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Preprint Seminar

# Descent of pseudocoherent and perfect complexes & vector bundles on analytic adic spaces
(locally Tate)

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(arxiv: 2105.12591, master thesis)

I. Algebraic case

II. Rigid-analytic / adic case

III. Analytic rings

IV. Descent of "quasicoherent" modules

V. Abstract manipulations: pluralisable, (pseudo) compact, nuclear objects } proof.

VI. Get discreteness - End of proof

## I. Algebraic case

**Def.** Let $A$ (ordinary) ring & $M \in D(A)$

1) $M$ is pseudocoherent if $M \overset{qiso}{\cong} ( \cdots \to P_{n-1} \to P_n \to 0)$

finite projective modules/$A$.

$\rightsquigarrow \mathrm{PCoh}_A, \ \mathrm{PCoh}_A^{\leq n}$

2) $M$ is perfect complex if $M \overset{qiso}{\cong} (0 \to P_a \to P_{a+1} \to \cdots \to P_b \to 0)$

$\rightsquigarrow \mathrm{Perf}(A), \ \mathrm{Perf}^{[a,b]}(A)$

**Thm** (Grothendieck, Lurie)
abelian cat. $\infty$-cat

The presheaves $\mathrm{AffSch}^{op} \longrightarrow \mathrm{Cat}_\infty$

$\mathrm{Spec}\, A \longmapsto \mathrm{PCoh}_A^{(\leq n)}, \ \mathrm{Perf}^{([a,b])}(A)$

satisfy fpqc descent.

$\rightsquigarrow$ ~~At~~ the abelian level: $\mathrm{Spec}\, A \longmapsto \mathrm{FinProj}_A, \ \mathrm{PCoh}_A$ satisfy fpqc descent.

$\mathrm{FinProj}_A = \mathrm{Perf}^{[0,0]}(A)$, $\mathrm{PCoh}_A = \mathrm{PCoh}_A \cap \mathrm{Mod}_A$

$$\rightsquigarrow \mathrm{FinProj}_A \xrightarrow{\cong} \mathrm{VB}(\mathrm{Spec}\, A)$$

$$M \longmapsto \tilde{M}$$

$$\Gamma(\mathrm{Spec}\, A, F) \longleftarrow F$$

Idea of proof:

1) Prove that $\mathrm{Spec}\, A \mapsto \mathcal{D}(A)$ satisfies fpqc descent

2) cut out the desired categories inside $\mathcal{D}(A)$
by conditions that localises & can be checked locally.

Applications:

(+ open-closed Excision Verdier sequence) + (non-connective) $\mathbb{K}(-) : \mathrm{Cat}^{\mathrm{idem}}_{\mathrm{stable}} \to \mathrm{Sp}$ localising invariant

$\Rightarrow \mathbb{K}(X) := \mathbb{K}(\mathrm{Perf}(X))$ satisfies Nisnevich descent.

## II. Analytic case

Geometric objects

K complete non-arch. field

- Tate : $\mathrm{Sp}(A)$ - max. spectrum of top. fin. type K-alg A
  ($\Leftrightarrow$ quotient topological K-alg of $K\langle T_1, -, T_n\rangle$)
  ring of convergent power series in $T_1, -, T_n$ with coefficients in K
  $$= \left\{ f = \sum_{I \in \mathbb{N}^n} a_I \underline{T}^I \;\middle|\; |a_I| \xrightarrow{|I| \to +\infty} 0 \right\}$$
  \+ admissible coverings (not all open coverings are admissible)
- Berkovich spaces : $\mathcal{M}(A) = \{$ cont. multiplicative ~~valuations~~ norms $\|\cdot\| : A \to \mathbb{R}_{\geq 0}\}$
  \+ admissible coverings
  $\mathcal{M}(A)$ is compact Hausdorff.
- Huber : consider higher rk valuations : $|\cdot|_v : A \longrightarrow \Gamma \cup \{0\}$, $\Gamma \ni 1$ (tot. ordered ab. grp.)

work with $(A, A^+)$ complete Huber pair.

- A complete Huber ring : topological ring. topology defined by I-adic topology on some open subring $A_0 \subseteq A$.
  st. $I \subseteq A_0$ f.g. ideal, $A_0$ is I-adically ~~cple~~ complete.
- $A^+ \subseteq A^0$ open subring + integrally closed ($A^0$ ← powerbdd elements.)

**Rmk**. $\forall$ complete Huber ring $A$, $\forall S \subseteq A$ subset.

$\rightsquigarrow$ complete Huber pair $(A, A^+)$ with $A^+ = \overline{A^{\circ\circ} + S}$ integrally closed subring gen. by $A^{\circ\circ} + S$.

**Ex**. complete Huber rings & pairs :

- $(\mathbb{Z}_p, \mathbb{Z}_p)$ not Tate, $(\mathbb{Q}_p, \mathbb{Z}_p)$ Tate
- $(K\langle T_1, -, T_n\rangle, \mathcal{O}_K\langle T_1, -, T_n\rangle)$ — $K$ non-arch field. $\overset{\Psi}{\pi} \neq 0$ pseudounif. } Tate
- $(C\langle T_1^{1/p^\infty}, -, T_n^{1/p^\infty}\rangle, \mathcal{O}_C\langle T_1^{1/p^\infty}, -, T_n^{1/p^\infty}\rangle)$ — $C = \widehat{\overline{K}}$.
- $(R, R^+)$ not Tate.
  $\uparrow$ discrete ring

**Def**. (Kedlaya-Liu) 1) A complete Huber pair $(A, A^+)$ is locally Tate ("analytic") if $A^{\circ\circ}$ generates the unit ideal of $A$.

2) $(A, A^+)$ is Tate if $\exists\, \varpi \in \underset{\text{unit}}{A^\times} \cap \underset{\text{top. nilp.}}{A^{\circ\circ}}$

**Fact**. Locally Tate $\Leftrightarrow$ "locally" Tate.

What does "locally" mean? Locally on affinoid adic space $\mathrm{Spa}(A, A^+)$ for analytic topology!

$$\mathrm{Spa}(A, A^+) \overset{\text{as set}}{=\!=} \{ \underset{\text{multiplicative}}{\text{cont. valuations}}\ |\cdot|_v : A \to \Gamma \cup \{0\} \mid |\cdot|_v \le 1 \text{ on } A^+ \}$$

$|\cdot|_v \rightsquigarrow \begin{cases} p = \ker(|\cdot|_v) \in \mathrm{Spec}\, A \\ |\cdot|_v : \kappa(p) \to \Gamma \cup \{0\} \text{ st. } |\cdot|_v \le 1 \text{ on image of } A^+. \end{cases}$

topology : analytic topology = coarsest topology st. $\forall f \in A$,

$$ev_f : A \to \prod_{v \in \mathrm{Spa}(A, A^+)} (\Gamma_v \cup \{0\}) \text{ continuous}$$

$\uparrow$ order topology $\cup\{0\}$

opens gen. by $\{x \in \Gamma \mid x < a\}$, $\{x \in \Gamma \mid x > a\}$ for $a \in \Gamma$.

