CLASSICAL DIEUDONNÉ THEORY

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Abstract. These are the notes for my two talks during the group de travail “p-divisible groups and Rapoport–Zink spaces”, on 11 November and 17 December 2025 respectively.

Contents

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Part 1. Classical Dieudonné Theory

Convention. All the group schemes considered in this talk will be *commutative*.

Notation. For any ring R, we denote by $\mathfrak{n}_R$ its nilradical, which consists of the nilpotent elements of R. For any perfect field k, we denote by $W(k)$ the associated ring of Witt vectors, and by $K \coloneqq W(k)[\frac{1}{p}]$ its fraction field. Whenever there is an extension k' of k, we use K' to denote the associated fraction field $W(k')[\frac{1}{p}]$.

1. Recollections on (formal) group schemes

1.1. **Cartier duality, Frobenius and Verschiebung.** Fixing a base *field* k (to avoid mentioning flatness), we will consider the following categories:

- Aff_k, the category of affine schemes over k.
- AffGrp_k, the category of affine commutative group schemes (abbr. affine groups) over k, which are defined as the group objects in Aff_k.
- FSch_k, the category of formal schemes over k.
- FGrp_k, the category of (affine) formal groups (abbr. (affine) formal groups) over k, which are defined as the group objects in FSch_k.

The categories AffGrp_k and FGrp_k both contain FinGrp_k, the category of finite group schemes over k, as a full subcategory. Any $G \in \mathrm{FinGrp}_k$ is a finite scheme, we call $\dim_k \mathcal{O}_G(G)$ the *rank* or the *order* of G, and denote it by $\mathrm{ord}(G)$.

The categories AffGrp_k, FGrp_k and FinGrp_k are all abelian categories.

For any $X \in \mathrm{Aff}_k$, we denote by $\widehat{X} \in \mathrm{FSch}_k$ its *formal completion*, i.e. the induced functor on Fin_k. Concretely, if $X = \mathrm{Spec}(A)$, then $\widehat{X} = \mathrm{Spf}(A^{\wedge,\mathrm{profin}})$ where $A^{\wedge,\mathrm{profin}} = \varprojlim_i A_i$ denotes the profinite completion of A[1], the A_i running through all possible finite quotient rings of A, which forms a projective system. *Caution*: the formal completion $\widehat{\mathbf{G}_a}$ is represented not by $k[\![T]\!]$, but by $k[T]^{\wedge,\mathrm{profin}}$, of which $k[\![T]\!]$ is the direct factor where T is topologically nilpotent.

For any affine (formal) group G over k, with ring of (continuous) functions B_G, one defines its *augmentation ideal* as

$$I_G \coloneqq \ker(e\colon B_G \to k),$$

where e is the ring homomorphism corresponding to the unit of the group $G(k)$. This unit induces a canonical splitting

$$B_G = k \oplus I_G$$

[1]When X is an affine group scheme, the finite subschemes $\mathrm{Spec}(A_i) \subset X$ are not necessarily group schemes; indeed, A_i are rings but A_i^* are not rings, the "ring structure" on A^* being rather $A_i^* \times A_j^* \to A_k^*$. So, each $\mathrm{Spec}(A_i)$ only carries a Frobenius, but does not Verschiebung.

as B_G-modules.

1.1.1. **Cartier duality.** The categories AffGrp_k and FGrp_k are related by the *Cartier duality*

$$(-)^D\colon \mathrm{AffGrp}_k \overset{\simeq}{\rightrightarrows} \mathrm{FGrp}_k, \quad (-)^D\colon \mathrm{FGrp}_k \overset{\simeq}{\rightrightarrows} \mathrm{AffGrp}_k,$$

mutually quasi-inverse to each other, which restrict to an auto-anti-equivalence of FinGrp_k [2]

More concretely:

- On the one direction, let $G = \mathrm{Spec}(A) \in \mathrm{AffGrp}_k$. The comultiplication map

$$\Delta_A\colon A \to A \otimes_k A$$

has the finiteness property that, there exists a filtered system of finite dimensional sub-k-vector spaces $(A_i)_{i\in I}$ of A such that $\bigcup_i A_i = \varinjlim_i A_i = A$ and

$$\Delta_A(A_i) \subset A_i \otimes A_i$$

(i.e. each A_i is a sub-k-coalgebra of A). But A_i are not k-algebras in general. Then, taking the topological dual

$$(-)^* \coloneqq \mathrm{Hom}_k(-, k)$$

for discrete k-vector spaces, endowing the dual with the natural product topology, one gets a cofiltered system of finite k-algebras $(A_i^*)_{i\in I}$, whose inverse limit is the topological dual k-algebra A^*. Thus, one obtains a formal scheme

$$G^D \coloneqq \mathrm{Spf}(A^*) = \varinjlim_i \mathrm{Spec}(A_i^*).$$

But *the finite schemes*

$$G_i^D \coloneqq \mathrm{Spec}(A_i^*)$$

do not necessarily have group law, because A_i^* does not have natural comultiplication structure in general, since A_i are not necessarily k-algebras. However, we have not used the original k-algebra structure on A. This structure induces a continuous comultiplication

$$\Delta_{A^*}\colon A^* \to A^* \hat{\otimes}_k A^*$$

by taking the topological dual. In other words, the continuity means that, given any $i, j \in I$, for $k \in I$ sufficiently large, there are canonically well-defined maps

$$G_i^D \times G_j^D \to G_k^D.$$

This endows the formal scheme

$$G^D = \varinjlim_i G_i^D$$

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[2] In order to emphasis the formal nature of formal groups, the iterature often denotes the duality functors by

$$\widehat{\mathbf{D}}(-)\colon \mathrm{AffGrp}_k \overset{\simeq}{\rightrightarrows} \mathrm{FGrp}_k, \quad \mathbf{D}(-)\colon \mathrm{FGrp}_k \overset{\simeq}{\rightrightarrows} \mathrm{AffGrp}_k.$$

But we will abuse the notation by simply denoting the duality functor in both directions by $(-)^D$, because the context will make it clear which functor is involved.

with a structure of formal groups over k.

- On the other direction, let $G = \mathrm{Spf}(B) \in \mathrm{FGrp}_k$. Then B is the topological inverse limit of (discrete) finite k-algebras:
$$B = \varprojlim_i B_i.$$
The continuous comultiplication map
$$\Delta_B \colon B \to B \hat{\otimes}_k B$$
translates to a "group law"
$$G_i \times G_j \to G_k$$
for $k \gg i, j$. However, $G_i^D \coloneqq \mathrm{Spec}(B_i^*)$ do not quite makes sense as schemes, let alone its inverse limit $G^D \coloneqq G_i^D$ as a scheme; indeed, $B_i^* \coloneqq \mathrm{Hom}_k(B_i, k)$ acquires only a k-coalgebra structure on B_i^* from the k-algebra structure on B_i, but B_i^* *does not acquire in general a k-algebra structure* simply because the comultiplcation on B does not induce a comultiplcation structure on B_i. Nevertheless, luckily, the above "group law" is equivalent to a "comultiplication"
$$\Delta_{i;j,k} \colon B_k \to B_i \otimes_k B_j$$
for $k \gg i, j$, or equivalently under the (topological) dual, a "multiplication"
$$B_i^* \otimes_k B_j^* \to B_k^*.$$
Therefore the inductive limit
$$B^* \coloneqq \varinjlim_i B_i^*$$
is naturally a (discrete) k-algebra, and
$$G^D \coloneqq \mathrm{Spec}(B^*)$$
is an affine scheme over k. Combined with the k-coalgebra structure on B^* obtained as the inductive limit of k-coalgebra structure on each B_i^*, one sees that G^D is an affine group scheme over k.

Convention. Assume from now on that $\mathrm{char}(k) = p > 0$.

1.1.2**. Frobenius and Verschiebung.** Each object in any of the above three categories is equipped with a Frobenius.

The Frobenius F (relative to k) on (formal) groups G is then transferred to the *Verschiebung* V on its Cartier dual G^D. Both operators are functorial in G, hence they induce naturally morphisms of affine (formal) groups over k.

We abuse the same notation to denote the corresponding (topological) ring homomorphisms.

By the construction of Cartier dual, V induces a morphism of augmentation ideals $I_G(p) \to I_G$.

Recall the fact that $FV = [p]_{G^{(p)}}$ and $VF = [p]_G$.

