_{i\geq 1}\frac{(\log[\varepsilon])^i}{i!}\partial_u^{i-1}$, and the u-component element $\gamma_{u,\varepsilon} \in \mathbf{Z}_p(1)$, alias of ε, acts on the B^+_{dR}-algebra D_n by $\gamma(U_n) = [\varepsilon]U_n$; similarly for the variable v part. Under this map, the class $[\mathrm{dlog}\, T]$ is mapped to

$$[(\log[\varepsilon],-\log[\varepsilon])] \in H^1\mathrm{Kosz}_{D_n}(\gamma_{u,\varepsilon}-1,\gamma_{v,\varepsilon}-1).$$

Finally, using the map $D_n \to \mathbf{B}_{\mathrm{dR}}(\widetilde{X}_{n,C}), U_n \mapsto [(p^nT)^\flat], V_n \mapsto [(p^nT^{-1})^\flat]$, we obtain the class

$$[(\log[\varepsilon],-\log[\varepsilon])] \in H^1\mathrm{Kosz}_{\mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{n,C})}(\gamma_{u,\varepsilon}-1,\gamma_{v,\varepsilon}-1). \tag{4.3.5.19}$$

To conclude, we claim that (4.3.5.19) recovers the formula (4.3.5.18) for $\rho''(\kappa)|_{X_{n,C}}$ under the identification $H^1\mathrm{Kosz}_{\mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{n,C})}(\gamma_u-1,\gamma_v-1) \simeq H^1(\underline{\mathbf{Z}_p(1)^2},\mathbf{B}^+_{\mathrm{dR}}(\widetilde{X}_{n,C})) \simeq H^1\,\underline{\mathrm{Hom}}((\underline{\mathbf{Z}_p(1)^2})^{\times\bullet},\mathbf{B}^+_{\mathrm{dR}}(\widetilde{\mathbf{G}}_{m,C}))$, which we summarised in the lemma 1.3.8.

□

4.3.6 - Theorem. *The syntomic and étale Chern classes are compatible, i.e. for $Z \in \mathcal{R}\mathrm{ig}_K$ and $i \in \mathbf{N}$, the following diagram commutes*

$$\begin{tikzcd}
 & H^{2i}_{\mathrm{syn}}(Z,i) \arrow[dd, "\rho^{\mathrm{arith}}_{\mathrm{syn}}"] \\
K_0^{\mathrm{naive}}(X) \arrow[ur, "c_i^{\mathrm{syn}}"] \arrow[dr, "c_i^{\mathrm{\acute{e}t}}"'] & \\
 & H^{2i}_{\mathrm{\acute{e}t}}(Z,\mathbf{Q}_p(i))
\end{tikzcd}$$

Proof. In view of projective bundle formulae defining general Chern classes, it suffices to show that the comparison map $\rho^{\mathrm{arith}}_{\mathrm{syn}} : R\Gamma_{\mathrm{syn}}(Z,i) \to R\Gamma_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(i))$ commutes with product structures, c_0 on the zeroth cohomology and c_1 on the first cohomology (i.e. $c_1(\mathcal{L})$ for line bundles $\mathcal{L}$). By construction of $\rho^{\mathrm{arith}}_{\mathrm{syn}}$ (3.1.3), it suffices to prove the corresponding statement for the composition

$$\begin{aligned}
R\Gamma_{\mathrm{syn}}(Z,i) \overset{\rho^{\mathrm{arith}}_{\mathrm{syn}}}{\to} R\Gamma_{\mathrm{pro\acute{e}t}}(Z,\mathbf{Q}_p(r)) &\to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \\
&\overset{\simeq}{\to} \left[R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\log}[\tfrac{1}{t}])^{\varphi=p^r,N=0} \to R\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{B}_{\mathrm{dR}}/t^r\mathbf{B}^+_{\mathrm{dR}})\right].
\end{aligned}$$

But it is clear that the latter preserves product structures by (4.2.1), commutes with c_0 by construction, and commutes with c_1 for line bundles by (4.3.5). □

4.4 Image of étale regulators

4.4.1. Étale regulators. For $Z \in \mathcal{R}\mathrm{ig}_K$ and $i \in \mathbf{N}$, let $\tilde{c}_i^{\text{ét}} : K_0^{\text{naive}}(Z) \to H^0(\underline{\mathscr{G}_K}, H^{2i}_{\text{ét}}(Z_C, \mathbf{Z}_p(i)))$ be the map induced by the i-th étale Chern class map and the Hochschild-Serre spectral sequence boundary map $\delta_0 : H^{2i}_{\text{ét}}(Z_C, \mathbf{Z}_p(i)) \to H^0(\underline{\mathscr{G}_K}, H^{2i}_{\text{ét}}(Z_C, \mathbf{Z}_p(i)))$. We define

$$K_0^{\text{naive}}(Z)_0 := \ker(K_0^{\text{naive}}(Z) \xrightarrow{\tilde{c}_i^{\text{ét}}} H^0(\underline{\mathscr{G}_K}, H^{2i}_{\text{ét}}(Z_C, \mathbf{Z}_p(i)))) \tag{4.4.1.1}$$

and the *i-th étale regulator map* as the map

$$r_i^{\text{ét}} : K_0^{\text{naive}}(Z) \to H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))) \tag{4.4.1.2}$$

induced from $c_i^{\text{ét}}|_{K_0^{\text{naive}}(Z)_0}$ and the Hochschild-Serre spectral sequence map

$$\delta_1 : H^{2i}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))_0 := \ker \delta_0 \to H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))).$$

4.4.2 - Theorem. *Let $Z \in \mathcal{R}\mathrm{ig}_K$ be proper. The étale regulator map $r_i^{\text{ét}}$ factors through the subgroup*

$$H^1_{\text{st}}(\underline{\mathscr{G}_K}, H^{2i-1}_{\text{proét}}(Z_C, \mathbf{Q}_p(i))) \subset H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\text{proét}}(Z_C, \mathbf{Q}_p(i))).$$

Proof. By (4.3.6) and (3.2.17), we have the following commutative diagram

$$\begin{tikzcd}
 & H^{2i}_{\text{syn}}(Z,i) \arrow[r, "\delta_0^{\text{syn}}"] \arrow[d, "\rho_{\text{syn}}^{\text{arith}}"] & H^0_{\text{st}}(\underline{\mathscr{G}_K}, H^{2i}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))) \arrow[d, "\text{can}", "\simeq"'] \\
K_0^{\text{naive}}(Z) \arrow[ur, "c_i^{\text{syn}}"] \arrow[dr, "c_i^{\text{ét}}"'] & & \\
 & H^{2i}_{\text{ét}}(Z, \mathbf{Q}_p(i)) \arrow[r, "\delta_0"] & H^0(\underline{\mathscr{G}_K}, H^{2i}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))).
\end{tikzcd}$$

Hence the image of $K_0^{\text{naive}}(Z)_0$ under c_i^{syn} is contained in $H^{2i}_{\text{syn}}(Z,i)_0 := \ker(\delta_0 \circ \rho_{\text{syn}}^{\text{arith}}) = \ker \delta_0^{\text{syn}}$. Consequently, again by (3.2.17), we obtain the following commutative diagram

$$\begin{tikzcd}
 & H^{2i}_{\text{syn}}(Z,i)_0 \arrow[r, "\delta_0^{\text{syn}}"] \arrow[d, "\rho_{\text{syn}}^{\text{arith}}"] & H^1_{\text{st}}(\underline{\mathscr{G}_K}, H^{2i-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))) \arrow[d, hook, "\text{can}"] \\
K_0^{\text{naive}}(Z)_0 \arrow[ur, "c_i^{\text{syn}}"] \arrow[dr, "c_i^{\text{ét}}"'] & & \\
 & H^{2i}_{\text{ét}}(Z, \mathbf{Q}_p(i))_0 \arrow[r, "\delta_0"] & H^1(\underline{\mathscr{G}_K}, H^{2i-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i)))
\end{tikzcd}$$

which shows the desired factorisation. □

4.5 Higher Chern class maps and étale regulators

4.5.1. Unstable $\mathbf{A}^1$-homotopy theory of rigid spaces. Let us first recall the related materials from [22, Section 3]. Let (R, R^+) be a uniform Huber pair with associated rigid space $S = \mathrm{Spa}(R, R^+)$. Let $\mathcal{C}$ be a presentable category of coefficients, e.g. $\mathcal{C} = \mathrm{pro}(\mathcal{S}\mathrm{pc})$, $\mathcal{C} = \mathrm{pro}^{\text{light}}(\mathcal{S}\mathrm{pc})$, $\mathcal{C} = \mathcal{C}\mathrm{ond}^{\text{light}}(\mathcal{S}\mathrm{pc})$ be the category of light condensed anima, or $\mathcal{C} = \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{pc})$ be the category of κ-condensed anima, where κ is some uncountable strong limit cardinal. Here, $\mathrm{pro}^{\text{light}}(-)$ denote the full subcategory of $\mathrm{pro}(-)$ consisting of those pro-objects that can be indexed by $\mathbf{N}$.