It has a basis consisting of rational open subsets. (closed under taking finite intersections)

Coverings = open coverings

## Rational (open) subsets of $X = \mathrm{Spa}(A, A^+)$ :

$$U = X\left(\frac{f_1, \dots, f_n}{g}\right) := \{ |\cdot|_v \in X \mid \forall i = 1, \dots, n, \ |f_i|_v \leq |g|_v \neq 0 \}$$

where $f_1, \dots, f_n, g$ generate ~~the~~ an open ideal of $A$.

① invert $g$

② require $\left|\frac{f_i}{g}\right| \leq 1$.

**Notation** : $U = \{ |f_i| \leq |g|, \forall i \}$

**Prop.** $U \cong \mathrm{Spa}(\overset{\exists \text{universal}}{A_U, A_U^+})$.

i.e. $\forall (B, B^+)$ complete Huber pair $\}$ s.t.
$\forall f : (A, A^+) \to (B, B^+)$

$$\begin{tikzcd}
\mathrm{Spa}(B,B^+) \arrow[d, "\exists!"'] \arrow[dr, "\tilde{f}"] & \\
U \arrow[r, phantom, "\subseteq"] & \mathrm{Spa}(A,A^+)
\end{tikzcd}$$

$$\begin{tikzcd}
(A_U, A_U^+) \arrow[rr, "\exists!"] \arrow[rr, bend left] & & (B, B^+) \\
& (A, A^+) \arrow[ul] \arrow[ur, "f"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

**Construction**. $(A_U, A_U^+) = \left(A\left[\frac{1}{g}\right], A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]\right)$ completion wrt $I$-adic top.

i.e. $\begin{cases} A_U = \left(A_0\left[\frac{f_i}{g}\right]\right)^{\wedge}_{I} \otimes_{A_0\left[\frac{f_i}{g}\right]} A\left[\frac{1}{g}\right] \\ A_U^+ = \text{topological closure of } A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right] \end{cases}$

## Example of rational ~~open~~ coverings :

- $\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}) = \underset{\{|T| \leq 1\}}{\underset{\parallel}{\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T])}} \cup \underset{\{1 \leq |T|\}}{\underset{\parallel}{\mathrm{Spa}(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])}}$

  More generally :

  Simple Laurent covering (SLC) :

  $$\mathrm{Spa}(A, A^+) = \underset{\{1 \leq |f|\}}{\mathrm{Spa}(A\langle \tfrac{1}{f} \rangle, A^+\langle \tfrac{1}{f} \rangle)} \cup \underset{\{|f| \leq 1\}}{\mathrm{Spa}(A\langle f \rangle, A^+\langle f \rangle)}$$

- Simple blanced covering (SBC) :

  $$\mathrm{Spa}(A, A^+) = \mathrm{Spa}(A\langle \tfrac{1}{f} \rangle, A^+\langle \tfrac{1}{f} \rangle) \cup \mathrm{Spa}(A\langle \tfrac{1}{1-f} \rangle, A^+\langle \tfrac{1}{1-f} \rangle)$$

Prop (Huber) Any open covering of $\mathrm{Spa}(A, A^+)$ can be refined to: a finite composition of (SLC) & (SBC).

Rmk. If $A$ is Tate (i.e. $\exists\, \varpi \in A^\times \cap A^{\circ\circ}$) or locally Tate then only (SLC) are needed.

Thm A. Let $(A, A^+)$ be an sheafy and locally Tate Huber pair,

↳ str. presheaf: $u \mapsto A_u$ is a sheaf.

then
$$\begin{tikzcd}[row sep=tiny] \mathrm{Spa}(A, A^+) \\ \rotatebox{90}{$\subseteq$}\ \text{rational open} \\ u \arrow[r, mapsto] & \mathrm{PCoh}^{(\leq n)}_{A_u},\ \mathrm{Perf}^{([a,b])}(A_u) \in \mathrm{Cat}_\infty \\ & \mathrm{PCoh}^{\circ}_{A_u},\ \mathrm{FinProj}(A_u) \in \mathrm{Cat} \end{tikzcd}$$

satisfy analytic descent!

$$\rightsquigarrow\quad \begin{tikzcd} \mathrm{FinProj}_A \arrow[r, shift left, "\cancel{M \mapsto M\otimes_A A_u}"] & VB(X) \arrow[l, shift left, "\Gamma(X, -)"] \end{tikzcd} \qquad X = \mathrm{Spa}(A, A^+)$$

$$M \longmapsto (\tilde{M} : u \mapsto M \otimes_A A_u)$$

Rmk. 2) Kedlaya-Liu: $\overset{\wedge}{\text{analytic}}$ descent of $\mathrm{FinProj}_A$ on locally Tate adic space $\mathrm{Spa}(A, A^+)$ $\Updownarrow$ sheafy

by direct attack

1) Bosch-Görtz-Gabler: flat descent for $\mathrm{Coh}(A)$, $A = K$-affd alg.

3) $K$ non arch field, $\pi$ pseudounif.

$$\{\mathcal{O}_K\text{-alg}, \pi\text{-tors. free}, \pi\text{-complete}\} \longrightarrow \mathrm{Cat}_\infty$$

$$R \longmapsto \begin{cases} \mathrm{PCoh}^{(\leq n)}(R[\tfrac{1}{\pi}]) \\ \mathrm{Perf}^{([a,b])}(R[\tfrac{1}{\pi}]) \\ VB(R[\tfrac{1}{\pi}]) \end{cases}$$

] Akhil Mathew

] Drinfeld

~~[illegible]~~ satisfies $\pi$-completely f. flat descent.

3') $\{K\text{-affd alg}\} \longrightarrow \mathrm{Cat}_\infty$

$A \longmapsto$ — " —

satisfies f. flat descent (maps of $A$ as ring maps)

Rmk. Akhil Mathew's proof :

- no good cat. of quasi-coherent modules $/(A,A^+)$
- use enlargement :

e.g. not abelian: $\bigoplus_{\mathbb{N}} \mathbb{Q}_p \to \widehat{\bigoplus_{\mathbb{N}}} \mathbb{Q}_p$

$$\begin{tikzcd} & M(A) \arrow[dr, "t\text{-exact, conservative}"] & \\ PCoh_A \arrow[ur, bend left, "f.\ faithful"] \arrow[rr, "f.\ faithful"'] & & \mathcal{D}(A) \end{tikzcd}$$

where $M(A)$ = stable $\infty$-cat. "$\pi$-isogeny cat." of the "cat. of bounded above $\pi$-complete complexes".

⚠ Not closed under colimits!