1.1.3. **p^∞-torsion groups.** For any formal group $G \in \mathrm{FGrp}_k$, we say that G is *of p^∞-torsion* if $\varinjlim_n G[p^n] \to G$ is an isomorphism. We denote by $\mathrm{FGrp}_k^{p^\infty\text{-tors}}$ the full subcategory of p^∞-torsion formal groups inside FGrp_k.

We have:

- $\mathrm{FGrp}_k^c \subset \mathrm{FGrp}_k^{p^\infty\text{-tors}}$
- $\mathrm{FGrp}_k^{\text{ét}} \cap \mathrm{FGrp}_k^{p^\infty\text{-tors}} = \{G \in \mathrm{FGrp}_k^{p^\infty\text{-tors}} \mid G(k^{\text{sep}}) \text{ is a } p^\infty\text{-torsion abelian group}\}$.

1.1.4. **p-divisible groups.** The category $\mathrm{BT}(k)$ of p-divisible groups over k can be seen as the full subcategory of $G \in \mathrm{FGrp}_k^{p^\infty\text{-tors}}$ such that $\mathrm{ord}(G[p^n]) = p^{nh}$ for some $h \in \mathbf{N}$ independent of $n \in \mathbf{N}$, or, equivalently, such that $[p]\colon G \to G$ is surjective and $G[p]$ is finite.

The relation between the classical notation $(G_n, i_n)_n$ and G is the following:

$$G_n = G[p^n], \quad i_n\colon G[p^n] \subset G[p^{n+1}].$$

The following short exact sequence will be useful:

$$0 \to G[p^n] \to G[p^{n+m}] \xrightarrow{[p^n]} G[p^m] \to 0.$$

Convention. Assume that k is a perfect field from now on. We have $F_k\colon \mathrm{Spec}(k) \xrightarrow{\simeq} \mathrm{Spec}(k)$, so we will often identify $G^{(p)} \xrightarrow{\simeq} G$ via F_k.

1.1.5. **Connected-étale decomposition.** Let $G \in \mathrm{AffGrp}_k$, then there is a canonical exact sequence $0 \to G^c \to G \to G^{\text{ét}} \to 0$. Since k is a perfect field, this sequence splits canonically as

$$G = G^c \times G^{\text{ét}}.$$

More explicitly, for any $R \in \mathrm{Fin}_k$, since k is perfect, we can write canonically $R = R^{\text{ét}} \oplus \mathfrak{n}_R$, where $\mathfrak{n}_R$ is the nilradical and $R^{\text{ét}} = R_{\text{red}} = R/\mathfrak{n}_R$. Then $G^{\text{ét}}(R) = G(R_{\text{red}})$ and $G^c(R) = \ker(G(R) \to G(R/\mathfrak{n}_R))$. In fact, G^c is always defined in this way even if k is not perfect, but $G^{\text{ét}}$ is usually defined as the quotient G/G^c. The perfectness is used in fact to split the exact sequence $0 \to \mathfrak{n}_R \to R \to R_{\text{red}} \to 0$, so that $G^{\text{ét}}$ can be indeed identified as $G^{\text{ét}}(R) = G(R_{\text{red}})$.

1.1.6. **Multiplicative-unipotent decomposition.** Under Cartier duality, étaleness corresponds to multiplicativity, while connected corresponds to unipotent groups. We have the canonical splitting

$$G = G^m \times G^u.$$

In general, if k is not a perfect field, one only has a canonical exact sequence $0 \to G^m \to G \to G^u \to 0$.

1.1.7. **Characterisation of components by Frobenius and Verschiebung.** Let $G \in \mathrm{FGrp}$ (or $G \in \mathrm{AffGrp}$). Then we have a canonical splitting (over a perfect field k):

$$G = G^{c,m} \times G^{c,u} \times G^{\text{ét},m} \times G^{\text{ét},u}.$$

Moreover, four components have the following characterisation by nilpotency of the Frobenius F and the Verschiebung V on G:

- $G = G^{\text{ét}}$ if and only if F is an monomorphism, if and only if F is an isomorphism.

- $G = G^m$ if and only if G^D is étale, if and only if V is an epimorphism, if and only if V is an isomorphism.
- $G = G^c$ if and only if G is local as a scheme, or if and only if $G(k^{\mathrm{sep}}) = \{0\}$, if and only if F is topologically nilpotent.
- $G = G^u$ if and only if G^D is connected, if and only if V is topologically nilpotent.

A further division yields, for $G \in \mathrm{FGrp}_k$, that

(i) $G = G^{c,m}$ if and only if F is topologically nilpotent and V is an isomorphism.
(ii) $G = G^{c,u}$ if and only if F and V are both topologically nilpotent.
(iii) $G = G^{\text{ét},m}$ if and only F and V are both isomorphisms, if and only $[p]\colon G \to G$ is an isomorphism.
(iv) $G = G^{\text{ét},u}$ if and only if F is an isomorphism and V is topologically nilpotent, if and only if F is an isomorphism and G is of p^∞-torsion.

The classes (i) and (iv) are dual to each other.

The classes (ii) and (iii) are both self-dual.

If one restricts to p^∞-torsion groups, the class (iii) will not appear, and this is the only obstruction for being of p^∞-torsion. In other words, it follows easily from above that

$$\mathrm{FGrp}_k^{p^\infty\text{-tors}} = \mathrm{FGrp}_k^c \times \mathrm{FGrp}_k^{\text{ét},u} = \mathrm{FGrp}_k^{c,m} \times \mathrm{FGrp}_k^{c,u} \times \mathrm{FGrp}_k^{\text{ét},u}. \tag{1.1.7.1}$$

1.1.8. **Example.** As typical examples of different types:

(i) μ_p is connected and multiplicative.
(ii) α_p is connected and unipotent.
(iii) $\mathbf{Z}/\ell\mathbf{Z}$ is étale and multiplicative.
(iv) $\mathbf{Z}/p\mathbf{Z}$ is étale and unipotent.

1.2. Group schemes of Witt vectors.

1.2.1. **Group schemes of Witt vectors.** Recall that W (*resp.* W_n) denotes the functor on commutative rings sending $R \in \mathrm{CRing}$ to the ring

$$W(R) \coloneqq \{(a_0, a_1, \dots) \mid a_m \in R, \forall m \in \mathbf{N}\},$$

$$\textit{resp. } W_{n+1}(R) \coloneqq \{(a_0, a_1, \dots, a_n) \mid a_m \in R, \forall m \in [\![0, n]\!]\},$$

where $\underline{a}$ is called a *Witt vector* (*resp.* an *n-truncated Witt vector*) and its coordinates a_n are called the *Witt coordinates* Here, the ring structure on $W(R)$ (*resp.* on $W_n(R)$) is such that the *ghost coordinates*

$$\gamma_n \coloneqq \sum_{i=0}^{n} p^{n-i} a_i^{p^i}, \quad n \in \mathbf{N}$$

verify the usual ring structure on R. With respect to Witt coordinates, the addition law is given by $\mathbf{Z}$-coefficient polynomials

$$S_n(X_0, X_1, \dots, X_n; Y_1, Y_2, \dots, Y_n), \quad n \in \mathbf{N},$$

and the multiplication law is given by $\mathbf{Z}$-coefficient polynomials

$$P_n(X_0, X_1, \dots, X_n; Y_1, Y_2, \dots, Y_n), \quad n \in \mathbf{N}.$$

If we endow X_i and Y_i with weight p^i, then S_n is homogeneous of degree p^n, and P_n is homogeneous of degree p^{2n}.

As a computational consequence, one sees that for any $r \in \mathbf{N}$, and $s \geq p$, we have

(1.2.1.1)

$$\begin{aligned} &S_{m+1}(X_{-n-m-1}, \dots, X_{-n}; Y_{-n-m-1}, \dots, Y_{-n}) \\ \equiv\ & S_m(X_{-n-m}, \dots, X_{-n}; Y_{-n-m}, \dots, Y_{-n}) \bmod (\dots, X_{-n-r-1}, X_{-n-r}; \dots, Y_{-n-r-1}, Y_{-n-r})^s, \quad \forall m \geq r - 1 + \frac{s-p}{p-1}. \end{aligned}$$

The trunctation map $W \to W_n$ is a morphism of ring schemes.