We define the *unstable rigid motivic homotopy category with coefficeints in* $\mathcal{C}$ as the reflective subcategory

$$\mathcal{R}\mathrm{igH}(S,\mathcal{C}) := \mathcal{S}\mathrm{hv}^{\mathbf{A}^1}_{\mathrm{Nis}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S,\mathcal{C}) \subset \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S,\mathcal{C}) \subset \mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S,\mathcal{C})$$

spanned by $\mathbf{A}^1$-invariant Nisnevich sheaves, with left adjoint the *motivic localisation* functor $L_{\mathrm{mot}} := L_{\mathrm{mot},\mathcal{C}}$. If $\mathcal{C}$ is also cocomplete, then the motivic localisation can be described by the colimit of functors $L_{\mathrm{mot}} \simeq \mathrm{colim}_n(L_{\mathrm{Nis}}L_{\mathbf{A}^1})^{\circ n}$[24], and the $\mathbf{A}^1$-localisation functor $L_{\mathbf{A}^1} : \mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S,\mathcal{C}) \to \mathcal{PS}\mathrm{hv}^{\mathbf{A}^1}(S,\mathcal{C}) \subset \mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S,\mathcal{C})$ can be described by the formula [22, Lemma 3.6]

$$(L_{\mathbf{A}^1}F)(X) \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} F(X \times \Delta^\bullet)$$

where the analytic n-simplex $\Delta^n := \{\sum_{i=0}^n X_i = 1\} \subset \mathbf{A}_K^{n+1}$ is the analytification of its algebraic analogue.

If $\mathcal{C}$ is moreover stable (for example the ∞-category of spectra Sp), then we define the *effective stable homotopy category with coefficients in* $\mathcal{C}$ as

$$\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{H}^{\mathrm{eff}}(S,\mathcal{C}) := \mathcal{R}\mathrm{igH}(S,\mathcal{C}).$$

4.5.2 - Example. According to previous results, for $r \in \mathbf{N}$, the syntomic cohomology $R\Gamma_{\mathrm{syn}}(-,r)$ and the integral étale cohomology $R\Gamma_{\text{ét}}(-,\mathbf{Z}_p(r))$ define objects of $\mathcal{R}\mathrm{igH}(S,\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p}))$ for $S \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$. Here, we have used canonical localisation embeddings

$$\mathscr{D}(\mathrm{Mod}^{\blacksquare}_{\mathbf{Z}_p}) \hookrightarrow \mathscr{D}(\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}) \simeq \mathcal{S}\mathrm{hv}^{\mathrm{hyp}}(*_{\text{proét}},\mathscr{D}(\mathcal{A}\mathrm{b})) \hookrightarrow \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})$$

which preserves limits to view solid cohomology theories as valued in $\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})$.

Another source of examples of objects in $\mathcal{R}\mathrm{igH}(S,\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc}))$ come from the analytic K-theory.

4.5.3. Analytic K-theory. Recall that the *connective analytic K-theory* of a rigid space $Z \in \mathcal{R}\mathrm{ig}_K$ is defined as

$$k^{\mathrm{an}}(Z) := \text{"}\lim\text{"}_j K_{\geq 0}(Z\langle\Delta_{\pi^j}\rangle) \in \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}); \tag{4.5.3.1}$$

here, we define

$$K_{\geq 0}(Z\langle\Delta_{\pi^j}\rangle) := \mathrm{colim}_{\Delta^{\mathrm{op}}} K_{\geq 0}(Z\langle\Delta^\bullet_{\pi^j}\rangle),$$

where $K_{\geq 0}(-)$ on the right is the usual connective algebraic K-theory à la Thomason-Trobaugh [56, Chapter IV, Remark 8.5.5][25], which is equivalent to the connective cover of the non-connective algebraic K-theory $K(-)$. Doing the same thing for its connected cover $K_{\geq 1}$[26], we define

$$k^{\mathrm{an}}_{\geq 1}(Z) := \text{"}\lim\text{"}_j K_{\geq 1}(Z\langle\Delta_{\pi^j}\rangle) \in \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}). \tag{4.5.3.2}$$

Notice that $k^{\mathrm{an}}_{\geq 1}(Z)$ is different in general from the connected cover $\tau_{\geq 1}k^{\mathrm{an}}(Z)$, but we still have a weak fibre sequence[27] in $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$ [37, Lemma 6.4] [22, Formula (5.3.1)]

$$k^{\mathrm{an}}_{\geq 1}(Z) \to k^{\mathrm{an}}(Z) \to \text{"}\lim\text{"}_j K_0(Z\langle\Delta_{\pi^j}\rangle). \tag{4.5.3.3}$$

[24] we need to take this colimit since neither Nisnevich sheafification nor $\mathbf{A}^1$-localisation of presheaves preserve the property of the other one.

[25] The original reference for Thomason-Trobaugh K-theory spectrum is [54, Definition 3.1]; it is a connective spectrum. One can also find a review and further discussion in an ∞-categorical account in [7, §7]. Furthermore, one can also refer to these references for the non-connective algebraic K-spectrum à la Bass-Thomason-Trobaugh via Bass construction [54, Definition 6.4] [56, Chapter IV, Definition 10.4] [7, Definition 9.6]; its connective cover recovers the above mentioned connective K-theory spectrum.

[26] The notation for the connected cover $\tau_{\geq 1}K$ in [37] is K^{GL}, while it is denoted by $K_{\geq 1}$ in [22]. We adopted the latter.

[27] Recall that [37, 22] there is a functor $\iota^* : \mathrm{pro}(\mathcal{S}\mathrm{p}) \to \mathrm{pro}(\mathcal{S}\mathrm{p}^+), \text{"}\lim\text{"}_I X_i \mapsto \text{"}\lim\text{"}_{\mathbf{N}\times I}\tau_{\leq n}X_i$. A morphism in $\mathrm{pro}(\mathcal{S}\mathrm{p})$ is said to be a *weak equivalence* if it is an equivalence after applying ι^*, a *weak fibre sequence* is a sequence which becomes a fibre sequence after applying ι^*.

The last term is isomorphic to $K_0(\mathcal{O}(Z))$ for $Z \in \mathcal{R}ig\mathcal{S}m_K$ under the condition ($\dagger_K$), or more specifically for $Z = \mathrm{Spa}(A, A^\bullet) \in \mathcal{R}ig\mathcal{S}m_K$ with A (regular) satisfying ($\dagger_A$) [37, Corollary 5.15, Lemma 6.4]. It gives rise to a fibre sequence in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$ after applying the limit preserving functor γ^{light} (1.2.1, vi, vii), and we see then that $\gamma^{\mathrm{light}} k^{\mathrm{an}}(Z) \in \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq -1})$[28]. Finally, the *non-connective analytic K-theory* on $\mathcal{R}\mathrm{ig}_K$ is defined objectwisely using the analytic Bass construction[29] as

$$K^{\mathrm{an}} := (k^{\mathrm{an}}_{\geq 1})^B; \tag{4.5.3.4}$$

it is $\mathbf{A}^1$-homotopy invariant by construction and satisfies Nisnevich descent [38, Theorem 2.10] [22, §5.1]; and it is homotopically bounded below, or more precisely $K^{\mathrm{an}}(Z) \in \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq -\dim Z - 1})$ for $Z \in \mathcal{R}\mathrm{ig}_K$, by [38, Theorem 2.4], since any affinoid Tate algebra A over K has a Noetherian ring of definition of dimension $\dim A + 1$.

We also have the KaroubiVillamayor analogue of k^{an}

$$KV^{\mathrm{an}} := \text{"}\lim\text{"}_\rho \mathrm{BGL}(\mathcal{O}(- \times \Delta_\rho)) \in \mathrm{pro}(\mathcal{S}\mathrm{pc}_*).$$

It actually belongs to $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{pc}^0_*)$ where $\mathcal{S}\mathrm{pc}^0_*$ denotes the ∞-category of pointed connected spaces.

4.5.4. Let A be a *Tate ring*, i.e. a Huber ring with a topologically nilpotent unit. We consider the following condition on A [37, §3.2]

($\dagger_A$) There exists a Noetherian ring of definition $A_0 \subset A$ and a *desingularisation*, i.e. a proper morphism of schemes $p : X \to \mathrm{Spec}\, A_0$ with X regular and such that p is an isomorphism over $\mathrm{Spec}\, A$.