Sketch of proof (Thm A):

- Define $D((A,A^+)_\square)$ for $(A,A^+)$ complete Huber pairs. (III)
- Prove that $D((A,A^+)_\square)$ satisfies analytic descent for sheafy and locally Tate $(A,A^+)$. (IV)
- Cut out several categories (V)

$$\begin{array}{ccc} D((A,A^+)_\square)^\omega & & \\ \cap & & \\ D((A,A^+)_\square)^{pc} & \subseteq & D((A,A^+)_\square) \\ & & \subseteq \\ & \mathrm{Nuc}(A,A^+) & \end{array}$$

which all satisfy analytic descent.

- Show that (VI)

$$\mathrm{Perf}(A) \xrightarrow{\cong} D((A,A^+)_\square)^\omega \cap \mathrm{Nuc}(A,A^+)$$
$$PCoh_A \xrightarrow{\cong} D((A,A^+)_\square)^{pc} \cap \mathrm{Nuc}(A,A^+).$$

main point: the right hand side $\Rightarrow$ discreteness (relative to $\underline{A}$)

Question / Rmk :

Zongze Liu: adic sheafiness of $\mathrm{Spa}(A_{inf}(R^+), A_{inf}(R^+))$

But not Tate (locally)

Can Thm A extend to this case?

# III. Analytic rings

**Goal**: do homological alg. on ~~top~~-quasi-coh. modules $/ (A, A^+)$

that localizes & globalizes on $\mathrm{Spa}(A, A^+)$.

**Condensed math**: record top. information on alg. str.

by testing on profinite sets using continuous functions.

$\mathrm{Cond}(\mathrm{Set}) = \mathrm{Shv}(\mathrm{ProFin}, \mathrm{Set})$ "top. space"

$\mathrm{Cond}(\mathrm{Ab}) = \mathrm{Shv}(\text{—"—}, \mathrm{Ab})$ "top. ab. grp"

$A$ top ring $\rightsquigarrow$ $\underline{A} : \mathrm{ProFin} \to \mathrm{Ring}$ "condensed ring"

$S \mapsto C(S, A)$

$\rightsquigarrow \mathrm{Mod}_{\underline{A}} = \mathrm{Mod}_{\underline{A}}(\mathrm{Cond}(\mathrm{Ab})) \rightsquigarrow D(\underline{A}) = D(\mathrm{Mod}_{\underline{A}})$

$\downarrow$

It's a Grothendieck abelian category

generated by compact projective objects $\underline{A}[S]$, $S \in$ ~~ProFin~~ EDS $\subseteq$ ProFin.

("free $\underline{A}$-mod gen. over $S$")

EDS $\rightarrow$ any surjection from profinite set splits

$(R)\underline{\mathrm{Hom}}_{\underline{A}}(\underline{A}[S], \underline{A}) = \underline{\mathrm{Hom}}(S, \underline{A}) = \underline{C(S, A)}$, $\forall S \in \mathrm{ProFin}$

$\exists - \otimes_{\underline{A}} -$, $\underline{\mathrm{Hom}}_{\underline{A}}(-,-)$ on $\mathrm{Mod}_{\underline{A}}$, and derived counterparts on $D(\underline{A})$.

**Not satisfying** for "$\underline{\mathrm{QCoh}(A, A^+)}$".

Want to consider "$A$-mod with complete $A^+$-lattice".

just like "$\mathbb{Q}_p$-mod with $p$-complete $\mathbb{Z}_p$-lattice" $\overset{⚠}{=}$ Banach

⚠ Will need more than $\mathbb{Q}_p$-Banach spaces – they are not enough.

**Analytic rings**: $(A, A^+)$ complete Huber pair

$\rightsquigarrow D((A, A^+)_\blacksquare) \xhookrightarrow{\text{fully faithful}} D(\underline{A})$ stable under $\Big\langle$ limits, colimits, ext. groups (internally), retracts.

$\underline{\underline{\text{s.t.}}}$
- It has a left adjoint $(-)^{L\blacksquare/A^+} := - \otimes^L_{\underline{A}} (A, A^+)_\blacksquare : D(\underline{A}) \to D((A, A^+)_\blacksquare)$
  $\supseteq \heartsuit$ def. $\supseteq \heartsuit$
- on hearts: has left adjoint $(-)^{\blacksquare/A^+} := - \otimes_{\underline{A}} (A, A^+)_\blacksquare : \mathrm{Mod}_{\underline{A}} \to \mathrm{Mod}(A, A^+)_\blacksquare$
- $M \in D((A, A^+)_\blacksquare)$ iff $H^i(M) \in \mathrm{Mod}_{(A, A^+)_\blacksquare}$, $\forall i \in \mathbb{Z}$.
- $\underline{A} \in D((A, A^+)_\blacksquare)$

- $\underline{R\mathrm{Hom}}_{\underline{A}}\left(\underset{\underset{D(\underline{A})}{\cap}}{M}, \underset{\underset{D((A,A^+)_\blacksquare)}{\cap}}{N}\right) \in D((A,A^+)_\blacksquare)$

$(-)^{(L)\blacksquare/A^+}$

- $\exists!\ - \otimes^{(L)}_{(A,A^+)_\blacksquare} - \cong \left(- \otimes_{\underline{A}} -\right)^{(L)\blacksquare/A^+}$ making solidification sym. monoidal.

Rmk. • $(-)^{(L)\blacksquare/A^+}$ being a left adjoint, preserves colimits,

so enough to know $\left(\underline{A}[S]\right)^{L\blacksquare/A^+}$

- will see $\left(\underline{A}[S]\right)^{L\blacksquare/A^+}$ concentrated in degree $0$ by formula,

$\rightsquigarrow\ - \otimes^{L}_{(A,A^+)_\blacksquare} -$ is left derived from $- \otimes_{(A,A^+)_\blacksquare} -$

- Stability under colim is counterintuitive :

$$\underset{\underset{\text{ind-Banach}}{\uparrow}}{\bigoplus_{\mathbb{N}} \mathbb{Q}_p}\ ,\ \underset{\underset{\text{Banach}}{\uparrow}}{\widehat{\bigoplus_{\mathbb{N}}} \mathbb{Q}_p} \in \mathrm{Mod}(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare$$

should think of solidification as completing only the comp. proj. generators $\underline{A}[S]$.

Ex. $(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare[S] = \left(\prod_I \mathbb{Z}_p\right)[\tfrac{1}{p}]$. $\quad C(S,\mathbb{Z}) = \bigoplus_I \mathbb{Z}$ (Speeker, Nöbeling)

More examples.

$$\begin{tikzcd} (\mathbb{Z},\mathbb{Z})_\blacksquare = \mathbb{Z}_\blacksquare \arrow[r, rightsquigarrow] & \mathrm{Solid} = \mathrm{Mod}_{\mathbb{Z}_\blacksquare} \arrow[r, phantom, "\subseteq"] & D(\underline{\mathbb{Z}}) \arrow[l, bend right, "(-)^{L\blacksquare/\mathbb{Z}}"'] \end{tikzcd}$$

$\mathbb{Z}_\blacksquare[S] = \varprojlim_i \mathbb{Z}[S_i]$ , $S = \varprojlim S_i$

$\cong \prod_I \mathbb{Z}$ $\quad$ | compact proj. generator of $D(\mathbb{Z}_\blacksquare)$
| flat for $- \otimes_{\mathbb{Z}_\blacksquare} -$

$\mathbb{Z}, \mathbb{Z}[T], \mathbb{Z}[[T]], \mathbb{Z}((T)) \in \mathrm{Solid}$. $\rightsquigarrow \mathbb{Z}_p \in \mathrm{Solid}$

$\forall M$ discrete $\mathbb{Z}$-mod, $M \otimes^{(L)}_{\mathbb{Z}} \mathbb{Z}_\blacksquare[S] \in \mathrm{Solid}$.