1.2.2**. Teichmüller lifts.** Let $R \in \mathrm{CRing}$. There is a multiplicative (but not additive) map

$$R \mapsto W(R), \quad x \mapsto [x] := (x, 0, 0, 0, \dots).$$

The element $[x] \in W(R)$ is called the *Teichmüller lift* of $x \in R$. Its action on Witt vectors satisfies the formula

$$[x] \cdot (a_0, a_1, a_2, \dots) = (xa_0, x^p a_1, x^{p^2} a_2, \dots).$$

1.2.3**. Witt Frobenius and Verschiebung.** For $R \in \mathrm{CRing}$, the Witt ring $W(R)$ (*resp.* $W_n(R)$) admits two operators: a (Witt) Frobenius F and a (Witt) Verschiebung V. The can be expressed as follows:

- For $\underline{a} = (a_0, a_1, a_2, \dots)$, we have

$$V\underline{a} = (0, a_0, a_1, \dots),$$

  that is, the Verschiebung shifts the Witt coordinates to the right. The Verschiebung V is a group homomorphism.
- For $\underline{b} = [x] + Vy$, we have

$$F\underline{b} = [x^p] + py.$$

It is then immediate to check that

$$FV = p, \quad (V\underline{a}) \cdot \underline{b} = V(\underline{a} \cdot F\underline{b}).$$

As a consequence, one gets $V(\underline{a})V(\underline{b}) = p \cdot V(\underline{ab})$, in particular the group homomorphism V does not preserve the ring structure.

Moreover, any Witt vector has a presentation as follows:

$$(a_0, a_1, a_2, \dots) = \sum_{i \in \mathbf{N}} V^i[a_i].$$

In the characteristic p case, i.e. in the case where $p \cdot R = 0$, the Frobenius has a simpler formula

(1.2.3.1) $$F(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)^3.$$

[3]This is because in general we have $p[x] - V[x^p] \in W(p \cdot R)$ by checking the ghost coordinates

We compute then to find that

$$VF = p.$$

Recall that we work over a field k of characteristic $p > 0$.

1.2.4. **Lemma.** *The Witt Frobenius and Verschiebung on W over k are indeed the Frobenius and Verschiebung for the affine (formal) groups schemes W over k.*

This lemma is important when one needs to make use of the functoriality of the Frobenius and Verschiebung for affine (formal) group schemes.

Proof. The scheme W over k has coordinate ring $k[X_0, X_1, X_2, \dots]$, and it is easy to check that the relative Frobenius F of W over k gives rise to the Witt Frobenius, since the formula (1.2.3.1), that both Frobenius are, on the coordinate ring, given by

$$F^\sharp \colon X_i \mapsto X_i^p.$$

This is an injective map.

Moreover, on the coordinate ring, the Witt Verschiebung is induced by

$$V^\sharp \colon X_i \mapsto X_{i-1}.$$

It is easy to check that the Witt Verschiebung $V^\sharp$ and the Verschiebung V^* both satisfy the relations:

$$F^\sharp V^\sharp = [p]^\sharp, \quad F^\sharp V^* = [p]^\sharp.$$

Since $F^\sharp$ is clearly injective, one concludes that $V^* = V^\sharp$. $\square$

2. Classical Dieudonné theory

The ring scheme W (*resp.* W_n) over k has a canonical $W(k)$-algebra structure.

Let k be a perfect field of characteristic $p > 0$. Let $\sigma \colon W(k) \to W(k)$ denote the Witt Frobenius on $W(k)$. Since k is perfect, σ is invertible.

Then the Frobenius F on W is σ-semilinear, and the Verschiebung V on W is σ^{-1}-semilinear, that is, for $a \in W(k)$ and $x \in W(R)$, we have

$$F(ax) = \sigma(a)F(x), \quad aV(x) = \sigma(a)x.$$

The *Dieudonné ring* is defined as the twisted non-commutative polynomial ring

$$D_k \coloneqq W(k)\{F, V\}/(FV = VF = p, Fa = \sigma(a)F, aV = V\sigma(a), \forall a \in W(k)).$$

Its elements can be written in a unique way as

$$x = \sum_{n \geq 1} w_{-n}F^n + w_0 + \sum_{n \geq 1} V^m w_m \in D_k.$$

We will often consider the *left* D_k-modules, which are called *Dieudonné modules* over k. They form a category which we denote by D_k-Mod.

In particular, for any k-algebra R, we have $W(R), W_n(R) \in D_k$-Mod.

2.0.1. **Lemma.** *The non-commutative ring D_k is integral and Noetherian. The left ideal D_kV is bilateral and the quotient ring D_k/D_kV is canonically identified with $k\{F\}$.*

2.1. **Dieudonné theory à la Demazure–Gabriel.** Let us present the original Dieudonné theory following Demazure–Gabriel [Dem72, DG70].

(A). *Unipotent Dieudonné theory.*

2.1.1**. Construction.** Consider the contravariant functor

$$\underline{M}\colon \mathrm{AffGrp}_k \to D_k\text{-Mod}, \quad G \mapsto \underline{M}(G) \coloneqq \varinjlim_{n,V} \mathrm{Hom}_{\mathrm{AffGrp}_k}(G, W_n),$$

where the transition maps are induced by $V\colon W_n \to W_{n+1}$, and where we endow $\underline{M}(G)$ with the natural left D_k-module structure coming from that of W_n. By identification of Witt Frobenius and Verschiebung (1.2.4) and by functoriality of the Frobenius and Verschiebung (1.1.2), the F and V actions on $\underline{M}(G)$ is also induced by the F and V on G.

The functor is clearly additive and left exact.

We call $\underline{M}(-)$ the *Dieudonné functor*.

2.1.2. **Example.** We have

$$\underline{M}(\mathbf{Z}/p\mathbf{Z}) = k$$

with $F = \sigma$ and $V = 0$, so one can identify

$$k \simeq D_k/(D_kV + D_k(F-1)) \xrightarrow{\simeq} \underline{M}(\mathbf{Z}/p\mathbf{Z}), \quad 1 \mapsto (\mathbf{Z}/p\mathbf{Z} \subset G_a).$$

2.1.3. **Example.** We have a natural isomorphism

$$k\{F\} \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{AffGrp}_k}(\mathbf{G}_a, \mathbf{G}_a).$$

More generally, by induction, we obtain natural isomorphisms

$$D_k/D_kV^n \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{AffGrp}_k}(W_n, W_n). \tag{2.1.3.1}$$

Since every map $W_n \to W_m$ $(m \geq n)$ factors uniquely through $W_n \to W_m[V^n] = W_n$, we obtain in particular

$$D_k/D_kV^n \xrightarrow{\simeq} \underline{M}(W_n).$$

Futhermore, by taking Frobenius kernels, we get

$$D_k/(D_kV^n + D_kF^m) \xrightarrow{\simeq} \underline{M}(W_n[F^m]), \quad 1 \mapsto (W_n[F^m] \subset W_n).$$

In particular, when $n = m = 1$, we obtain

$$k \xrightarrow{\simeq} D_k/(D_kV + D_kF) \xrightarrow{\simeq} \underline{M}(\alpha_p), \quad 1 \mapsto (\alpha_p \subset \mathbf{G}_a).$$

2.1.4. **Unipotency of W_n.** The ring scheme W_n (viewed as a group scheme) is killed by V^n, hence is killed by $p^n = (FV)^n = F^n V^n$ (the last equality holds in characteristic p). Therefore, any morphism $G \to W_n$ in AffGrp_k factors through the unipotent component $G \to G^u \to W_n$. This induces an isomorphism

$$\underline{M}(G^u) \xrightarrow{\simeq} \underline{M}(G)$$

in the category D_k-Mod, and even in its full subcategory $D_k\text{-Mod}^{V^\infty\text{-tors}}$ of topologically V-nilpotent D_k-modules.

Let us state the theorem:

2.1.5. **Theorem** (Dieudonné, Demazure–Gabriel)**.** *The contravariant functor $\underline{M}(-)$ induces an equivalence of abelian categories*

$$\underline{M} \colon \mathrm{AffGrp}_k^u \xrightarrow{\simeq} D_k\text{-Mod}^{V^\infty\text{-tors}}.$$

For any perfect field extension k' of k, the canonical morphism

$$W(k') \otimes_{W(k)} \underline{M}(G) \to \underline{M}(G \otimes_k k')$$

of $D_{k'}$-modules is an isomorphism.