This condition makes

$$K^{\mathrm{an}}(A) = (k^{\mathrm{an}}_{\geq 1})^B(A) \to (k^{\mathrm{an}})^B(A) \tag{4.5.4.1}$$

an weak equivalence in $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$ and also induces weak equivalences in $\mathrm{pro}(\mathcal{S}\mathrm{p})$

$$\pi_0(K^{\mathrm{an}}(A)) \simeq K_0(A), \quad \tau_{\geq 0}(K^{\mathrm{an}}(A)) \simeq k^{\mathrm{an}}(A), \quad KV^{\mathrm{an}}(A) \simeq \Omega^\infty \tau_{\geq 1}(k^{\mathrm{an}}(A)) \simeq \Omega^\infty \tau_{\geq 1}(K^{\mathrm{an}}(A)) \tag{4.5.4.2}$$

by main results of [37, Corollary 6.20 and Lemma 7.5].

Consider also the following condition on some base Tate ring R:

($\dagger_K$) Every regular and topologically finite type K-algebra A satisfies ($\dagger_A$).

4.5.5 - Remark. Although γ^{light} is only *left t-exact* [22, Lemma A.19], we claim that the fundamental groups and truncations in (4.5.4.2) can either be taken in $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$ or in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$, using the following lemma:

4.5.5.1 - Lemma. *Let* $X = \text{"}\lim\text{"}_{i \in \mathbf{N}} X_i \in \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$.

(i) Let $n \in \mathbf{Z}$ *such that* $\pi_n X = \text{"}\lim\text{"}_{i \in \mathbf{N}} \pi_n X_i \in \mathrm{pro}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ *is a Mittag-Leffler system. Then we have canonical equivalences*

$$\gamma^{\mathrm{light}}(\tau_{\geq n} X) \xrightarrow{\simeq} \tau_{\geq n}(\gamma^{\mathrm{light}} X), \quad \gamma^{\mathrm{light}}(\tau_{\leq n-1} X) \xleftarrow{\simeq} \tau_{\leq n-1}(\gamma^{\mathrm{light}} X).$$

(ii) If moreover $\pi_{n+1} X = \text{"}\lim\text{"}_{i \in \mathbf{N}} \pi_{n+1} X_i \in \mathrm{pro}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ *is a Mittag-Leffler system. Then we have canonical equivalences*

$$\gamma^{\mathrm{light}}(\tau_{\geq n+1} X) \xrightarrow{\simeq} \tau_{\geq n+1}(\gamma^{\mathrm{light}} X), \quad \gamma^{\mathrm{light}}(\tau_{\leq n} X) \xleftarrow{\simeq} \tau_{\leq n}(\gamma^{\mathrm{light}} X), \quad \gamma^{\mathrm{light}}(\pi_n X) \simeq \pi_n(\gamma^{\mathrm{light}} X).$$

[28] Although $k^{\mathrm{an}} \in \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq 0})$, it is *a priori* not clear whether $\gamma^{\mathrm{light}}(k^{\mathrm{an}}(Z)) \in \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq 0})$ or not, because of the fact that γ^{light} is in general *only left t-exact*, but not right t-exact. Nevertheless, as the projective system is $\mathbf{N}$-indexed, we have $\gamma^{\mathrm{light}} k^{\mathrm{an}}(Z) \in \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq -1})$ and, by the Milnor sequence, that $\pi_{-1}(\gamma^{\mathrm{light}} k^{\mathrm{an}}(Z)) \simeq \lim^1_j \underline{K_0(Z\langle \Delta_{\pi^j} \rangle)}$.

[29] See [37, §6.1] or [22, §5.3] for details of the analytic Bass construction.

Proof. (i) Consider the (weak) fibre sequence $\tau_{\geq n}X \to X \to \tau_{\leq n-1}X$ in $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$. Applying γ^{light}, we obtain a fibre sequence
$$\gamma^{\mathrm{light}}(\tau_{\geq n}X) \to \gamma^{\mathrm{light}}(X) \to \gamma^{\mathrm{light}}(\tau_{\leq n-1}X)$$
in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$. As γ^{light} is left t-exact, the third term $\gamma^{\mathrm{light}}(\tau_{\leq n-1}X)$ lies in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})_{\leq n-1}$. As for the first term $\gamma^{\mathrm{light}}(\tau_{\geq n}X)$, *a priori* it lies in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq n-1})$ as the projective system is $\mathbf{N}$-indexed, and by Milnor sequence, we have $\pi_{n-1}(\gamma^{\mathrm{light}}(\tau_{\geq n}X)) = \lim^1_n \underline{\pi_0 X_n}$. Now the Mittag-Leffler condition implies that this last $\lim^1$ vanishes, whence $\gamma^{\mathrm{light}}(\tau_{\geq n}X) \in \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})_{\geq n} \simeq \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}_{\geq n})$; in particular the above fibre sequence gives the truncations $\tau_{\geq n}$ and $\tau_{\leq n-1}$ of $\gamma^{\mathrm{light}}X$.

(ii) Consider this time the fibre sequence
$$\gamma^{\mathrm{light}}(\tau_{\geq n+1}X) \to \gamma^{\mathrm{light}}(X) \to \gamma^{\mathrm{light}}(\tau_{\leq n}X)$$
in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$. By (i) applied to $n+1$, we obtain equivalences
$$\gamma^{\mathrm{light}}(\tau_{\geq n+1}X) \xrightarrow{\simeq} \tau_{\geq n+1}(\gamma^{\mathrm{light}}X), \quad \gamma^{\mathrm{light}}(\tau_{\leq n}X) \xleftarrow{\simeq} \tau_{\leq n}(\gamma^{\mathrm{light}}X).$$
Finally, consider the fibre sequence
$$\gamma^{\mathrm{light}}(\pi_n X) \to \gamma^{\mathrm{light}}(\tau_{\leq n}X) \to \gamma^{\mathrm{light}}(\tau_{\leq n-1}X)$$
in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$. The Mittag-Leffler property of $\pi_n X$ again implies that $\gamma^{\mathrm{light}}(\pi_n X) \in \mathcal{C}\mathrm{ond}(\mathrm{Sp})^{\heartsuit}[n]$, so we get
$$\gamma^{\mathrm{light}}(\pi_n X) \simeq \tau_{\geq n}\gamma^{\mathrm{light}}(\tau_{\leq n}X) \xleftarrow{\simeq} \tau_{\geq n}\tau_{\leq n}(\gamma^{\mathrm{light}}X) \simeq \pi_n(\gamma^{\mathrm{light}}X).$$
□

Let us deduce the above claim from the lemma. Let us check the conditions:

- Under the regularity and ($\dagger_A$) condition, we have $\pi_0(K^{\mathrm{an}}(A)) = \text{"}\lim\text{"}_j K_0(A\langle\Delta_{\pi^j}\rangle) \simeq K_0(A)$; in particular, this tower is Mittag-Leffler. So the condition (i) of the above lemma is verified for $n = 0$.
- Next, before continuing, recall that for any affinoid Tate algebra A over K (or more generally a complete normed ring, not necessarily commutative nor unital), we have $\pi_1(KV^{\mathrm{an}}(A)) \simeq \text{"}\lim\text{"}_\rho \mathrm{GL}(A)/\mathrm{GL}(A)_\rho$ in $\mathrm{pro}(\mathcal{A}\mathrm{b})$ [37, Lemma 7.3], where $\mathrm{GL}(A)_\rho \subset \mathrm{GL}(A)$ is the subgroup generated by matrices g such that $\lim_{n\to+\infty} \|(g-1)^n\|\rho^n = 0$, cf. *loc. cit.*, which is a normal subgroup; so $\mathrm{GL}(A)_\rho$ decreases as ρ tends to $+\infty$. In particular, the tower $\pi_1(KV^{\mathrm{an}}(A))$ is a Mittag-Leffler system and has a countable cofinal subsystem, hence the condition (ii) of the lemma is also verified for $n = 0$.

Now we deduce from the lemma the canonical natural equivalences
$$\tau_{\geq 0}(\gamma^{\mathrm{light}}K^{\mathrm{an}}(A)) \simeq \gamma^{\mathrm{light}}k^{\mathrm{an}}(A), \quad \gamma^{\mathrm{light}}(\tau_{\geq 1}(K^{\mathrm{an}}(A))) \xrightarrow{\simeq} \tau_{\geq 1}(\gamma^{\mathrm{light}}(K^{\mathrm{an}}(A))), \quad \gamma^{\mathrm{light}}(\pi_0(K^{\mathrm{an}}(A))) \simeq \pi_0(K^{\mathrm{an}}(A)).$$
Our claim then follows.