$(\mathbb{R})^{L\blacksquare/\mathbb{Z}} = 0$

$\mathbb{Z}_p \otimes^{L}_{\mathbb{Z}_\blacksquare} \mathbb{Z}_\ell \cong \begin{cases} \mathbb{Z}_p , & \ell = p \\ 0 . & \ell \neq p \end{cases}$

- $(\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}$ "alg. affine line"

$\overset{\nearrow}{\text{discrete.}}$ $$D((\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}) = \mathrm{Mod}_{\mathbb{Z}[T]}(D(\mathbb{Z}_{\blacksquare})) \subseteq D(\underline{\mathbb{Z}[T]})$$ $$\xleftarrow{(-)^{L\blacksquare/\mathbb{Z}}}$$

$M^{L\blacksquare/\mathbb{Z}}$ <u>here</u> is the same as $M^{L\blacksquare/\mathbb{Z}}$ (considering $M \in D(\underline{\mathbb{Z}})$)
(considering $M \in D(\mathbb{Z}[T])$)

$\rightsquigarrow$ $(-)^{L\blacksquare/\mathbb{Z}}$ is well-defined, does not evoke confusions.

- $(\mathbb{Z}[T], \mathbb{Z}[T])_{\blacksquare}$ "solid affine line".
$\overset{\shortparallel}{\mathbb{Z}[T]_{\blacksquare}}$

$$D((\mathbb{Z}[T])_{\blacksquare}) \subseteq D(\mathbb{Z}[T])$$ $$\xleftarrow{(-)^{L\blacksquare/\mathbb{Z}[T]}}$$

$\forall M$ discrete $\mathbb{Z}[T]$-mod, $M \in D(\mathbb{Z}[T]_{\blacksquare})$

$\forall M = \varprojlim_i (\cdots \to M_2 \twoheadrightarrow M_1 \twoheadrightarrow M_0) \in \mathbb{Z}$-mod,
discrete.

$$\left\langle \begin{array}{l} M \otimes_{\mathbb{Z}} \mathbb{Z}[T] \in D((\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}) \\ (M \otimes_{\mathbb{Z}} \mathbb{Z}[T])^{L\blacksquare/\mathbb{Z}[T]} \simeq M\langle T\rangle = \varprojlim_i (M_i \otimes_{\mathbb{Z}} \mathbb{Z}[T]). \\ \in D(\mathbb{Z}[T]_{\blacksquare}) \end{array} \right.$$

- $(R, R^+)$ discrete Huber pair

$$(R, R^+)_{\blacksquare}[S] := \varinjlim_{\substack{B \subset R^+ \\ \text{f.g.}/\mathbb{Z}}} (\underline{R[S]})^{L\blacksquare/B} \simeq \varinjlim_{\substack{B \subset R^+ \\ \text{f.g.}/\mathbb{Z}}} R \otimes_B B_{\blacksquare}[S].$$

- $(A, A^+)$ complete Huber pair, $A^{\delta} =$ ($A$ with discrete topology).

$$(A, A^+)_{\blacksquare}[S] := (\underline{A}[S])^{L\blacksquare/(A^+)^{\delta}}$$

$$= \varinjlim_{\substack{B \subset A^+ \\ \text{f.g.}/\mathbb{Z}}} (\underline{A}[S])^{L\blacksquare/B^{\delta}}$$

<u>Idea</u>: check solidness wrt. individual elements $f \in A^+$
($\Leftarrow$ wrt. f.g. $\mathbb{Z}$-subalg. of $A^+$)

$$= \varinjlim_{\substack{B \subset A^+ \\ \text{f.g.}/\mathbb{Z}}} \varinjlim_{M = \varprojlim M_i} \underline{M} \otimes_B B_{\blacksquare}[S] \simeq \prod_I \underline{M}$$

$(M_i)_i$: some surjective family of f.g. $B$-mod discrete.

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~~Prof~~

<u>Prop</u> $M \in D((A,A^+)_\blacksquare) \iff M \in D(\mathbb{Z}[T]_\blacksquare)$
$\forall\ \mathbb{Z}[T] \to A^+$
$T \mapsto f.$

<u>Maps of analytic rings</u>: $(A,A^+)_\blacksquare \to (B,B^+)_\blacksquare$

$\iff$ map $\underline{A} \to \underline{B}$ st. $\underline{B} \in \mathrm{Mod}_{(A,A^+)_\blacksquare}$

(true by <u>prop.</u> above)

$\rightsquigarrow$ base change: $- \otimes_{(A,A^+)_\blacksquare} (B,B^+)_\blacksquare$

<u>Pushout</u> of <u>analytic rings</u>: makes sense in the larger class of the <u>analytic</u> <u>animated</u> <u>rings</u>.

<u>Ex</u>. $(A,A^+)$ complete Huber pair, $f,g$ generating open ideal of $A$.

$$\underbrace{(A,A^+)_\blacksquare \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare}}_{\substack{g \mapsfrom T \\ \text{invert } g}} \underbrace{(\mathbb{Z}[T^{\pm1}],\mathbb{Z})_\blacksquare \otimes^L_{(\mathbb{Z}[u],\mathbb{Z})_\blacksquare} \mathbb{Z}[u]_\blacksquare}_{\substack{fT^{-1} \mapsfrom u \\ \text{require } |\frac{f}{g}| \leq 1.}} \cong \underline{A\langle T\rangle}/^{\mathbb{L}}_{gT-f} \xrightarrow{\sim} \underline{A\langle \tfrac{f}{g}\rangle}$$

if. $A\langle T\rangle \xrightarrow{gT-f} A\langle T\rangle$ is closed embedding.

(<u>e.g.</u> $A$ <u>discrete</u>
or $A$ <u>sheafy + analytic</u>)

⚠ If. $(A,A^+)_\blacksquare \to (B,B^+)_\blacksquare$ pushout,

$$\begin{tikzcd} (A,A^+)_\blacksquare \arrow[r] \arrow[d] & (B,B^+)_\blacksquare \arrow[d] \\ (C,C^+)_\blacksquare \arrow[r] & (D,\mathcal{M}_D) \end{tikzcd}$$

then $D(\underline{D}) \xrightarrow{\text{solidify}} D(D,\mathcal{M}_D)$ is $\varinjlim\Big((-)^{L\blacksquare/B^+} \to ((-)^{L\blacksquare/B^+})^{L\blacksquare/C^+} \to (((-)^{L\blacksquare/B^+})^{L\blacksquare/C^+})^{L\blacksquare/B^+} \to \cdots\Big)$

since $(-)^{L\blacksquare/B^+}$ & $(-)^{L\blacksquare/C^+}$ do not commute in general!