Moreover, this equivalence $\underline{M}(-)$ induces equivalences between the following full subcategories:

(i) *The unipotent affine groups that are of finite type (as schemes) over k correspond to the V^∞-torsion D_k-modules that are finitely generated:*

$$\underline{M} \colon \mathrm{AffGrp}_k^{u,\mathrm{ft}} \xrightarrow{\simeq} D_k\text{-Mod}^{V^\infty\text{-tors,fg}}.$$

(ii) *The unipotent finite group schemes correspond to the V^∞-torsion D_k-modules that are of finite length as $W(k)$-modules:*

$$\underline{M} \colon \mathrm{FinGrp}_k^u \xrightarrow{\simeq} D_k\text{-Mod}^{V^\infty\text{-tors,flen}}.$$

The strategy of proof goes as follows:

- *Limit argument*: if $G \in \mathrm{AffGrp}_k^u$ is an inverse limit $G = \varprojlim_i G_i$ of $G_i \in \mathrm{AffGrp}_k^u$, then $\underline{M}(G) \xleftarrow{\simeq} \varinjlim_i \underline{M}(G)$. By a limit procedure, it is easy to reduce to the case of unipotent affine groups G that are finite type over k, that is, to reduce to the statement (i).
- *Exactness*: let us show that $\underline{M}$ is an exact functor, i.e. that it preserves short exact sequences. Let By definition, the functor $\underline{M}$ is left exact, hence to prove the right exactness, it suffices to show that

$$\varinjlim_{n,V} \mathrm{Ext}^1_{\mathrm{AffGrp}_k}(G, W_n) = 0. \tag{2.1.5.1}$$

By a similar limit argument as above, one can assume that G is of finite type over k. Now, since G is unipotent, by dévissage one can further assume G to be a closed subgroup of the additive group $\mathbf{G}_a$. Under these assumptions, the proof of (2.1.5.1) would be a consequence of the following explicit computations in AffGrp_k:

  – The 1-trunctation map $W_n \to \mathbf{G}_a$ induces a bijection $\mathrm{Ext}^1(G, W_n) \xrightarrow{\simeq} \mathrm{Ext}^1(G, \mathbf{G}_a)$.
  – The inclusion $G \subset \mathbf{G}_a$ induces a surjection $\mathrm{Ext}^1(\mathbf{G}_a, W_n) \twoheadrightarrow \mathrm{Ext}^1(G, W_n)$.

– The group $\mathrm{Ext}^1(\mathbf{G}_a, W_n)$, as a right module over $\mathrm{Hom}(\mathbf{G}_a, \mathbf{G}_a) = k\{F\}$, is free over the class $[e_n]$ of the extension
$$e_n : 0 \to W_n \xrightarrow{V} W_{n+1} \to \mathbf{G}_a \to 0.$$
– The map $V\colon\ \mathrm{Ext}^1(G, W_n) \to \mathrm{Ext}^1(G, W_{n+1})$ is equal to the zero map.

- A consequence of above the compuations is that, for any $G \in \mathrm{AffGrp}_k^u$, there exists a resolution
$$0 \to G \to W_n^{\oplus r} \to W_n^{\oplus s} \tag{2.1.5.2}$$
for some $n, r, s \in \mathbf{N}$.
- *Fully faithfulness*: for $G, G' \in \mathrm{AffGrp}_k^u$, the map
$$\mathrm{Hom}_{\mathrm{AffGrp}_k}(G, G') \to \mathrm{Hom}_{D_k}(\underline{M}(G'), \underline{M}(G))$$
is a bijection. By the previous point, one sees that this holds if $G' = W_n$. Let us now pass to general G'. For $G' \in \mathrm{AffGrp}_k^{u,\mathrm{ft}}$, the fully faithfulness follows by dévissage from the case where $G' = W_n$, using (2.1.5.2). For general $G' \in \mathrm{AffGrp}_k^u$, an easy limit argument suffices.
- *Essential surjectivity*: for any $M \in D_k\text{-}\mathrm{Mod}^{V^\infty\text{-tors,fg}}$, it is killed by some V^n, and has a resolution
$$(D_k/D_kV^n)^{\oplus s} \xrightarrow{\varphi} (D_k/D_kV^n)^{\oplus r} \to M \to 0$$
for some $r, s \in \mathbf{N}$. By fully faithfulness and $D_k/D_kV^n \xrightarrow{\simeq} \underline{M}(W_n)$ (2.1.3), the map φ is of the form $\varphi = \underline{M}(f)$ for some $f \in \mathrm{Hom}_{\mathrm{AffGrp}_k}(W_n, W_n)$. The exactness implies then that $M \xrightarrow{\simeq} \underline{M}(\ker f)$.

This finishes the proof sketch. The main technical point is the computations done for proving the exactness.

Many statements can be reduced using the limit argument and the resolution (2.1.5.2).

Still need to show that the statement (ii) commutes with the Cartier duality and the Dieudonné duality, so that it is reasonable to define $\underline{M}(-)$ on FinGrp_k^m by $\underline{M}((-)^D)^*$.

(B). *Duality theory.*

2.1.6. **Dual Dieudonné modules.** Set
$$W_\infty(k) := \varinjlim_n W_n(k) = K/pW(k) = \{(\dots, a_{-n}, \dots, a_{-1}, a_0) \mid a_{-m} \in k, \forall m \in \mathbf{N}\}.$$
For $M \in D_k$-Mod, we define its dual as $M^* := \mathrm{Hom}_{D_k}(M, W_\infty)$ with the induced Frobenius and Verschiebung actions switched[4]. The operation $(-)^*$ induces an autoduality on the full subcategory $D_k\text{-}\mathrm{Mod}^{\mathrm{flen}}$.

2.1.7. For $G \in \mathrm{FinGrp}_k^{c,u}$, there is a canonical functorial morphism
$$\varepsilon_G\colon \underline{M}(G^D) \to \underline{M}(G)^* \tag{2.1.7.1}$$

[4]More precisely, for $u \in M^*$, we define
$$(Fu)(m) := \sigma(u(V(m))), \quad (Vu)(m) := \sigma^{-1}(u(F(m))), \quad \forall m \in M.$$

in D_k-Mod. In order to define this, one uses the pairing $W_n[F^m] \times W_m[F^n] \to \mathbf{G}_m$ via Artin-Hasse exponential. This induces an isomorphism

$$W_n[F^m] \xrightarrow{\simeq} (W_m[F^n])^D.$$

Then (2.1.7.1) is defined as follows: let $\varphi \in \underline{M}(G^D)$.

- For any $\varphi_n \colon G^D \to W_n$ representing φ, one considers the following composition map

$$W_n[F^n] \simeq W_n[F^n]^D \xrightarrow{\varphi_n^D} (G^D)^D \simeq G.$$

- Under the Dieudonné functor, this becomes a D_k-linear map

$$\underline{M}(G) \to \underline{M}(W_n[F^n]).$$

- We have natural D_k-linear maps

$$\underline{M}(W_n[F^n]) \subset \varinjlim_{m,V} \mathrm{Hom}_{\mathrm{Sch}_k}(W_n[F^n], W_m) \to \varinjlim_{m,V} \mathrm{Hom}_{\mathrm{Sch}_k}(\mathrm{Spec}(k), W_m) = W_\infty(k),$$

  whose composition can also be identified as the D_k-linear composition map

$$\underline{M}(W_n[F^n]) \xleftarrow{\simeq} D_k/(D_k V^n + D_k F^n) \to D_k/(D_k V^n + D_k F) \simeq W_n(k).$$

  Composed with the map $\underline{M}(G) \to \underline{M}(W_n[F^n])$, one gets the desired D_k-linear map

$$\varepsilon_G(\varphi_n) \colon M(G) \to W_\infty(k).$$

One checks that $\varepsilon_G(\varphi_n)$ depends only on $\varphi \in \underline{M}(G)$, thus defines an element $\varepsilon_G(\varphi) \in \underline{M}(G)^*$.

2.1.8. **Lemma.** *The duality morphism (*2.1.7.1*) is D_k-linear.*

2.1.9. **Theorem.** *The duality morphism (*2.1.7.1*) is an isomorphism.*

Proof idea. First check that this an isomorphism if $G = W_n[F^n]$. Then show that any $G \in \mathrm{FinGrp}_k^{c,u}$ is a subgroup of $W_n[F^n]$ for some $n \in \mathbf{N}$ (see (2.1.5.2)). □

Thanks to (2.1.9), it is now reasonable to extend the Dieudonné theory to the whole $\mathrm{FinGrp}_k^{p^\infty\text{-tors}}$.

(C). *Dieudonné theory for multiplicative groups.*

2.1.10**. Construction.** Recall the decomposition of p^∞-torsion formal groups (1.1.7.1); in particular, the component $G^{\text{ét},m}$ will never show up. For $G \in \mathrm{FinGrp}_k^{p^\infty\text{-tors}}$, decompose uniquely $G = G^m \times G^u$ (we have $G^m = G^{c,m}$ since it has no component $G^{\text{ét},m}$), and define

$$\underline{M}(G) \coloneqq \underline{M}((G^m)^D)^* \oplus \underline{M}(G^u) \in D_k\text{-Mod}.$$

By construction, for $G \in \mathrm{FinGrp}_k^{p^\infty\text{-tors}}$, there is a functorial isomorphism

$$\underline{M}(G^D) \simeq \underline{M}(G)^*$$

of D_k-modules.