4.5.6 - Example. If the condition ($\dagger_K$) holds, then by representability of analytic K-theory [22, Theorem 5.7], there is a canonical equivalence

$$L_{\mathrm{mot}}(\mathbf{Z}\times \mathrm{BGL}) \xrightarrow{\simeq} \Omega^\infty\tau_{\geq 0}L_{\mathrm{mot}}k^{\mathrm{an}} \xrightarrow{\simeq} \Omega^\infty\tau_{\geq 0}L_{\mathrm{mot}}(k^{\mathrm{an}})^B \xleftarrow{\simeq} \Omega^\infty\tau_{\geq 0}K^{\mathrm{an}}\text{[30]} \tag{4.5.6.1}$$

in the category $\mathcal{R}\mathrm{igH}(K,\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc}))$; in fact, there is already an equivalence $L_{\mathbf{A}^1}(\mathbf{Z}\times\mathrm{BGL}) \simeq \Omega^\infty\tau^{\mathrm{pre}}_{\geq 0}K^{\mathrm{an}}$ on $\mathcal{A}\mathrm{ffd}\mathcal{S}\mathrm{m}_K$ under this assumption, whence an equivalence $L_{\mathrm{Nis}}L_{\mathbf{A}^1}(\mathbf{Z}\times\mathrm{BGL}) \simeq \Omega^\infty\tau_{\geq 0}K^{\mathrm{an}}$ by Nisnevich

[30] Here, by abuse of notation, we denoted by the same notation K^{an} the image of K^{an} via the small limits preserving and conservative functor $\gamma^{\mathrm{light}} : \mathrm{pro}^{\mathrm{light}}(\mathcal{S}p^+) \to \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$, cf. [?, Lemma A.8]dahlhausenyaylali2024A1htpyRig

descent of K^{an} [22, Lemma 5.6]. Its proof will be recalled with more details in the proof of the following (4.6.3), though treated there with respect to the étale topology. Without assumption ($\dagger_K$), there are still natural maps (4.5.6.1), but they are not necessarily equivalences.

This example motivates the following definition.

4.5.7. Connective motivic analytic K-theory. Let BGL be the analytic sheafification of the presheaf $\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K^{\mathrm{op}} \to \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{pc}), Z \mapsto \mathrm{BGL}(\mathcal{O}_Z(Z))$. Alternatively, $\mathrm{BGL} = L_{\mathrm{Nis}} \varinjlim_n |B_\bullet \tilde{Y}_{\mathrm{GL}_n}|$; this is also the analytification of its algebraic analogue. We define the *connective motivic analytic K-theory over K* as the object

$$k^{\mathrm{an,mot}} := L_{\mathrm{mot}}(\mathbf{Z} \times \mathrm{BGL}) \in \mathcal{R}\mathrm{igH}(K, \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{pc})). \tag{4.5.7.1}$$

For $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$, we define its non-negative motivic K-groups as the homotopy groups of $k^{\mathrm{an,mot}}(Z) \in \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})$

$$k_j^{\mathrm{an,mot}}(Z) := \pi_j(k^{\mathrm{an,mot}}(Z)) \in \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}.$$

Essentially by construction, it is clear that there is a natural map

$$K_0^{\mathrm{naive}}(Z) \to k_0^{\mathrm{an,mot}}(Z)(*). \tag{4.5.7.2}$$

4.5.8. Enriched Yoneda lemma. Putting ourselves first into a general setting, let $\mathcal{E} \in \mathcal{CA}\mathrm{lg}(\mathrm{Pr}^{L,\otimes})$ be a presentably symmetric monoidal presentable ∞-category (i.e. a symmetric monoidal ∞-category which is presentable and such that the tensor product preserves colimits in each factor), with canonical unit $i_{\mathcal{E}} : \mathcal{S}\mathrm{pc} \to \mathcal{E}$. On the one hand, for any $F, G \in \mathrm{Fun}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K^{\mathrm{op}}, \mathcal{E})$, there is a $\mathcal{E}$-enriched internal mapping object $\underline{\mathrm{Hom}}_{\mathcal{E}}(F,G) \in \mathcal{E}$ such that $\underline{\mathrm{Hom}}_{\mathcal{E}}(F,-) : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{E}$ is right adjoint to pointwise $-\otimes F : \mathcal{E} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$. On the other hand, we denote $\tilde{Y}_c = i_{\mathcal{E}}(\mathrm{Hom}_{\mathcal{C}}(-,c)) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ the *$\mathcal{E}$-enriched Yoneda embedding* of $c \in \mathcal{C}$, which is the usual Yoneda embedding valued in $\mathcal{S}\mathrm{pc}$ composed with $i_{\mathcal{E}}$. Our key ∞-categorical claim will be the *$\mathcal{E}$-enriched Yoneda lemma* [22, Lemma B.3], namely there is a natural[31] equivalence $F(c) \simeq \underline{\mathrm{Hom}}_{\mathcal{E}}(\tilde{Y}_c, F)$ in $\mathcal{E}$ for any $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$.

Specialised in our situation where $\mathcal{C} = \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S$ and $\mathcal{E} = \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$, the unit functor $\mathcal{S}\mathrm{pc} \to \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$ factoring as $\mathcal{S}\mathrm{pc} \xrightarrow{\Sigma^\infty} \mathcal{S}\mathrm{p} \xrightarrow{\mathrm{const}} \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})$ in $\mathcal{P}\mathrm{r}^{L,\otimes}$, the enriched Yoneda lemma implies that for $\mathcal{F} \in \mathrm{Fun}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_S^{\mathrm{op}}, \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p}))$, there is a natural equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty \tilde{Y}_Z, \mathcal{F}) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})}(\Sigma^\infty \tilde{Y}_Z, \mathcal{F}) \simeq \mathcal{F}(Z) \tag{4.5.8.1}$$

in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$. Indeed, the first equivalence holds as the functor L_{mot} is compatible with the $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$-module structure by base change of the motivic localisation functor [22, Remark 4.1]; the second follows from the enriched Yoneda lemma.

4.5.9. Higher syntomic and étale Chern class maps. To explain the idea, let us first take some $\mathcal{F} \in \mathcal{R}\mathrm{igH}(K, \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p}))$. The evaluation on $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ (i.e. pullback along the embedding of ∞-categories $(\{Z\}, \mathrm{id}_Z) \hookrightarrow \mathcal{C}$) induces a map

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty k^{\mathrm{an,mot}}, \mathcal{F}) &\to \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty k^{\mathrm{an,mot}}(Z), \mathcal{F}(Z)) \\ &\to \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(\Sigma^\infty k^{\mathrm{an,mot}}(Z), \mathcal{F}(Z)) \end{aligned}$$

in $\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})$. Let $i \in \mathbf{Z}$. Any element

$$\mathfrak{c} \in \pi_{-i}\, \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty k^{\mathrm{an,mot}}, \mathcal{F})(*)$$

[31] It is natural in objects of $\mathcal{C}$ for any given $\mathcal{C}$ by Hinich's work [33], but the naturality in $\mathcal{C}$ of this transformation have only been proven later by Shay Ben-Moshe in [5].

determines a class in $\pi_0 \operatorname{Hom}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(\Sigma^\infty k^{\mathrm{an,mot}}, \Sigma^i \mathcal{F})$, whence morphisms

$$\mathfrak{c}_j : \pi_j(\Sigma^\infty k^{\mathrm{an,mot}}(Z)) \to \pi_{j-i}(\mathcal{F}(Z)), \quad j \in \mathbf{N}$$

in $\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$. As $\pi_j(\Sigma^\infty k^{\mathrm{an,mot}}(Z)) \simeq \varinjlim_{m\to+\infty} \pi_{j+m}\Sigma^m k^{\mathrm{an,mot}}(Z)$, we extract from above morphisms (by abuse of notation)

$$\mathfrak{c}_j : k_j^{\mathrm{an,mot}}(Z) \to \pi_{j-i}(\mathcal{F}(Z)), \quad j \in \mathbf{N}$$

in $\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$. Then, let us give an expression of $\underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty k^{\mathrm{an,mot}}, \mathcal{F})$. As L_{mot} and Σ^∞ commute with colimits, we can write $L_{\mathrm{mot}}\Sigma^\infty k^{\mathrm{an,mot}} = \coprod_{\mathbf{Z}} \varinjlim_n |L_{\mathrm{mot}}\Sigma^\infty \tilde{Y}_{B_\bullet \mathrm{GL}_n}|$. Therefore, we can compute

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty k^{\mathrm{an,mot}}, \mathcal{F}) &\simeq \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(\coprod_{\mathbf{Z}} \varinjlim_n |L_{\mathrm{mot}}\Sigma^\infty \tilde{Y}_{B_\bullet \mathrm{GL}_n}|, \mathcal{F}) \\ &\simeq \prod_{\mathbf{Z}} \varprojlim_n \lim_{\Delta} \underline{\mathrm{Hom}}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})}(L_{\mathrm{mot}}\Sigma^\infty \tilde{Y}_{\mathrm{GL}_n^{\times\bullet}}, \mathcal{F}) \\ &\simeq \prod_{\mathbf{Z}} \varprojlim_n \lim_{\Delta} \mathcal{F}(\mathrm{GL}_n^{\times\bullet}) =: \prod_{\mathbf{Z}} \varprojlim_n \mathcal{F}(B_\bullet \mathrm{GL}_n)\end{aligned} \tag{4.5.9.1}$$

where the last equivalence is due to the enriched Yoneda lemma (4.5.8.1).