When the morphisms $(A,A^+)_\blacksquare \to (B,B^+)_\blacksquare$ & $(C,C^+)_\blacksquare$ are <u>steady</u>, the $(D,\mathcal{M}_D)$-solidification $\cong ((-)^{L\blacksquare/B^+})^{L\blacksquare/C^+} \cong ((-)^{L\blacksquare/C^+})^{L\blacksquare/B^+}$.

<u>Steady</u> maps are stable under base change, composition, colimits.
($\longleftrightarrow$ <u>adic</u> morphisms of Huber pairs)

Ex (Steady maps)

$$\begin{tikzcd}
 & & (\mathbb{Z}[T^{\pm 1}], \mathbb{Z})_■ \arrow[dr] & \\
\mathbb{Z}[T]_■ = (\mathbb{Z}[T], \mathbb{Z}[T])_■ & (\mathbb{Z}[T], \mathbb{Z})_■ \arrow[l, "j"'] \arrow[ur] \arrow[rr, "j'"] & & (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_■ \\
 & \mathbb{Z}_■ = (\mathbb{Z}, \mathbb{Z})_■ \arrow[ul, "f"] \arrow[u, "\bar{f}"'] \arrow[urr, "f'"'] & &
\end{tikzcd}$$

Base change along $\mathbb{Z}[T] \to \overset{\overset{A}{\cup}}{A^+}$ gives:
$T \longmapsto f$

$$\begin{tikzcd}
 & & \text{"}(A[\tfrac{1}{f}], A^+)_■\text{"} \arrow[dr] & \\
(A\langle f\rangle, A^+\langle f\rangle)_■ & (A, A^+)_■ \arrow[l] \arrow[ur] \arrow[rr] & & (A\langle \tfrac{1}{f}\rangle, A^+\langle \tfrac{1}{f}\rangle)_■
\end{tikzcd}$$

More generally: $(A, A^+)_■ \longrightarrow$ ~~$A$~~ $(A_u, A_u^+)_■$ steady

sheafy & locally Tate $\qquad \forall u$ rational open $\subseteq \mathrm{Spa}(A, A^+)$.

$\Longrightarrow \forall u, v$ rat'l open $\subseteq \mathrm{Spa}(A, A^+)$.

$$\begin{tikzcd}
D((A, A^+)_■) \arrow[r, "L_u"] \arrow[d, "L_v"] & D((A_u, A_u^+)_■) \arrow[d, "L_v"] \\
D((A_v, A_v^+)_■) \arrow[r, "L_u"] & D((A_{u\cap v}, A_{u\cap v}^+)_■)
\end{tikzcd}$$

commutes.

well-defined

## IV. Descent of $D((A, A^+)_■)$

**Thm B.** The presheaf $u \overset{\text{rat'l open}}{\longmapsto} D((A_u, A_u^+)_■)$ satisfies analytic descent on $\mathrm{Spa}(A, A^+)$ for $(A, A^+)$ $\left\{ \begin{array}{c} \text{sheafy} \\ + \\ \text{locally Tate} \end{array} \right.$.

First recall the proof of Zariski descent of $u (= D(f)) \mapsto D(u) := D(A_u)$ for schemes.

a) Reduce to affine covering $\overset{= D(f)}{u} \sqcup \overset{= D(1-f)}{V} \to X = \mathrm{Spec}(A)$

b) Since $L_u$ & $L_v$ commutes & $L_{u \cap v} \cong L_u \cdot L_v \cong L_v \cdot L_u$,

it's enough to prove

$$\begin{tikzcd} D(X) \arrow[r, "L_u"] \arrow[d, "L_v"'] & D(u) \arrow[d, "L_v"] \\ D(V) \arrow[r, "L_u"] & D(u \cap V) \end{tikzcd}$$

Cartesian

or equiv. $D(X) \xrightarrow{F} D(u) \times_{D(u \cap v)} D(v)$ is equiv.

c) Since $\begin{tikzcd} D(u) \arrow[r, hook, "\text{f.faithful}"'] \arrow[r, phantom, "\perp", shift left=2] & D(X) \arrow[l, bend right, "L_u"'] \end{tikzcd}$, $\begin{tikzcd} D(V) \arrow[r, hook, "\text{f.faithful}"'] \arrow[r, phantom, "\perp", shift left=2] & D(X) \arrow[l, bend right, "L_V"'] \end{tikzcd}$

get $F \dashv G$. $F : M \longmapsto (L_u M, L_V M, L_V(L_u M) \overset{\alpha}{\cong} L_u(L_V M))$

$G : \underset{L_V M_u \overset{\alpha}{\cong} L_u M_V}{M_u \times M_V} \longleftarrow (M_u, M_V, L_V M_u \overset{\alpha}{\cong} L_u M_V)$

Enough to show $id \xrightarrow{\text{unit}} GF$, $FG \xrightarrow{\text{counit}} id$ equivalences.

d) $D(X) \longrightarrow D(u) \times D(V)$ conservative

$\rightsquigarrow$ check equivalences $id \xrightarrow{\text{unit}} GF$ on $D(u)$ & $D(V)$.

$M \longmapsto L_u M \times_{L_V L_u M \cong L_u L_V M} L_V M$ $\rightsquigarrow L_u M \xrightarrow{\cong} L_u M$ on $u$

$\rightsquigarrow L_V M \xrightarrow{\cong} L_V M$ on $V$

e) Similarly for $FG \xrightarrow{\text{counit}} id$.

pf (Thm B). Produce analogue of a) — d).

a) Reduce to check coverings $u \sqcup V \longrightarrow X$

(SLC) $\text{Spa}(A\langle \frac{1}{f} \rangle, A^+\langle \frac{1}{f} \rangle) \sqcup \text{Spa}(A\langle \frac{f}{1} \rangle, A^+\langle \frac{f}{1} \rangle) \to \text{Spa}(A, A^+)$

(SBC) $\text{Spa}(A\langle \frac{1}{f} \rangle, A^+\langle \frac{1}{f} \rangle) \sqcup \text{Spa}(A\langle \frac{1}{1-f} \rangle, A^+\langle \frac{1}{1-f} \rangle) \to \text{Spa}(A, A^+)$

b) Rational localizations are steady since:

multiplication $A\langle T \rangle \xrightarrow{x} A\langle T \rangle$ by $x = T - f$, $fT - 1$, $(1-f)T - 1$

~~[illegible]~~ are closed embedding if $A$ is (locally) Tate & sheafy.

c). $L_u \dashv$ fully faithful embedding, $L_V \dashv$ — " —, by construction of analytic rings.
$(j^*, j'^*)$

d) Conservativity: reduce to that of ① $D((\mathbb{Z}[T], \mathbb{Z}))_{\blacksquare} \to D(\mathbb{Z}[T], \mathbb{Z}[T])_{\blacksquare}) \times D((\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_{\blacksquare})$

& ② $D((\mathbb{Z}[T], \mathbb{Z}))_{\blacksquare} \to D((\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_{\blacksquare}) \times D((\mathbb{Z}[T, (1-T)^{-1}], \mathbb{Z}[(1-T)^{-1}])_{\blacksquare})$

① $\mathrm{Mod}_{\mathbb{Z}((T^{-1}))}(-\text{''}-) \hookrightarrow D((\mathbb{Z}[T],\mathbb{Z})_\square) \xrightarrow{j^*} D((\mathbb{Z}[T],\mathbb{Z}[T])_\square)$ exact seq.