2.1.11. **Example.** We have

$$\underline{M}(\mu_p) = k$$

with $F = 0$ and $V = \sigma^{-1}$, so one can identify

$$k \simeq D_k/(D_k(V-1) + D_k F) \xleftarrow{\simeq} \underline{M}(\mathbf{Z}/p\mathbf{Z})^* = \underline{M}(\mu_p),$$

where the middle isomorphism is given by $1 \leftarrow\!\shortmid (\mathbf{Z}/p\mathbf{Z} \subset \mathbf{G}_a)^\vee$.

2.1.12. **Theorem** (Dieudonné, Demazure–Gabriel)**.** *The above construction $\underline{M}(-)$ induces an equivalence of abelian categories*

$$\underline{M} \colon \mathrm{FinGrp}_k^{p^\infty\text{-tors}} \xrightarrow{\simeq} D_k\text{-}\mathrm{Mod}^{\mathrm{flen}}.$$

For any perfect field extension k' of k, the canonical morphism

$$W(k') \otimes_{W(k)} \underline{M}(G) \to \underline{M}(G \otimes_k k')$$

of $D_{k'}$-modules is an isomorphism.

2.1.13. **Corollary.** *For $G \in \mathrm{FinGrp}_k^{p^\infty\text{-tors}}$, we have*

$$\mathrm{ord}(G) = p^{\mathrm{length}\,\underline{M}(G)}.$$

This identity can be shown by reducing to the simple groups μ_p, α_p and $\mathbf{Z}/p\mathbf{Z}$ (and eventually by finite Galois base change for the étale part).

The theorem (2.1.12) is enough for the application to p-divisible groups in the next section. But we will first introduce Fontaine's uniform construction of $\underline{M}(-)$ without resorting to (2.1.9).

2.2. **Example: abelian varieties over k.** No time to discuss. See [Dem72, Chapter V].

Rough ideas: since an abelian variety A over k, the p-divisible group $A[p^\infty]$ has order p^{2g}, where $g = \dim(A)$; since $A[p^\infty]$ admits a polarisation with pairing valued in μ_{p^∞}, its Newton slopes (counted with multiplicities) are symmetric with respect to $\frac{1}{2}$, so they sum up to g.

3. Fontaine's uniform construction

We have seen that the approach of Demazure–Gabriel using the system $W_1 \xrightarrow{V} W_2 \xrightarrow{V} \cdots$ only treats the unipotent part at first, and seems slightly artificial in the multiplicative case. In the following, we are going to present Fontaine's uniform construction of $\underline{M}$ [Fon77] for formal groups, which makes use of an enlargement (and formal completion) of (a variant of) the ind-affine group $\mathrm{CW}^u \coloneqq \varinjlim_{n,V} W_n$.

We will work mostly with formal groups over k.

Various topology on D_k can be considered, the p-adic topology, the V-adic topology, etc., but without special mention, a *topological* D_k-module will involves the p-adic topology on D_k.

3.1. **Construction.**

(A). *Witt covectors.*

3.1.1. **Unipotent Witt covectors.** We define the formal group over k of *unipotent Witt covectors*

$$\widehat{\mathrm{CW}}^u := \varinjlim (W_0 \xrightarrow{V} W_1 \xrightarrow{V} \cdots),$$

where we put an u in the subscript to indicate its unipotency, or equivalently its topological V-nilpotency.

For any $R \in \mathrm{Fin}_k$, we have

$$\widehat{\mathrm{CW}}^u(R) = \{(\ldots, a_{-n}, \ldots, a_{-1}, a_0) \mid a_{-m} \in R, \forall m \in \mathbf{N}; a_{-m} = 0, \forall m \gg 0.\}$$

The (formal) group $\widehat{\mathrm{CW}}^u$ over k has a natural action of D_k, but $\widehat{\mathrm{CW}}^u$ does not have a natural ring structure since the group homomorphism V does not preserve the multiplication (1.2.3).

3.1.2. **Witt covectors.** Let us define the formal group $\widehat{\mathrm{CW}}$ of *Witt covectors.* For any $R \in \mathrm{Fin}_k$, define

$$\mathrm{CW}(R) := \{(\ldots, a_{-n}, \ldots, a_{-1}, a_0) \mid a_{-m} \in R, \forall m \in \mathbf{N}; a_{-m} \in \mathfrak{n}_R, \forall m \gg 0.\}$$

Clearly $\mathrm{CW}(R)$ is of p^∞-torsion and $\mathrm{CW}(R) \supset \mathrm{CW}^u(R)$. Thanks to the congruences (1.2.1.1), $\widehat{\mathrm{CW}}^u(R)$ has a natural (commutative) group law extending that on $\mathrm{CW}^u(R)$; more precisely,

$$S_{-n}(\underline{a};\underline{b}) := \lim_{m\to+\infty} S_m(a_{-n-m}, \ldots, a_{-n}; b_{-n-m}, \ldots, b_{-n})$$

stabilises and defines the $(-n)$-th component $(\underline{a}+\underline{b})_{-n}$. The formal group $\widehat{\mathrm{CW}}$ admits then a Forbenius and a Verschiebung, which is given on $\mathrm{CW}(R)$ by

$$F((a_{-n})_{n\in\mathbf{N}}) = (a_{-n}^p)_{n\in\mathbf{N}}, \quad V((a_{-n})_{n\in\mathbf{N}}) = (a_{-n-1}^p)_{n\in\mathbf{N}}.$$

Moreover, since k is perfect, there is also a $W(k)$-linear structure on $\widehat{\mathrm{CW}}$ such that for $x \in k$, its Teichmüller lift $[x] \in W(k)$ acts by

$$[x](a_{-n})_{n\in\mathbf{N}} = (\sigma^{-n}(x)a_{-n})_{n\in\mathbf{N}}.$$

These actions make $\mathrm{CW}(R)$ a D_k-module.

As a set, we can identify

$$\mathrm{CW}(R) \simeq {\prod_{n\le 0}}'(R, \mathfrak{n}_R),$$

where the latter is the restricted product of R with respect to the subset $\mathfrak{n}_R$. We endow $\mathrm{CW}(R)$ with the natural topology such that the above is a homeomorphism. One checks directly that $\mathrm{CW}(R)$ becomes a topological group, and even a topological D_k-module.

3.1.3. **Remark.** For $R \in \mathrm{Fin}_k$, the topological group $\mathrm{CW}(R)$ is (separated and) complete, and the subset $\mathrm{CW}^u(R) \subset \mathrm{CW}(R)$ is dense.

3.1.4. **Variants of Witt covectors.** For $R \in \mathrm{Fin}_k$, we define

$$\mathrm{CW}^{\text{ét}}(R) := \mathrm{CW}(R^{\text{ét}}) = \mathrm{CW}(R_{\text{red}}) = \mathrm{CW}(R/\mathfrak{n}_R),$$

$$\mathrm{CW}^c(R) := \ker(\mathrm{CW}(R) \to \mathrm{CW}(R/\mathfrak{n}_R)) \subset \mathrm{CW}(R),$$

which are naturally topological D_k-modules. Since k is a perfect field, $R = R^{\text{ét}} \oplus \mathfrak{n}_R$, hence the canonical splitting

$$\mathrm{CW}(R) = \mathrm{CW}^{\text{ét}}(R) \oplus \mathrm{CW}^c(R)$$

of topological D_k-modules.

3.1.5. **Definition.** (i) A topological $W(k)$-modules is said to be *profinite* (*resp. pro-Artinian*) if it can be written as an inverse limit of $W(k)$-modules which are of finite length (*resp.* Artinian).

(ii) Let $A \subset D_k$ be a sub-$W(k)$-algebra. A topological D_k-module is said to be *A-profinite* (*resp. A-pro-Artinian*) if it can be written as an inverse limit of A-modules which are of finite length (*resp.* Artinian) over $W(k)$ (or equivalently, if it is profinite (*resp.* pro-Artinian) and the open sub-A-modules form a fundamental system of neighbourhoods of 0).

If $u\colon M \to M'$ is a morphism of pro-Artinian $W(k)$-modules, then its image is closed in M'. In particular, $\text{Top-}D_k\text{-Mod}^{A\text{-pro-Art}}$ is an abelian category and contains $\text{Top-}D_k\text{-Mod}^{A\text{-profin}}$ as a thick full subcategory.