We are now ready to construct rigid-analytic syntomic and étale Class maps. By computation of (4.3.3), there are universal syntomic Chern classes

$$C_i^{\mathrm{syn}} := (c_i)_{n\in\mathbf{Z}} \in \prod_{\mathbf{Z}} \varprojlim_n H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_n, i)(*), \quad i \in \mathbf{N}$$

for $\mathcal{F} = R\Gamma_{\mathrm{syn}}(Z, i)$ and universal étale Chern classes

$$C_i^{\mathrm{\acute{e}t}} := (c_i)_{n\in\mathbf{Z}} \in \prod_{\mathbf{Z}} \varprojlim_n H^{2i}_{\mathrm{\acute{e}t}}(B_\bullet \mathrm{GL}_n, \mathbf{Z}_p(i))(*),, \quad i \in \mathbf{N}$$

for $\mathcal{F} = R\Gamma_{\mathrm{\acute{e}t}}(Z, \mathbf{Z}_p(i))$. Applying the above construction to these classes, one obtains natural higher syntomic Chern class maps

$$c_{i,j}^{\mathrm{syn}} : k_j^{\mathrm{an,mot}}(Z) \to H^{2i-j}_{\mathrm{syn}}(Z, i), \quad i, j \in \mathbf{N} \tag{4.5.9.2}$$

and natural higher étale Chern class maps

$$c_{i,j}^{\mathrm{\acute{e}t}} : k_j^{\mathrm{an,mot}}(Z) \to H^{2i-j}_{\mathrm{\acute{e}t}}(Z, \mathbf{Z}_p(i)), \quad i, j \in \mathbf{N} \tag{4.5.9.3}$$

in $\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$. Since both this construction and that of (4.3.1) and (4.3.2) rely on the projective bundle formula, by checking various compatibilities, one verifies that these Chern class maps are compatible through the comparison map (4.5.7.2).

4.5.10. Higher étale regulators. For $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ and $i, j \in \mathbf{N}$, let $\tilde{c}_{i,j}^{\mathrm{\acute{e}t}} : k_j^{\mathrm{an,mot}}(Z) \to H^0(\underline{\mathscr{G}_K}, H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)))$ be the map induced by the higher étale Chern class map $c_{i,j}^{\mathrm{\acute{e}t}}$ and the Hochschild-Serre spectral sequence boundary map $\delta_0 : H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)) \to H^0(\underline{\mathscr{G}_K}, H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)))$. We define

$$k_j^{\mathrm{an,mot}}(Z)_0 := \ker(k_j^{\mathrm{an,mot}}(Z) \xrightarrow{\tilde{c}_{i,j}^{\mathrm{\acute{e}t}}} H^0(\underline{\mathscr{G}_K}, H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)))) \tag{4.5.10.1}$$

and the higher étale regulator map

$$r_{i,j}^{\mathrm{\acute{e}t}} : k_j^{\mathrm{an,mot}}(Z)_0 \to H^1(\underline{\mathscr{G}_K}, H^{2i-j-1}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i))) \tag{4.5.10.2}$$

induced from $c^{\text{ét}}_{i,j}|_{k_j^{\text{an,mot}}(Z)_0}$ and the Hochschild-Serre spectral sequence map

$$\delta_1 : H^{2i-j}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))_0 := \ker\delta_0 \to H^1(\underline{\mathcal{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))).$$

4.5.11 - Theorem. *Let $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ be proper. The étale regulator map $r^{\text{ét}}_{i,j}$ (4.5.10.2) factors through the condensed subgroup*

$$H^1_{\text{st}}(\underline{\mathcal{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))) \subset H^1(\underline{\mathcal{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))).$$

Proof. By (4.3.6) and (3.2.17), we have the following commutative diagram

$$\begin{tikzcd}
 & H^{2i-j}_{\text{syn}}(Z,i) \arrow[r, "\delta_0^{\text{syn}}"] \arrow[d, "\rho^{\text{arith}}_{\text{syn}}"] & H^0_{\text{st}}(\underline{\mathcal{G}_K}, H^{2i-j}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))) \arrow[d, "\simeq"', "\text{can}"] \\
k_j^{\text{an,mot}}(Z) \arrow[ur, "c^{\text{syn}}_{i,j}"] \arrow[dr, "c^{\text{ét}}_{i,j}"'] & & \\
 & H^{2i-j}_{\text{ét}}(Z, \mathbf{Q}_p(i)) \arrow[r, "\delta_0"] & H^0(\underline{\mathcal{G}_K}, H^{2i-j}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))).
\end{tikzcd}$$

Hence the image of $k_j^{\text{an,mot}}(Z)_0$ under $c^{\text{syn}}_{i,j}$ is contained in $H^{2i-j}_{\text{syn}}(Z,i)_0 := \ker(\delta_0 \circ \rho^{\text{arith}}_{\text{syn}}) = \ker\delta_0^{\text{syn}}$. Consequently, again by (3.2.17), we obtain the following commutative diagram

$$\begin{tikzcd}
 & H^{2i-j}_{\text{syn}}(Z,i)_0 \arrow[r, "\delta_0^{\text{syn}}"] \arrow[d, "\rho^{\text{arith}}_{\text{syn}}"] & H^1_{\text{st}}(\underline{\mathcal{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))) \arrow[d, hook, "\text{can}"] \\
k_j^{\text{an,mot}}(Z)_0 \arrow[ur, "c^{\text{syn}}_{i,j}"] \arrow[dr, "c^{\text{ét}}_{i,j}"'] & & \\
 & H^{2i-j}_{\text{ét}}(Z, \mathbf{Q}_p(i))_0 \arrow[r, "\delta_0"] & H^1(\underline{\mathcal{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i)))
\end{tikzcd}$$

which shows the desired factorisation. □

4.6 Towards étale analytic K-theory

The natural map (4.5.6.1) is only valid under the condition ($\dagger_K$), which has not been proven to be true yet.

However, in the scope of an étale local theory, ($\dagger_A$) is true by Temkin's altered local uniformisation theorem [53, Corollary 3.3.2], so that we may obtain an *unconditional* representability of the étale analytic K-theory by going through the arguments in [22, §5], as we shall explain below in this subsection.

4.6.1. Étale analytic K-theory. Recall that for a Tate ring A, there is a natural map in $\text{pro}^{\text{light}}(\mathcal{S}\text{p})$

$$K^{\text{an}}(A) = (k^{\text{an}}_{\geq 1})^B(A) \to (k^{\text{an}})^B(A), \tag{4.6.1.1}$$

and we view it by application γ^{light} as a map in $\mathcal{C}\text{ond}^{\text{light}}(\mathcal{S}\text{p})$. If moreover A is regular and satisfies ($\dagger_A$), then (4.6.1.1) becomes an equivalence [37, Lemma 6.18], and we have equivalences [37, Corollary 6.20]

$$k^{\text{an}}(A) \xrightarrow{\simeq} \tau_{\geq 0}(k^{\text{an}})^B(A) \xleftarrow{\simeq} \tau_{\geq 0}K^{\text{an}}(A).$$

Let the *étale analytic K-theory over K* be the étale sheaf

$$K^{\text{an,ét}} := L_{\text{ét}}K^{\text{an}} \in \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{R}\text{ig}\mathcal{S}\text{m}_K, \mathcal{C}\text{ond}^{\text{light}}(\mathcal{S}\text{p}))$$

which is the étale sheafification of the presheaf $K^{\mathrm{an}} \in \mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}))$. It might not be $\mathbf{A}^1$-homotopy invariant due to non-quasi-compactness of $\mathbf{A}^1$. We could also consider further eh-sheafification $K^{\mathrm{an,\acute{e}h}} := L_{\mathrm{\acute{e}h}} K^{\mathrm{an}} \in \mathcal{S}\mathrm{hv}_{\mathrm{\acute{e}h}}(\mathcal{R}\mathrm{ig}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}))$; but we will not pursue this in the following.