$\mathrm{Mod}_{\mathbb{Z}[[T]]}(-\text{''}-) \hookrightarrow D((\mathbb{Z}[T],\mathbb{Z})_\square) \xrightarrow{j'^*} D((\mathbb{Z}[T^{\pm 1}],\mathbb{Z}[T^{-1}])_\square)$ exact seq.

$$\Rightarrow \ker(j^*, j'^*) = \mathrm{Mod}_{\underbrace{\mathbb{Z}((T^{-1})) \otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square} \mathbb{Z}[[T]]}_{\overset{\shortparallel}{0}}}(-\text{''}-) \qquad 1 = (T^{-1})^n \cdot (T)^n \xrightarrow{n\to+\infty} 0$$

② $$\ker = \mathrm{Mod}_{\underbrace{\mathbb{Z}[[T]] \otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square} \mathbb{Z}[[1-T]]}_{\overset{\shortparallel}{0}}}(-\text{''}-) \qquad 1 = (T+(1-T))^n \longrightarrow 0.$$

$\square$

~~IV~~

## V. Cutting out specific subcategories that descend.

**Def.** $(\mathcal{C}, \otimes, 1)$ sym. monoidal

$M \in \mathcal{C}$ dualisable if $\exists$ dual $M'$, $ev_M : M' \otimes M \to 1$

$coev_M : 1 \longrightarrow M \otimes M'$

s.t.
$$\begin{tikzcd}
M \arrow[r, "coev_M \otimes id"] & (M\otimes M')\otimes M \arrow[r, phantom, "\cong"] \arrow[r, bend right] & M \otimes (M'\otimes M) \arrow[r, "id \otimes ev_M"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
M' \arrow[r] & M' \otimes (M\otimes M') \arrow[r, phantom, "\overset{id}{\cong}"] \arrow[r, bend right, "id"'] & (M'\otimes M)\otimes M' \arrow[r] & M'
\end{tikzcd}$$

**Lem.** $(\mathcal{C}, \otimes, 1)$ closed $(\Rightarrow \exists\, R\underline{\mathrm{Hom}}(-,-))$

$M \in \mathcal{C}$ dualisable $\Longrightarrow$ $M' \otimes N \cong R\underline{\mathrm{Hom}}(M, N)$

Not clear whether $D((A,A^+)_\square)^{dual}$ satisfies descent.

Motivation: $D(A)^{dual} = \mathrm{Perf}(A)$

$$\begin{matrix} D(A) \\ \shortparallel \\ D((A,A^+)_\square)^{dual + discrete} \end{matrix}$$

Not true that $D(A)$ satisfies descent.

$\nwarrow$ classical derived cat

Resort to other categories that descend :

**Def / Prop.** $M \in \mathcal{C} \; \left( = D((A,A^+)_\square) \right)$ is compact

$\overset{\text{def}}{\iff}$ $R\mathrm{Hom}(M,-) : \mathcal{C} \to D(\mathbb{Z})$ preserves $\bigoplus_I$

$\iff$ $R\mathrm{Hom}(M,-) : \mathcal{C} \to D(\mathbb{Z})$ preserves filtered colim.

$\iff$ $M$ retract of a finite complex with terms in a chosen family of compact $\overset{\wedge}{}$ projective generators (e.g. $(A,A^+)_\square[S]$, $S \in \mathrm{ProFin}$)

$\rightsquigarrow D((A,A^+)_\square)^\omega$

It satisfies analytic descent (formal proof).

**Def / Prop.** $M \in \mathcal{C} \; \left( = D((A,A^+)_\square) \right)$ is pseudocompact

$\overset{\text{def}}{\iff}$ $\left[ R\mathrm{Hom}(M,-) : \mathcal{C}^{\geq j} \to D(\mathbb{Z}) \text{ preserves } \bigoplus_I , \ \forall j . \right]$

$\cancel{\Updownarrow}$ for fixed $j \in \mathbb{Z}$.

$\iff$ [ ——— " ——— filtered colim, $\forall j$ ]

$\iff$ $M \simeq$ bdd above complex with terms in ..

——— " ——— (e.g. $(A,A^+)_\square[S]$, $S \in \mathrm{ProFin}$).

$\rightsquigarrow$ ~~Perf~~ $D((A,A^+)_\square)^{pc}$

It also satisfies analytic descent (same formal proof).

nuclearity : some "orthogonal concept to (pseudo) compactness"

e.g. $\mathbb{Q}_p$-Banach spaces e.g. $(\prod \mathbb{Z}_p)[\tfrac{1}{p}]$

**Def.** Denote $(-)^\vee = R\underline{\mathrm{Hom}}_{\mathcal{C}}(-,1)$, $(\mathcal{C}, \otimes, 1)$ stable closed.

$(-)(*) = R\mathrm{Hom}_{\mathcal{C}}(1,-)$ $\rightsquigarrow R\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$

1) Say $f : M \to N$ is trace-class if it comes from $\phi : 1 \to M^\vee \otimes N$,

i.e. $$\begin{tikzcd} M \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[rr, bend right, "f"'] & M \otimes M^\vee \otimes N \arrow[r, "ev_M \otimes \mathrm{id}"] & N \end{tikzcd}$$

i.e. $\pi_0(M^\vee \otimes N)(*) \to \pi_0 R\mathrm{Hom}_{\mathcal{C}}(M,N)$

$\exists \phi \longmapsto f$

2) Say $N \in \mathcal{C}$ is <u>nuclear</u> if for $P \in$ a family of comp. proj. gen.

~~BHom~~ $(P^\vee \otimes N)(*) \xrightarrow{\cong} R\underline{Hom}(P, N)$ isom. of spectra.

$\rightsquigarrow$ $\mathrm{Nuc}(\mathcal{C})$

<u>Ex.</u> $\mathrm{Nuc}(\mathbb{Q}_p, \mathbb{Z}_p) =$ generated under colimits in $D((\mathbb{Q}_p, \mathbb{Z}_p)_\square)$ by $\mathbb{Q}_p$-Banach spaces.