3.1.6. **Proposition.** *Let $R \in \mathrm{Fin}_k$.*

(i) *The topological D_k-module $\mathrm{CW}^c(R)$ is $W(k)\{F\}$-profinite and is killed by some p^n, and we have*

$$\mathrm{CW}^c(R) = \mathrm{CW}(R)^{\text{top-}F\text{-nilp}} \subset \mathrm{CW}(R).$$

(ii) *The topological D_k-module $\mathrm{CW}^{\text{ét}}(R)$ is discrete and Artinian, and we have*

$$\mathrm{CW}^{\text{ét}}(R) = \bigcap_{n\geq 0} F^n\mathrm{CW}(R) = \bigcap_{n\geq 0} p^n\mathrm{CW}(R) \subset \mathrm{CW}(R).$$

Proof idea. (i) The sub-$W(k)\{F\}$-modules

$$U(R, \mathfrak{n}_R, s) \coloneqq \{(a_{-n})_{n\geq 0} \mid a_{-n} \in \mathfrak{n}_R^{p^{s-n}}, \forall n < s\} \subset \mathrm{CW}^c(R)$$

are open and form a fundamental system of neighbourhoods of 0[5]. Their quotients are of finite length over $W(k)$ since R is finite-dimensional over k. In particular, they are killed by certain p^n.

(ii) We may reduce to the case where $R^{\text{ét}} = k'$ is a finite (separable) extension of k. Then

$$\mathrm{CW}^{\text{ét}}(R) = \mathrm{CW}(k') = \mathrm{CW}^u(k') = K'/pW(k'), \tag{3.1.6.1}$$

which is discrete and Artinian. ☐

3.1.7. **Witt covectors over profinite rings.** Let R be a profinite topological ring over k, or, equivalently, consider $\mathrm{Spf}(R) \in \mathrm{FSch}_k$. Let $R = \varprojlim_i R_i$ be a profinite presentation, with $R_i \in \mathrm{Fin}_k$, then

$$\widehat{\mathrm{CW}}(R) \coloneqq \widehat{\mathrm{CW}}(\mathrm{Spf}(R)) = \mathrm{Hom}_{\mathrm{FSch}_k}(\mathrm{Spf}(R), \widehat{\mathrm{CW}}) = \varprojlim_i \mathrm{CW}(R_i).$$

We endow $\widehat{\mathrm{CW}}(R)$ with the inverse limit topology. Alternatively, R is the product of its (finite) local rings $R_{\mathfrak{m}}$, hence

$$\widehat{\mathrm{CW}}(R) = \prod_{\mathfrak{m}} \mathrm{CW}(R_{\mathfrak{m}}). \tag{3.1.7.1}$$

[5]Notice that they are *not necessarily* stable under the V-action.

Then the topology on $\widehat{\mathrm{CW}}(R)$ coincides with the product topology.

Similarly, we can define $\widehat{\mathrm{CW}}^{\text{ét}}(R)$ and $\widehat{\mathrm{CW}}^{c}(R)$, and we have the canonical decomposition

$$\mathrm{CW}(R) = \mathrm{CW}^{\text{ét}}(R) \oplus \mathrm{CW}^{c}(R)$$

as topological D_k-modules.

All of above are naturally topological D_k-modules.

3.1.8. **Corollary.** *Let R be a profinite topological ring over k.*

(i) *The topological D_k-module $\widehat{\mathrm{CW}}^{\text{ét}}(R)$ is $W(k)\{F\}$-profinite, and we have*

$$\mathrm{CW}^{\text{ét}}(R) = \mathrm{CW}(R)^{\text{top-}F\text{-nilp}} \subset \mathrm{CW}(R).$$

(ii) *The topological D_k-module $\widehat{\mathrm{CW}}^{\text{ét}}(R)$ is pro-discrete and pro-Artinian, and we have*

$$\mathrm{CW}^{\text{ét}}(R) = \bigcap_{n \geq 0} F^n \mathrm{CW}(R) = \bigcap_{n \geq 0} p^n \mathrm{CW}(R) \subset \mathrm{CW}(R).$$

(B). *Dieudonné module functor.*

3.1.9. **Definition.** Let $G \in \mathrm{FGrp}_k$. Define

$$\underline{M}(G) \coloneqq \mathrm{Hom}_{\mathrm{FGrp}_k}(G, \widehat{\mathrm{CW}}) \subset \widehat{\mathrm{CW}}(G),$$

which is a closed topological sub-D_k-module of $\widehat{\mathrm{CW}}(G)$.

Though our construction changes from group schemes to their formal scheme theoretic analogues, the classical examples in the subsection 2.1 still hold with no big difference.

For unipotent finite groups, we have $\mathrm{CW}^u(G) = \mathrm{CW}(G)$, therefore their Dieudonné modules coincide directly with those à la Demazure–Gabriel.

3.1.10. **Example.** We have

$$k \xrightarrow{\simeq} D_k/(D_kV + D_kF) \xrightarrow{\simeq} \underline{M}(\alpha_p), \quad 1 \mapsto (\alpha_p \subset \widehat{\mathbf{G}}_a = \widehat{W_1} \to \widehat{\mathrm{CW}}).$$

More generally, we have

$$D_k/(D_kV^n + D_kF^m) \xrightarrow{\simeq} \underline{M}(W_n[F^m]), \quad 1 \mapsto (W_n[F^m] \subset \widehat{W_n} \to \widehat{\mathrm{CW}}),$$

which can be show by dévissage.

3.1.11. **Example.** We have

$$\underline{M}(\mathbf{Z}/p\mathbf{Z}) = k$$

with $F = \sigma$ and $V = 0$, so one can identify

$$k \simeq D_k/(D_kV + D_k(F-1)) \xrightarrow{\simeq} \underline{M}(\mathbf{Z}/p\mathbf{Z}), \quad 1 \mapsto (\ldots, 0, 0, e_0) \in \mathrm{CW}^u(\mathbf{Z}/p\mathbf{Z}).$$

For non-unipotent p^∞-torsion finite group schemes, our computation is now direct:

3.1.12. **Example.** We have

$$\underline{M}(\mu_p) = k$$

with $F = 0$ and $V = \sigma^{-1}$. In fact, writing $\mu_p = \mathrm{Spec}(k[T]/(T-1)^p)$, one can identify

$$k \simeq D_k/(D_k(V-1) + D_k F) \xrightarrow{\simeq} \underline{M}(\mu_p), \quad 1 \mapsto (\dots, T-1, T-1, T-1) \in \mathrm{CW}(\mu_p).$$

More generally, we have

$$W_n(k) \simeq D_k/(D_k(V-1) + D_k(F-p) + p^n D_k) \xrightarrow{\simeq} \underline{M}(\mu_{p^n}), \quad 1 \mapsto (\dots, T-1, T-1, T-1) \in \mathrm{CW}(\mu_p),$$

hence the D_k-module structure on $W_n(k)$ is given by $F = p\sigma$ and $V = \sigma^{-1}$.

Concerning $\widehat{W_n}$, one should be more careful in transcribing results from the subsection 2.1, since a *profinite completion* must be involved, just as appears in the formal completion procedure from schemes to formal schemes.

3.1.13. **Example.** As a formal group theoretic analogue of (2.1.3), we have canonical isomorphisms of topological D_k-modules:

$$\begin{aligned}(D_k/D_k V^n)^{\wedge,\mathrm{profin}} &\xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{W}_n, \widehat{W}_m)\\ &= \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{W}_n, \widehat{\mathrm{CW}}^u)\\ &= \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{W}_n, \widehat{\mathrm{CW}}), \quad m \geq n,\end{aligned} \tag{3.1.13.1}$$

where $(-)^{\wedge,\mathrm{profin}}$ of $W(k)$-modules means taking the $W(k)\{F\}$-profinite completion, i.e.,

$$M^{\wedge,\mathrm{profin}} := \varprojlim_i M_i,$$

where M_i runs through all sub-$W(k)\{F\}$-modules that are of finite length over $W(k)$[6], which form a inverse system. In particular,

$$(D_k/D_k V^n)^{\wedge,\mathrm{profin}} \xrightarrow{\simeq} \underline{M}(\widehat{W}_n). \tag{3.1.13.2}$$

Exercise: Check that $k\{F\}^{\wedge,\mathrm{profin}} \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{\mathbf{G}_a}, \widehat{\mathbf{G}_a})$, using that $\widehat{\mathbf{G}_a} = \mathrm{Spf}(k[T]^{\wedge,\mathrm{profin}})$.