4.6.2. For $\mathrm{Spa}(A, A^+) \in \mathcal{A}\mathrm{ffd}_K$, there are natural maps

$$\mathbf{Z}(A) \times \mathrm{BGL}(A) \to \Omega^\infty K_{\geq 0}(A) \to \Omega^\infty \tau_{\geq 0}(k^{\mathrm{an}})^B(A) \leftarrow \Omega^\infty \tau_{\geq 0} K^{\mathrm{an}}(A) \tag{4.6.2.1}$$

by construction. Moreover, since $(k^{\mathrm{an}})^B$ and K^{an} are $\mathbf{A}^1$-homotopy invariant, so are $\Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 0} (k^{\mathrm{an}})^B$ and $\Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 0} K^{\mathrm{an}}$. Therefore, we obtain maps in $\mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p}))$

$$L_{\mathbf{A}^1}(\mathbf{Z} \times \mathrm{BGL}) \to \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 0} (k^{\mathrm{an}})^B \leftarrow \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 0} K^{\mathrm{an}}. \tag{4.6.2.2}$$

4.6.3 - Theorem (Representability of étale analytic K-theory). *The map (4.6.2.2) induces equivalences after étale sheafification, that is, we have equivalences in* $\mathcal{S}\mathrm{hv}_{\mathrm{\acute{e}t}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc}))$

$$L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1}(\mathbf{Z} \times \mathrm{BGL}) \xrightarrow{\simeq} \Omega^\infty \tau_{\geq 0} L_{\mathrm{\acute{e}t}} (k^{\mathrm{an}})^B \xleftarrow{\simeq} \Omega^\infty \tau_{\geq 0} L_{\mathrm{\acute{e}t}} K^{\mathrm{an}} = \Omega^\infty \tau_{\geq 0} K^{\mathrm{an,\acute{e}t}}.$$

Proof. Recall that sheafification commutes with Ω^∞ and $\tau_{\geq 0}$, namely we have equivalence of functors $L_\tau \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 0} \xrightarrow{\simeq} \Omega^\infty \tau_{\geq 0} L_\tau$ for any topology τ. So we are reduced to showing that the maps (4.6.2.2) becomes equivalences on certain basis of $\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_{\mathrm{\acute{e}t}}$ or on stalks (i.e. on strict henselian local rings).

By Temkin's altered local uniformisation theorem [53, Corollary 3.3.2], any $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ is étale locally of the form $\mathrm{Spa}(A, A^\circ)$ with $\mathrm{Spf}\, A^\circ$ being a semistable formal scheme over $\mathcal{O}_L$ for some finite extension L/K, hence A is regular and satisifies ($\dagger_A$). The second equivalence of (4.6.2.2) is true for such $\mathrm{Spa}(A, A^\circ) \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ by (4.5.4.1), hence after étale sheafification we obtain $L_{\mathrm{\acute{e}t}}(k^{\mathrm{an}})^B \xleftarrow{\simeq} L_{\mathrm{\acute{e}t}} K^{\mathrm{an}}$. So we are left to prove the first equivalence.

We now proceed exactly as in the proof of [22, Theorem 5.7]. Using the fibre sequence "$\tau_{\geq 1} \to \tau_{\geq 0} \to \pi_0$" in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$, which stays a fibre sequence in $\mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc})$ after applying Ω^∞, it is enough to examine respectively the connected covers and in degree zero, namely the maps in $\mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{pc}))$ *resp.* in $\mathcal{PS}\mathrm{hv}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b})$

$$\mathrm{BGL} \to \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 1} (k^{\mathrm{an}})^B \leftarrow \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 1} K^{\mathrm{an}}, \quad \textit{resp. } \mathbf{Z} \to \pi^{\mathrm{pre}}_0 (k^{\mathrm{an}})^B \leftarrow \pi^{\mathrm{pre}}_0 K^{\mathrm{an}}.$$

In degree zero, we have $K^{\mathrm{an}}_0(A') \simeq K_0(A')$ for strict henselian local rings A' by Temkin's result and (4.5.4.2), and $K_0(A') \simeq \mathbf{Z}$ for any local rings A', so we obtain equivalences $L_{\mathrm{\acute{e}t}} \mathbf{Z} \xrightarrow{\simeq} L_{\mathrm{\acute{e}t}} K^{\mathrm{an}}_0$. Noticing that $\mathbf{Z} \xrightarrow{\simeq} L_{\mathbf{A}^1} \mathbf{Z}$, we have shown that $L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1} \mathbf{Z} \xrightarrow{\simeq} L_{\mathrm{\acute{e}t}} \Omega^\infty \pi^{\mathrm{pre}}_0 K^{\mathrm{an}}$ in $\mathcal{S}\mathrm{hv}_{\mathrm{\acute{e}t}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{pc}))$.

On connected covers, for $\mathrm{Spa}(A, A^\circ) \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ with A regular and satisfying ($\dagger_A$), we compute as follows:

$$\begin{aligned}
(L_{\mathbf{A}^1}\mathrm{BGL})(\mathrm{Spa}(A, A^\circ)) &\simeq \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{BGL}(\mathrm{Spa}(A, A^\circ)\langle \Delta^n \rangle) \\
&\simeq \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{Hom}(\mathrm{colim}_\rho \mathrm{Spa}(A, A^\circ)\left\langle \Delta^n_\rho \right\rangle, \mathrm{BGL}) \\
&\simeq \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} \lim_\rho \mathrm{Hom}(\mathrm{Spa}(A, A^\circ)\left\langle \Delta^n_\rho \right\rangle, \mathrm{BGL}) \\
&\simeq \lim_\rho \mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}} \mathrm{Hom}(\mathrm{Spa}(A, A^\circ)\left\langle \Delta^n_\rho \right\rangle, \mathrm{BGL}) && \text{by (4.6.4) below} \\
&\simeq \lim_\rho \mathrm{BGL}(A\left\langle \Delta_\rho \right\rangle) \\
&= KV^{\mathrm{an}}(A) && \text{cf. [37, Definition 7.1]} \\
&\simeq \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 1} K^{\mathrm{an}}(A) && \text{by [37, Lemma 7.5];}
\end{aligned}$$

here, the regularity and ($\dagger_A$) conditions are used only in the last isomorphism. Taking étale sheafification, by

Temkin's results, we obtain an equivalence in $\mathcal{S}\mathrm{hv}_{\mathrm{\acute{e}t}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}))$

$$L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1} \mathrm{BGL} \xrightarrow{\simeq} L_{\mathrm{\acute{e}t}} \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 1}.$$

To conclude, we deduce from above that

$$L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1}(\mathbf{Z} \times \mathrm{BGL}) \simeq (L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1} \mathbf{Z}) \times (L_{\mathrm{\acute{e}t}} L_{\mathbf{A}^1} \mathrm{BGL}) \xrightarrow{\simeq} L_{\mathrm{\acute{e}t}} \Omega^{\infty,\mathrm{pre}} \tau^{\mathrm{pre}}_{\geq 0} K^{\mathrm{an}} \simeq \Omega^{\infty} \tau_{\geq 0} L_{\mathrm{\acute{e}t}} K^{\mathrm{an}},$$

where the first equivalence uses the fact that $L_{\mathrm{\acute{e}t}}$ and $L_{\mathbf{A}^1}$ commute both with finite products. □

We have used a variant of [38, Lemma 2.8] on exchanging geometric realisations and derived limits.

4.6.4 - Lemma. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure which is left complete [43, Proposition 1.2.1.17]. Let I be an ∞-category such that $\lim_I$ sends $\mathrm{Fun}(I, \mathcal{C}^{\leq 0})$ into $\mathcal{C}^{\leq m_I}$ for certain $m_I \in \mathbf{N}$. Let $(X_{i,\bullet})_{i \in I}$ be a diagram of simplicial objects in $\mathcal{C}_{\geq 0}$, i.e. $X_{i,[n]}$ is connective for all $i \in I, [n] \in \Delta$, then the limit over I commutes with geometric realisation, i.e.*

$$\left| \lim_{i \in I} X_{i,\bullet} \right| \simeq \lim_{i \in I} |X_{i,\bullet}|$$

where we denote by $\lim_{i \in I} X_{i,\bullet}$ the levelwise limit of simplicial objects in $\mathcal{C}$.