$\Rightarrow$ $\mathbb{Q}_p$-Banach spaces, $\prod_{\mathbb{N}} \mathbb{Q}_p$, $\mathbb{Q}_p$-Fréchet spaces $\in \mathrm{Nuc}(\mathbb{Q}_p, \mathbb{Z}_p)$

<u>In fact</u>, $\mathrm{Nuc}(A, A^+)$ is closed under colimits.

& closed under <u>countable</u> products/limits & $- \otimes^L_{(A,A^+)_\square} -$

if $A$ nuclear solid alg./non-arch. local field $K$.

Formal Lemma:

<u>Lem.</u> 1) dualisable $M$ + $1 \in \mathcal{C}$ compact $\Rightarrow$ $M$ compact

2) dualisable $M$ $\Rightarrow$ $M$ nuclear

3) nuclear + compact $\Rightarrow$ dualisable

<u>Pf.</u> 1) $R\mathrm{Hom}(M, \bigoplus_I -) \cong R\mathrm{Hom}(1, R\underline{\mathrm{Hom}}(M, \bigoplus_I -)) \cong R\mathrm{Hom}(1, M' \otimes (\bigoplus_I -))$

$\wr$ 1 compact

$\bigoplus_I R\mathrm{Hom}(M, -) \cong \bigoplus_I R\mathrm{Hom}(1, R\underline{\mathrm{Hom}}(M, -)) \cong \bigoplus_I R\mathrm{Hom}(1, M' \otimes -)$

2) $\underline{R\mathrm{Hom}}(M, M) \cong M' \otimes M \cong M^\vee \otimes M$

$\Rightarrow R\mathrm{Hom}(M, M) \cong (M^\vee \otimes M)(*) \Rightarrow \mathrm{id}_M$ is trace-class

$\Rightarrow M = \mathrm{colim}(M \xrightarrow{\mathrm{id}_M} M \xrightarrow{\mathrm{id}_M} \cdots)$

$$\begin{tikzcd}
\Rightarrow P^\vee \otimes M = \mathrm{colim}( P^\vee \otimes M \arrow[r] \arrow[d] & P^\vee \otimes M \arrow[r] \arrow[d] & {} ) \\
\cong \mathrm{colim}( \underline{R\mathrm{Hom}}(P, M) \arrow[r] \arrow[ur, dashed, "\exists"] & \underline{R\mathrm{Hom}}(P, M) \arrow[r] \arrow[ur, dashed, "\exists"] & \cdots )
\end{tikzcd}$$

$\cong \underline{R\mathrm{Hom}}(P, M)$

3) $(M^\vee \otimes M)(*) \cong R\mathrm{Hom}(M, M)$ + Canonical $ev_M : M^\vee \otimes M \to 1$

compact $\quad$ nuclear

$\exists\, coev_M \longmapsto \mathrm{id}$ $\quad\rightsquigarrow$ $M$ dualisable with dual $M^\vee$. $\square$

**Cor C.** $D((A,A^+)_\blacksquare)^{\text{dual}} = D((A,A^+)_\blacksquare)^{\omega} \cap \mathrm{Nuc}(A,A^+)$

satisfies analytic descent.

**Rmk.** $(A,A^+)$ complete Huber ring.

Then $\mathrm{Nuc}(A,A^+) = \mathrm{Nuc}(A,\mathbb{Z}) =: \mathrm{Nuc}(A)$

depends only on $A$. (cf. Andreychev's PhD thesis)

But we will denote it by $\mathrm{Nuc}(A,A^+)$.

## VI. Get discreteness

**Condensification:** $(A,A^+)$

$$\mathrm{Cond}_A : D(A) \overset{\text{f. faithful, exact, preserves filt. colim.}}{\hookrightarrow} D(\underline{A}^\delta) \longrightarrow D(\underline{A})$$

$$M \longmapsto \underline{M}^\delta \longmapsto \underline{M}^\delta \otimes^L_{\underline{A}^\delta} \underline{A}$$

**Def.** $M \in D(\underline{A})$ is discrete (relatively to $\underline{A}$) if $M \in \mathrm{Ess.Im.}(\mathrm{Cond}_A)$.

**Lem.** 1) $\mathrm{Cond}_A$ is fully faithful, exact, preserves filt. colim.

& $H^i(\mathrm{Cond}_A M) = 0 \Rightarrow H^i(M) = 0$ (tor-amplitude control)

2) $\mathrm{Cond}_A$ lands in $D((A,A^+)_\blacksquare) \subseteq D(\underline{A})$

$\subseteq$

even in $\mathrm{Nuc}(A,A^+)$. (since $\underline{A} \in \mathrm{Nuc}(A,A^+)$ )

stable under colim.

3) $(A,A^+)$ ~~steady~~ locally Tate & $M \in D(\underline{A})$

$\overset{\text{qiso}}{\simeq}$ complex of finite free $A$-mod.

then $H^i(\mathrm{Cond}_A M) = 0 \Leftarrow H^i(M) = 0$.

$\uparrow$

Open mapping theorem for complete + first countable top. $A$-mod (under the above assumption)

**Rem.**
- $\mathrm{Cond}_A$ compatible with base change of $(A,A^+)$
- discreteness is closed under retracts
- $M \in D((A,A^+)_\blacksquare)$, $M$ retract of $\underline{A}^n \Leftrightarrow M \overset{= \mathrm{Cond}_A(M_0)}{\phantom{x}}$ discrete & $M_0 \in \mathrm{FinProj}_A$.

~~Cor~~/**Fact.** $M \in D(A)$.

Then • $\mathrm{Cond}_A(M)$ is dualisable

$\Leftrightarrow$ $M$ dualisable $\Leftrightarrow$ $M$ compact $\Leftrightarrow$ $M \in \mathrm{perf}(A)$

• $\mathrm{Cond}_A(M)$ is pseudocompact

$\Leftrightarrow$ $M$ pseudocompact $\Leftrightarrow$ $M \in \mathrm{PCoh}_A$.

Functional analysis : $\underset{\text{"complete"}}{\overset{}{\text{Banach}}}$ + nuclear $\Rightarrow$ fin. dim.