3.1.14. **Example.** Consider the p-adically completed Dieudonné module

$$(D_k)^{\wedge,\mathrm{profin},p\text{-nilp}} := \varprojlim_n (D_k)^{\wedge,\mathrm{profin}}/p^n(D_k)^{\wedge,\mathrm{profin}},$$

which is also the component of $(D_k)^{\wedge,\mathrm{profin}}$ where p is topologically nilpotent, in other words, the topologically p-nilpotent profinite completion of D_k.

As an extension of the previous example, we have a canonical (ordinary) ring isomorphism:

$$(D_k)^{\wedge,\mathrm{profin},p\text{-nilp}} \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{FGrp}_k}(\widehat{\mathrm{CW}}, \widehat{\mathrm{CW}}) \subset \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{\mathrm{CW}}, \widehat{\mathrm{CW}}).$$

[6] Specifically, we are taking limits along all sub-$W(k)\{F\}$-modules, but not sub-D_k-modules, of finite length over $W(k)$; cf. footnote (1) for similar non-stability under the V-action.

Indeed, since $\widehat{\mathrm{CW}}^u \subset \widehat{\mathrm{CW}}$ is dense (3.1.3), $\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{FGrp}_k}(\widehat{\mathrm{CW}}, \widehat{\mathrm{CW}})$, which is defined as pointwise (i.e. functorially on each $R \in \mathrm{Fin}_k$) continuous group homomorphisms. They correspond uniquely (by restriction to $\widehat{\mathrm{CW}}^u$) to the elements of $\mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{\mathrm{CW}}^u, \widehat{\mathrm{CW}}^u)$ that verify the continuity condition. By the previous example, (3.1.13), we have a ring isomorphism

$$(D_k)^{\wedge,\mathrm{profin},V\text{-nilp}} := \varprojlim_n (D_k)^{\wedge,\mathrm{profin}}/(D_k)^{\wedge,\mathrm{profin}}V^n \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{\mathrm{CW}}^u, \widehat{\mathrm{CW}}^u),$$

which is identified with the component of $(D_k)^{\wedge,\mathrm{profin}}$ where V is topologically nilpotent. The map

$$(D_k)^{\wedge,\mathrm{profin},p\text{-nilp}} \to (D_k)^{\wedge,\mathrm{profin},V\text{-nilp}}$$

is injective and an element of $(D_k)^{\wedge,\mathrm{profin},V\text{-nilp}}$ defines a continuous morphism if and only if it lies in $(D_k)^{\wedge,\mathrm{profin},p\text{-nilp}}$.

However, it seems that $\underline{M}(\widehat{\mathrm{CW}}) = \mathrm{Hom}_{\mathrm{FGrp}_k}(\widehat{\mathrm{CW}}, \widehat{\mathrm{CW}})$ is more mysterious and would be certain profinite completion with respect to not only V-nilpotent quotients but also F-nilpotent quotients.

Let us have a closer look at $\underline{M}(G)$.

3.1.15. **Lemma.** *If $G \in \mathrm{FGrp}_k^{p^\infty\text{-tors}}$, then $\underline{M}(G)$ is $W(k)\{F\}$-profinite.*

Proof. Since G is of p^∞-torsion, we have $G = \bigcup_{n\geq 0} G[p^n]$, hence $\underline{M}(G) = \varprojlim_n \underline{M}(G[p^n])$ in the category Top-D_k-Mod. We know that $\underline{M}(G[p^n])$ is $W(k)\{F\}$-pro-Artinian (3.1.6) and is killed by p^n, hence $\underline{M}(G[p^n])$ is $W(k)\{F\}$-profinite[7]. □

3.1.16. **Example.** We have $\widehat{\mathrm{CW}} \in \mathrm{FGrp}_k^{p^\infty\text{-tors}}$. Indeed, by definition (3.1.2), for any $R \in \mathrm{Fin}_k$ and any $a \in \mathrm{CW}(R)$, a suitable power of Frobenius F^m kills the nilpotent coordinates, so that $F^m(a) \in \mathrm{CW}^u(R)$, whence $V^nF^m(a) = 0$ for sufficiently large n; in particular, $p^na = F^{n-m}(V^nF^m(a)) = 0$.

3.1.17**.** Let us provide a more explicit description of $\underline{M}(G)$, for future usage in relating with tangent and cotangent spaces.

Recall that by definition, $\widehat{\mathrm{CW}}(G)$ consists of morphisms of formal schemes while $\underline{M}(G) \subset \widehat{\mathrm{CW}}(G)$ consists of those which respect the group laws.

Let $G = \mathrm{Spf}(B_G) \in \mathrm{FGrp}_k$. Let I_G be the augmentation ideal of B_G and let $V_G\colon B_G \to B_G$ be the (continuous) morphism of topological k-algebras corresponding to the Verschiebung $G \xleftarrow{\simeq} G^{(p)} \xrightarrow{V} G$. We have $V_G(I_G) \subset I_G$ (hence $V_G(I_G^r) \subset I_G^r$ as V_G is a ring homomorphism), since the Verschiebung is a group homomorphism hence sends the unit to the unit.

Let us denote by B_G^W the subset of $\widehat{\mathrm{CW}}(G)$ consisting of all elements of the form $a^W := (V_G^n(a))_{n\geq 0}$ for some $a \in I_G$. By definition, the map

$$B_G^W \to I_G, \quad a^W \mapsto a \tag{3.1.17.1}$$

[7]Let M_i be a discrete (quotient $W(k)\{F\}$-module which is an) Artinian $W(k)$-module. Since M_i is killed by p^n, it must be of finite length over $W(k)$.

is injective, and is compatible with the natural F- and V-actions[8]. This map is not additive.

It is surjective if $G \in \mathrm{FGrp}_k^{p^\infty\text{-tors}}$. Indeed, if G is connected, then I_G is topologically nilpotent; if G is étale and of p^∞-torsion, then the Verschiebung is topologically nipotent (1.1.7.1), hence V_G is topologically nilpotent on I_G. Therefore, the map

$$I_G \to B_G^W, \quad a \mapsto a^W = (V_G^n(a))_{n\geq 0}$$

defines a well-defined inverse to (3.1.17.1).

Moreover, $B_G^W \subset \widehat{\mathrm{CW}}(G)$ is a closd subset because it is equivalently defined by the equations $a_{-n-1} = V_G(a_{-n})$, $n \geq 0$, or in terms of the Witt covector Verschiebung:

$$V(a_{-n})_{n\geq 0} = (V_G(a_{-n}))_{n\geq 0}. \tag{3.1.17.2}$$

It is even a sub-D_k-module because the equation is compatible with the D_k-actions by definition (3.1.2). Thus the bijection (3.1.17.1) is a homeomorphism.

3.1.18. **Lemma.** *Let $G = \mathrm{Spf}(B_G) \in \mathrm{FGrp}_k$. Then inside $\widehat{\mathrm{CW}}(G)$ we have the inclusion*

$$\underline{M}(G) \subset B_G^W.$$

In particular, the map

$$\underline{M}(G) \to I_G, \quad (a_{-n})_{n\geq 0} \mapsto a_0 \tag{3.1.18.1}$$

is injective, continuous, compatible with the F- and V-actions.

Proof. Since $\underline{M}(G)$ consists of morphisms of formal groups, which preserve in particular the Verschiebung operators, hence (3.1.17.2) holds, so $\underline{M}(G) \subset B_G^W$. The rest follows by (3.1.17.1) and the remarks after it. ☐

(C). *Candidate for the quasi-inverse functor.*

3.1.19. **Definition.** Let $M \in \text{Top-}D_k\text{-Mod}$. For $R \in \mathrm{Fin}_k$, define

$$\underline{G}(M)(R) := \mathrm{Hom}_{D_k}^{\mathrm{cont}}(M, \mathrm{CW}(R)) := \mathrm{Hom}_{\text{Top-}D_k\text{-Mod}}(M, \mathrm{CW}(R)). \tag{3.1.19.1}$$

The functor $\underline{G}(M)\colon \mathrm{Fin}_k \to Set$ is left exact and thus defines a formal group $\underline{G}(M) \in \mathrm{FGrp}_k$.