Proof. The ∞-categorical Dold-Kan correspondence for stable ∞-categories [43, Theorem 1.2.4.1] says that $\mathrm{Fun}(N(\Delta)^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(N(\mathbf{N}), \mathcal{C})$, and under this equivalence, a simplicial object $X_\bullet$ is identified with the sequence of geometric realisations of its skeleta

$$|\mathrm{sk}_0 X_\bullet| \to |\mathrm{sk}_1 X_\bullet| \to \cdots.$$

Moreover, we have

$$|X_\bullet| \simeq \varinjlim_n |\mathrm{sk}_n X_\bullet|.$$

Thus the equivalence stated in the lemma is equivalent to

$$\varinjlim_n \left| \mathrm{sk}_n \lim_{i \in I} X_{i,\bullet} \right| \simeq \lim_{i \in I} \varinjlim_n \left| \mathrm{sk}_n X_{i,\bullet} \right|.$$

By left completeness (so-called "convergence of Postnikov tower"), it suffices to check isomorphisms on their k-truncations $\tau_{\leq k}$. By $(-m_I)$-connectivity assumption on $\lim_I$ and right-t-exactness of $\varprojlim_n$, it is the same as checking π_k of the n-th term with $n := k + m_I$; hence we are reduced to proving this equivalence after dropping out $\varinjlim_n$, i.e. to prove

$$\left| \mathrm{sk}_n \lim_{i \in I} X_{i,\bullet} \right| \simeq \lim_{i \in I} \left| \mathrm{sk}_n X_{i,\bullet} \right|$$

for $n = k + m_I$. But this follows from the commutativity between limits and finite colimits in a stable ∞-category [43, Proposition 1.1.4.1]. □

4.6.5 - Example. Here are some main cases where the connectivity condition of the lemma (4.6.4) is satisfied:

(i) The limit is a product, i.e. $\lim_I$ can be realised as a product $\prod_J$ for some set J, but under the condition the $\prod_J$ is t-exact on $\mathcal{C}$. This is the case for example where $\mathcal{C} = \mathcal{C}\mathrm{ond}_\kappa(\mathcal{S}\mathrm{p})$ and J is a κ-small set. In this case, we can choose $m_I = 0$.

(ii) The index category I is (the nerve of) $\mathbf{N}$, so that $m_I = 1$ by vanishing of higher derived limits in degrees > 1.

Now, we are going to relate the analytic K-theory à la Kerz-Saito-Tamme to the continuous K-theory. Before that, let us recall the two versions of continuous K-theory and their relations.

4.6.6. Nuclear-continuous K-theory. Recall that in Andreychev's thesis [1], for any Tate complete Huber ring A, he applied Efimov's K-theory to the dualisable category $\mathrm{Nuc}(A)$ to define the *non-connective nuclear-continuous K-theory spectrum* $\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A)$, which agrees with the continuous K-theory $\mathbf{K}_{\mathrm{cont}}(A)$ à la Morrow (defined using integral models) thanks to Efimov's Continuity Theorem [1, Satz 5.8] (cf. [24, Example 1.30]). Let us recall the proof of their identification.

4.6.7 - Theorem (Efimov, Andreychev). *For Tate complete Huber rings A, there is a natural isomorphism of spectra*

$$\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A) \xrightarrow{\simeq} \mathbf{K}_{\mathrm{cont}}(A).$$

Proof. Let A_0 be a ring of definition of A with a pseudouniformiser $\varpi \in A_0$, so that $A_0[\frac{1}{\varpi}] = A$. First, observe the following diagram:

(4.6.7.1)
$$\begin{tikzcd}
\mathrm{Tor}(\varpi^\infty) \arrow[r] \arrow[d] & \mathscr{D}(A_0) \arrow[r, "L"] \arrow[d] & \mathscr{D}(A_0[\frac{1}{\varpi}]) \arrow[d] \\
\mathrm{Tor}^{\mathrm{nuc}}(\varpi^\infty) \arrow[r] & \mathrm{Nuc}(A_0) \arrow[r, "L'"] & \mathrm{Nuc}(A_0[\frac{1}{\varpi}]).
\end{tikzcd}$$

Here L and L' denote the canonical localisation functors and $\mathrm{Tor}(\varpi^\infty)$ and $\mathrm{Tor}^{\mathrm{nuc}}(\varpi^\infty)$ denote their kernels. Then the function $\mathrm{Tor}(\varpi^\infty) \to \mathrm{Tor}^{\mathrm{nuc}}(\varpi^\infty)$ is an equivalence according to [1, Satz 4.11], hence applying the Efimov's K-theory, we obtain a pullpack-pushout square

(4.6.7.2)
$$\begin{tikzcd}
\mathbf{K}(A_0) \arrow[r, "L"] \arrow[d] & \mathbf{K}(A_0[\frac{1}{\varpi}]) \arrow[d] \\
\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A_0) \arrow[r, "L'"] & \mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A_0[\frac{1}{\varpi}])
\end{tikzcd}$$

in the stable ∞-category of spectra. By Efimov's Continuity Theorem, we have a natural isomorphism of spectra $\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A_0) \xrightarrow{\simeq} \mathbf{K}_{\mathrm{cont}}(A_0)$, hence as pushouts we get an isomorphism of spectra $\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A) \xrightarrow{\simeq} \mathbf{K}_{\mathrm{cont}}(A)$. □

4.6.8. Condensed nuclear-continuous K-theory. It is possible to upgrade the isomorphism of spectra (4.6.7) to an isomorphism of condensed spectra. For this, we first upgrade the concerned spectra to condensed spectra.

On the one hand, the continuous K-theory spectrum $\mathbf{K}_{\mathrm{cont}}(A_0)$ as above underlies the condensed spectrum

$$\underline{\mathbf{K}_{\mathrm{cont}}}(A_0) := \varprojlim_n \underline{\mathbf{K}(A_0/\varpi^n)},$$

where the K-theory spectra $\mathbf{K}(A_0/\varpi^n)$ are endowed with the discrete topology; then we define the *condensed continuous K-theory spectrum* $\underline{\mathbf{K}_{\mathrm{cont}}}(A)$ as the pushout

(4.6.8.1)
$$\begin{tikzcd}
\mathbf{K}(A_0) \arrow[r] \arrow[d] & \mathbf{K}(A) \arrow[d] \\
\underline{\mathbf{K}_{\mathrm{cont}}}(A_0) \arrow[r] & \underline{\mathbf{K}_{\mathrm{cont}}}(A)
\end{tikzcd}$$

within the stable ∞-category $\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p})$.

On the other hand, $\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(A)$ can be upgraded to a condensed spectrum $\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A)$ as follows. Let (A, A^+) be a complete Huber pair. Recall that for any profinite set S, the object $\mathrm{Spa}(A, A^+) \times \underline{S}$ is the well-defined adic space associated with the complete Huber pair $(C(S, A), C(S, A^+))$. Consider the presheaf

$$\begin{aligned}
\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A) : \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in} &\to \mathcal{S}\mathrm{p} \\
S &\mapsto \mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(C(S, A))
\end{aligned}$$

It sends finite disjoint unions to products, so defines a sheaf on the site of extremally disconnected sets. We are going to show that $\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A)$ defines a sheaf on the whole site of profinite sets, and call it the *condensed nuclear-continuous K-theory spectrum of A*, and globalise it to any locally Tate adic spaces.

4.6.9 - Theorem. *Let A be a Tate complete Huber ring with ring of definition A_0 and a pseudouniformiser $\varpi \in A_0$. Let (A_0, I) be an adic pair with $I \subset A_0$ weakly pro-regular ideal. The presheaf $\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A_0)$ is a sheaf on $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$. More precisely, for any profinite set S and any hypercovering $S_\bullet \to S$ by extremally disconnected sets, we have a natural isomorphism*

$$\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(C(S, A_0)) \xrightarrow{\simeq} \varprojlim_{\Delta} \mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(C(S_\bullet, A_0)).$$

Proof. The ideal $I = \varpi A_0 \subset A_0$ is weakly pro-regular by [1, Lemma 4.1, Lemma 3.5], so that the problem is well-posed. Consider the $\mathbf{Z}_\blacksquare$-solid algebra $\underline{A_0} = \varprojlim_n \underline{A_0/I^n}$. Since the natural map

$$\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathbf{Z}[S_\bullet] \to \mathbf{Z}[S]$$

becomes an equivalence after $\mathbf{Z}_\blacksquare$-solidification [12, Proposition 5.6], after taking (external) $R\,\mathrm{Hom}(-, \underline{A_0})$ we obtain a cosimplicial resolution (of rings)

$$C(S, A_0) \xrightarrow{\simeq} \lim_{\Delta} C(S_\bullet, A_0).$$

By Efimov's Continuity Theorem, there is a natural isomorphism of spectra

$$\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(C(S, A_0)) \xrightarrow{\simeq} \varprojlim_{n} \mathbf{K}(C(S, A_0/I^n)).$$

We claim that

$$\mathbf{K}(C(S, A_0/I^n)) \xrightarrow{\simeq} \lim_{\Delta} \mathbf{K}(C(S_\bullet, A_0/I^n)), \quad n \in \mathbf{N}.$$

Indeed, by writing the hypercovering $S_\bullet \to S$ as a cofiltered limit of hypercoverings $S_{\bullet,j} \to S_j$ (indexed by $j \in J$) of finite sets by finite sets, which in particular splits, we have

$$\mathbf{K}(C(S_j, A_0/I^n)) \xrightarrow{\simeq} \lim_{\Delta} \mathbf{K}(C(S_{\bullet,j}, A_0/I^n)).$$