Here :

$$\begin{matrix} \text{compact} & + \text{nuclear} & \Rightarrow & \text{fin. dim} \\ \cap\!\shortmid & & & \cap\!\shortmid \\ \text{pseudocompact} & + \text{nuclear} & \Rightarrow & \text{discrete} \end{matrix}$$

**Thm D.** Pseudo~~coh~~compact + nuclear in $D(A,A^+)_\square) \Rightarrow$ discrete

( do <u>not</u> need locally Tate <u>nor</u> sheafy condition )

**Pf.** <u>Step 1.</u> $M$ pseudocoherent + nuclear $\Rightarrow$ $M = \varinjlim_n M_n$

with $M_n$ dualisable , successive ext. of $\mathrm{cone}(P \xrightarrow{1-f} P)$

$f : \underset{(A,A^+)_\square[S]}{\overset{\shortparallel}{P}} \to P$ trace-class

& st. $M_n \to M \to \underset{\widehat{D}^{\le -n}((A,A^+)_\square)}{\overset{\in}{\mathrm{cone}}}$

<u>Pf.</u>

$$\begin{tikzcd}
M = ( \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 ) \\
 & & & P_0 \arrow[u, "id"] \arrow[lll, bend left, "g"]
\end{tikzcd}$$

$P_i = (A,A^+)_\square[S_i]$

$S_i \in \mathrm{ProFin}$

$$(R)\mathrm{Hom}(P_0[0], M) \underset{\text{M nuclear}}{\cong} \Big( (R)\underline{\mathrm{Hom}}(P_0[0], \underline{A}) \otimes^{(L)} M \Big)(*)$$

$$(R)\underline{\mathrm{Hom}}(P_0[0], \underline{A}) \cong \underline{C(S_0, A)}$$

Concentrated in degree 0 $\Rightarrow$ surjective in $\pi_0$

$$\Big( (R)\underline{\mathrm{Hom}}(P_0[0], A) \otimes^{L} P_0 \Big)(*) \twoheadrightarrow \Big( (R)\underline{\mathrm{Hom}}(P_0[0], \underline{A}) \otimes^{(L)} M \Big)(*)$$

$g \in (R)\mathrm{Hom}(P_0[0], M)$, $\exists \phi \in \Big( (R)\underline{\mathrm{Hom}}(P_0[0], A) \otimes^{L} P_0 \Big)(*)$, $g \longleftarrow\!\shortmid \phi$

Hence: $(R)\mathrm{Hom}(P_0, P_0) \leftarrow \Big( (R)\underline{\mathrm{Hom}}(P_0[0], A) \otimes^{L} P_0 \Big)(*)$

$f . \longleftarrow\!\shortmid \phi$, $\quad f \mapsto g$

$$M = ( \cdots \to P_1 \to P_0 )$$

$\Rightarrow$

$$\begin{tikzcd}
M & & P_0 \\
& P_0 \arrow[ul, bend left, "g"] \arrow[ur, dashed, "\mathrm{id}"] & \\
& & P_0 \arrow[ul, "f"] \arrow[uu, "\mathrm{id}"'] \arrow[uull, bend left, "g"]
\end{tikzcd}$$

$$\Rightarrow \quad \underbrace{\mathrm{Cone}( P_0 \xrightarrow{1-f} P_0 )}_{\overset{\shortparallel}{M_0}} \xrightarrow{g} M \to \underset{\overset{\rotatebox{90}{$\in$}}{D^{\leq -1}((A,A^+)_\square)}}{\mathrm{cone}}$$

Induction $\rightsquigarrow$ $M_1, M_2,$ etc.

Step 2. ‖ $\forall f : (A,A^+)_\square[S] \to (A,A^+)_\square[S]$ trace-class

‖ $\mathrm{cone}(1-f) \in \mathrm{Perf}(A)$.

Proof idea is similar to show:

$T : V \to V$ trace-class (or more generally compact operator) of $\mathbb{C}$-Banach spaces

$\Rightarrow$ $\mathrm{id}-T$ is Fredholm, i.e. $\ker(\mathrm{id}-T)$, $\mathrm{coker}(\mathrm{id}-T)$ are fin. dim.

Pf. For simplicity, $S = \varprojlim_{n\in\mathbb{N}} S_n$, $(A,A^+) = (\mathbb{Q}_p, \mathbb{Z}_p)$

$$f : \Big(\prod_I \mathbb{Z}_p\Big)[\tfrac{1}{p}] \longrightarrow \Big(\prod_I \mathbb{Z}_p\Big)[\tfrac{1}{p}]$$

$$m \longmapsto \sum f_i(m) \otimes y_i \quad \text{with} \begin{cases} y_i \in p^{-N}\mathbb{Z}_p \\ f_i \in C(S,\mathbb{Q}_p) \\ f_i \to 0. \end{cases}$$

Write the matrix repr. $f$:

$$f = \begin{array}{c} \\ \\ \end{array}\!\!\begin{array}{cc|c} J & I\setminus J & \\ \hline F_{11} & F_{12} & J \\ F_{21} & F_{22} & I\setminus J \end{array}$$

Choose $J$ finite, "big" st. $\|F_{22}\| < 1$ $\Rightarrow$ $1 - F_{22}$ invertible

$$\Rightarrow 1 - f = \begin{pmatrix} 1-F_{11} & -F_{12} \\ -F_{21} & \underbrace{1-F_{22}}_{\text{invertible}} \end{pmatrix}$$

$$\Rightarrow \mathrm{cone}(1-f) \simeq \mathrm{cone}(1-F_{11}) = \Big( \mathbb{Q}_p^J \xrightarrow{1-F_{11}} \mathbb{Q}_p^J \Big)$$

$$\Big(\prod_J \mathbb{Z}_p\Big)[\tfrac{1}{p}] \to \Big(\prod_J \mathbb{Z}_p\Big)[\tfrac{1}{p}] \quad \in \mathrm{Perf}(\mathbb{Q}_p)$$

$$\overset{\shortparallel}{\mathbb{Q}_p^J} \to \overset{\shortparallel}{\mathbb{Q}_p^J}$$

$\square$

# Conclusion:

- $\mathrm{Perf}(A)$ = discrete + dualizable/compact

  = discrete + compact + nuclear

  = compact + nuclear

  $= \mathcal{D}((A,A^+)_a)^{\omega} \cap \mathrm{Nuc}(A,A^+)$

- $\mathrm{PCoh}_A$ = discrete + pseudocompact

  = discrete + pseudocompact + nuclear

  = ~~discrete~~ pseudocompact + nuclear

  $= \mathcal{D}((A,A^+)_a)^{pc} \cap \mathrm{Nuc}(A,A^+)$

⇒ They all satisfy analytic descent!

---

Thm. (Andreychev, Ph.D. thesis)

- $$\begin{tikzcd} \mathrm{Nuc}(B) \\ A \arrow[u, "\text{fét. Tate Huber rings}"] \end{tikzcd} \simeq \mathrm{Mod}_B(\mathrm{Nuc}(A)) \simeq \mathrm{Nuc}(A) \otimes_{\mathrm{Perf}(A)} \mathrm{Perf}(B)$$

  $$\simeq \mathrm{Nuc}(A) \otimes_{\mathcal{D}(A)} \mathcal{D}(B)$$

- ~~Nuc~~ $\mathrm{Nuc}(A)$ dualizable in $\mathrm{Pr}^L_{st}$ w/ Lurie's $\otimes$.

- $\mathrm{Nuc}(A)$ satisfies fét-descent for Tate Huber rings $A$

⇒ $\mathrm{Nuc}(A)$ satisfies étale descent for sheafy+loc Tate $(A,A^+)$.