Now take the filtered colimit with respect to $j \in J^{\mathrm{op}}$: as the algebraic K-theory $\mathbf{K}(-)$ commutes with filtered colimits of rings and $C(S, A_0/I^n) = \varinjlim_j C(S_j, A_0/I^n)$, similarly termwisely $C(S_\bullet, A_0/I^n) = \varinjlim_j C(S_{\bullet,j}, A_0/I^n)$, we obtain

$$\begin{aligned}\mathbf{K}(C(S, A_0/I^n)) \simeq \varinjlim_{j} \mathbf{K}(C(S_j, A_0/I^n)) &\xrightarrow{\simeq} \varinjlim_{j} \lim_{\Delta} \mathbf{K}(C(S_{\bullet,j}, A_0/I^n)) \\ &\xrightarrow{\simeq} \lim_{\Delta} \varinjlim_{j} \mathbf{K}(C(S_{\bullet,j}, A_0/I^n)) \\ &\simeq \lim_{\Delta} \mathbf{K}(C(S_\bullet, A_0/I^n))\end{aligned}$$

The second to last isomorphism can be seen by comparing the *convergent* spectral sequences

$$E^{p,q}_{1,(j)} = \pi_{-q}(\mathbf{K}(C(S_{p,j}, A_0/I^n))) \Rightarrow \pi_{-(p+q)}(\mathbf{K}(C(S_j, A_0/I^n)))$$

$$E^{p,q}_{1} = \pi_{-q}(\mathbf{K}(C(S_p, A_0/I^n))) \Rightarrow \pi_{-(p+q)}(\mathbf{K}(C(S, A_0/I^n))).$$

Indeed, $\pi_{-(p+q)}$ commutes with filtered colimits, and

$$\pi_{-q}\mathbf{K}(C(T, A_0/I^n)) \simeq \mathrm{Map}(T, \pi_{-q}\mathbf{K}(A_0/I^n)) \simeq C(T, \mathbf{Z}) \otimes_{\mathbf{Z}} \pi_{-q}\mathbf{K}(A_0/I^n)$$

for any finite set T, so the the above spectral sequences degenerates already at the page E_1 by exactness of the complexes $0 \to C(S_j, \mathbf{Z}) \to C(S_{1,j}, \mathbf{Z}) \to C(S_{2,j}, \mathbf{Z}) \to \cdots$ for any $j \in J$. $\square$

4.6.10 - Corollary. *For any Tate complete Huber ring A, we have a natural isomorphism of spectra*

$$\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(C(S,A)) \xrightarrow{\simeq} \varprojlim_{\Delta} \mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}(C(S_\bullet, A_0)).$$

In particular, $\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A)$ is a sheaf on $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$ valued in spectra; in other words, $\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A)$ is a condensed spectrum.

Proof. This follows from the theorem as in the proof of (4.6.7), using the diagram (4.6.7.2). $\square$

4.6.11 - Theorem. *For any Tate complete Huber ring A, there is a natural morphism of condensed spectra*

$$\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(A) \xrightarrow{\simeq} \underline{\mathbf{K}_{\mathrm{cont}}}(A).$$

Proof. For any profinite set S, the topological ring $C(S,A)$ is still a Tate complete Huber ring, so we may apply theorem (4.6.7) to conclude. $\square$

We are ready to relate the analytic K-theory à la Kerz-Saito-Tamme to the nuclear-continuous K-theory.

4.6.12 - Theorem. *For qcqs $Z \in \mathcal{R}\mathrm{ig}_K$, there are natural isomorphisms of condensed spectra*

$$L_{\mathbf{A}^1}\underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}(Z) \xrightarrow{\simeq} L_{\mathbf{A}^1}\underline{\mathbf{K}_{\mathrm{cont}}}(Z) \xrightarrow{\simeq} K^{\mathrm{an}}(Z).$$

Proof. By analytic descent, we are easily reduced to the case where $Z = \mathrm{Sp}\, A$ is affinoid. Then the first isomorphism holds by (4.6.11), even before taking $\mathbf{A}^1$-localisation. The second isomorphism is [22, Theorem 5.18]. $\square$

Finally, let us come back to higher étale Chern class maps and regulators.

4.6.13. Higher syntomic and étale Chern class maps (continued). Using (4.6.3) and following the steps of (4.5.9), for $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$, we obtain natural higher syntomic Chern class maps

(4.6.13.1) $$c^{\mathrm{syn}}_{i,j} : K^{\mathrm{an,\acute{e}t}}_j(Z) \to H^{2i-j}_{\mathrm{syn}}(Z, i), \quad i,j \in \mathbf{N}$$

and natural higher étale Chern class maps

(4.6.13.2) $$c^{\mathrm{\acute{e}t}}_{i,j} : K^{\mathrm{an,\acute{e}t}}_j(Z) \to H^{2i-j}_{\mathrm{\acute{e}t}}(Z, \mathbf{Z}_p(i)), \quad i,j \in \mathbf{N}$$

in $\mathcal{C}\mathrm{ond}\mathcal{A}\mathrm{b}$ on the étale analytic K-groups of Z. They are compatible through the comparison map (4.5.7.2).

Recall that we have maps in $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K, \mathcal{C}\mathrm{ond}^{\mathrm{light}}(\mathcal{S}\mathrm{p}))$

$$k^{\mathrm{an,mot}} = L_{\mathrm{mot}}(\mathbf{Z} \times \mathrm{BGL}) \to L_{\mathrm{\acute{e}t}}(\mathbf{Z} \times \mathrm{BGL}) \simeq K^{\mathrm{an,\acute{e}t}} \leftarrow K^{\mathrm{an}} \leftarrow \underline{\mathbf{K}^{\mathrm{nuc}}_{\mathrm{cont}}}.$$

Therefore, these newly defined higher Chern class maps refine (4.5.9.2) and (4.5.9.3), and induces higher Chern class maps on the analytic K-groups $K^{\mathrm{an}}_j(Z), j \in \mathbf{N}$, in so particular on the nuclear-continuous K-groups.

4.6.14. Higher étale regulators (continued). For $Z \in \mathcal{R}\mathrm{ig}\mathcal{S}\mathrm{m}_K$ and $i,j \in \mathbf{N}$, let $\tilde{c}^{\mathrm{\acute{e}t}}_{i,j} : K^{\mathrm{an,\acute{e}t}}_j(Z) \to H^0(\underline{\mathcal{G}_K}, H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)))$ be the map induced by the higher étale Chern class map $c^{\mathrm{\acute{e}t}}_{i,j}$ (4.6.13.2) and the Hochschild-Serre spectral sequence boundary map $\delta_0 : H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)) \to H^0(\underline{\mathcal{G}_K}, H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i)))$. We define

(4.6.14.1) $$K^{\mathrm{an,\acute{e}t}}_j(Z)_0 := \ker(K^{\mathrm{an,\acute{e}t}}_j(Z) \xrightarrow{\tilde{c}^{\mathrm{\acute{e}t}}_{i,j}} H^0(\underline{\mathcal{G}_K}, H^{2i-j}_{\mathrm{\acute{e}t}}(Z_C, \mathbf{Z}_p(i))))$$

and the higher étale regulator map

$$r^{\text{ét}}_{i,j} : K^{\text{an,ét}}_j(Z) \to H^1(\underline{\mathscr{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))) \tag{4.6.14.2}$$

induced from $c^{\text{ét}}_{i,j}|_{K^{\text{an,ét}}_j(Z)_0}$ and the Hochschild-Serre spectral sequence map

$$\delta_1 : H^{2i-j}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))_0 := \ker \delta_0 \to H^1(\underline{\mathscr{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Z}_p(i))).$$

This newly defined higher étale regulator map $r^{\text{ét}}_{i,j}$ refines that of (4.5.10.2), and induces higher étale regulator map on $K^{\text{an}}_0(Z)_0$. We shall use the same notation for the rational coefficients $\mathbf{Q}_p(i)$.

4.6.15 - Theorem. *Let $Z \in \mathcal{R}\text{ig}\mathcal{S}\text{m}_K$ be proper. The étale regulator map $r^{\text{ét}}_{i,j}$ (4.6.14.2) factors through the condensed subgroup*

$$H^1_{\text{st}}(\underline{\mathscr{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))) \subset H^1(\underline{\mathscr{G}_K}, H^{2i-j-1}_{\text{ét}}(Z_C, \mathbf{Q}_p(i))).$$

Proof. The proof goes *verbatim* as that of (4.5.11). □

4.6.16 - Remark. Since the syntomic cohomology and (pro)étale cohomology satisfy even éh-descent in $\mathcal{R}\text{ig}_K$, one may replace the site $\mathcal{R}\text{ig}\mathcal{S}\text{m}_{K,\text{ét}}$ by $\mathcal{R}\text{ig}_{K,\text{éh}}$ thanks to Haoyang Guo's description of local nature of the éh-topology, so as to extend the above theorem to the case of (possibly singular) proper $Z \in \mathcal{R}\text{ig}_K$.

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