3.1.20. **Lemma.** *If M is $W(k)\{F\}$-profinite, then $\underline{G}(M)$ is of p^∞-torsion.*

Proof. For $R \in \mathrm{Fin}_k$, the (3.1.6) shows that $\mathrm{CW}^c(R)$ is killed by some p^n, and $\mathrm{CW}^{\text{ét}}(R)$ is discrete. Hence $\mathrm{Hom}_{D_k}^{\mathrm{cont}}(M, \mathrm{CW}^c(R))$ is killed by p^n. As for $\mathrm{Hom}_{D_k}^{\mathrm{cont}}(M, \mathrm{CW}^{\text{ét}}(R))$, by writing $M = \varprojlim_i M_i$ as the inverse limit of finite-length $W(k)$-modules M_i, which are killed by some p^{n_i}, we see that $\mathrm{Hom}_{D_k}^{\mathrm{cont}}(M, \mathrm{CW}^{\text{ét}}(R)) = \varinjlim_i \mathrm{Hom}_{D_k}^{\mathrm{cont}}(M_i, \mathrm{CW}^{\text{ét}}(R))$ is of p^∞-torsion. Therefore, $\mathrm{Hom}_{D_k}^{\mathrm{cont}}(M, \mathrm{CW}(R))$ is of p^∞-torsion. ☐

3.1.21. **Remark.** The D_k-module D_k does not seem to have a topological D_k-module structure to become a profinite $W(k)$-module, let alone $W(k)\{F\}$-profinite.

[8]But this map is not D_k-linear, since I_G is not a D_k-module: in fact, on I_G, we have $F_G V_G = [p] \neq 0 = p$. Even worse, this map is not additive.

3.1.22. **Theorem** (Fontaine)**.** *The functor $\underline{M}(-)$ induces an equivalence*

$$\underline{M}\colon \mathrm{FGrp}_k^{p^\infty\text{-tors}} \xrightarrow{\simeq} \text{Top-}D_k\text{-}\mathrm{Mod}^{W(k)\{F\}\text{-profin}}, \tag{3.1.22.1}$$

and $\underline{G}(-)$ provides a quasi-inverse.

In particular, this equivalence restricts to an equivalence

$$\mathrm{FinGrp}_k^{p^\infty}\text{-tors} \simeq D_k\text{-}\mathrm{Mod}^{\mathrm{flen}},$$

and the (2.1.13) holds, i.e., for $G \in \mathrm{FinGrp}_k^{p^\infty\text{-tors}}$, we have

$$\mathrm{ord}(G) = p^{\mathrm{length}\,\underline{M}(G)}. \tag{3.1.22.2}$$

3.1.23. **Remark.** By a limit procedure, we see that the equivalence (3.1.22.1) also restricts to an equivalence between the category of inductive limits of finite k-group schemes of p^∞-torsion, and the cateogry of topological D_k-modules that are D_k-profinite.

3.1.24. **Remark.** There is also a theory of duality, which we omit here.

3.2. **Application to p-divisible groups.**

3.2.1. **Theorem.** *The equivalence (3.1.22.1) restricts to an equivalence*

$$\underline{M}\colon \mathrm{BT}(k) \xrightarrow{\simeq} D_k\text{-}\mathrm{Mod}^{\text{finite free}}. \tag{3.2.1.1}$$

Moreover, for $G \in \mathrm{BT}(k)$ of height h and dimension $d \leq h$, we have

$$h = \mathrm{rank}_{W(k)}\,\underline{M}(G),$$
$$d = \dim_k t_G(k) = \dim_k(\underline{M}(G)/F\underline{M}(G)),$$
$$h - d = \dim_k t_{G^D}(k) = \dim_k \ker(\underline{M}(G) \xrightarrow{V} \underline{M}(G)).$$

Proof. Using the defining properties of p-divisible groups, the equivalance follows from (3.1.22.1). The rank identities will follow from the proof of Theorem (3.1.22), especially (3.4.3) and (3.5.1). □

3.2.2. **Corollary.** *The contravariant functor from the category of isogeny classes of p-divisible groups over k to the category of isocrystals over k of Newton slope in $[0,1]$:*

$$\mathrm{BT}(k)^0 \to \mathrm{Isoc}^{[0,1]}_{W(k)\frac{1}{p}}, \quad G \mapsto \underline{M}(G)[\frac{1}{p}]$$

is fully faithful, and essentially surjecive if k is algebraically closed.

Proof. The slope condition is due to the existence of V such that $FV = p$. The fully faithfulness follows easily from (3.2.1). Assume now $k = \overline{k}$. By Dieudonné-Manin classfication, it is enough to show that any crystal M over k that is irreducible and isoclinic of slope $\frac{d}{h}$ (for some integers $h > 0$ and $0 \leq d \leq h$) comes from a p-divisible group $G \in \mathrm{BT}(k)$. By (3.2.1), it is enough to construct a p^∞-torsion formal group G such that $\underline{M}(G) \simeq M$. For this, we use resolutions; but unlike $\underline{M}(\mathrm{CW}^u) \simeq (\mathbf{D}_k)^\wedge_V$ in the classical theory à la Demazure–Gabriel in the subsection 2.1, the $\underline{M}(\widehat{\mathrm{CW}})$ in Fontaine's theory is more complicated (3.1.14) and would involve other profinite completion than simply V-adic completion (for example, the

$D_k/(D_kF^n + D_kP(V)) \in D_k\text{-Mod}^{\text{flen},c,m}$ where $P(V) \in W(k)\{V\}$ with constant term $a_0 \in W(k)^\times$ provides one possible discrete quotient; cf. (3.4.3.3)). For this reason, we distinguish the (topologically) unipotent case and the (connected) multiplicative case:

- *Unipotent case*: $0 \leq d < h$. There is a resolution in Top-D_k-$\text{Mod}^{W(k)\{F\}\text{-profin}}$:

$$(D_k)^{\wedge,\text{profin},V\text{-nilp}} \xrightarrow{F^{h-d}-V^d} (D_k)^{\wedge,\text{profin},V\text{-nilp}} \to M \to 0.$$

  Notice that $(D_k)^{\wedge,\text{profin},V\text{-nilp}} \simeq \underline{M}(\widehat{\text{CW}}^u)$ (3.1.13). So we have

$$M \simeq \underline{M}(\ker(\widehat{\text{CW}}^u \xrightarrow{F^{h-d}-V^d} \widehat{\text{CW}}^u)).$$

- *Multiplicative case*: $d = h$, hence $M \simeq W(k)$ with $F = p\sigma$ and $V = \sigma^{-1}$. Then by (3.1.12), we obtain

$$M \simeq \underline{M}(\mu_{p^\infty}).$$

It might be interesting to know whether one has

$$M \simeq \underline{M}(\ker(\widehat{\text{CW}} \xrightarrow{F^{h-d}-V^d} \widehat{\text{CW}})).$$

in both cases. □

3.3. **Proofs, I: preparation.** We are going to pave the way for the proof, which will be divided into two parts, the étale part and the connected part.

(A). *Adjointness.* First, let us remark some generalities. It is clear that the contravariant functors $\underline{M}(-)$ and $\underline{G}(-)$ are both additive and left exact (i.e. send right exact sequences to left exact ones).

One observes the following lemma, which follows immediately from the definitions.

3.3.1. **Lemma.** *The contravariant functors $\underline{M}(-)$ and $\underline{G}(-)$ are mutually adjoint to each other, that is, for any $G \in \text{FGrp}_k^{p^\infty\text{-tors}}$ and $M \in$ Top-D_k-$\text{Mod}^{W(k)\{F\}\text{-profin}}$, we have*

$$\text{Hom}_{\text{FGrp}_k^{p^\infty\text{-tors}}}(G, \underline{G}(M)) \xrightarrow{\simeq} \text{Hom}_{\text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-profin}}}(M, \underline{M}(G)).$$

Let us denote the corresponding units for the adjunction as

$$\alpha\colon G \mapsto \underline{GM}(G), \quad \beta\colon M \mapsto \underline{MG}(M).$$

Thanks to the existence of these adjunction units, it remains to show that they are in fact equivalences.

By definition of units, we see that $\underline{M}(\alpha)$ and $\underline{G}(\beta)$ are epimorphisms.

(B). *Splitting into étale and connected parts.* Recall that

$$\text{FGrp}_k^{p^\infty\text{-tors}} = \text{FGrp}_k^{p^\infty\text{-tors,ét}} \times \text{FGrp}_k^{p^\infty\text{-tors},c}.$$

Recall also that $G \in \text{FGrp}_k^{p^\infty\text{-tors}}$ is étale if and only if $F\colon G \to G$ is a monomorphism (i.e. $\ker(F) = 0$, or equivalently, if F an isomorphism), and that G is connected if and only if F is topologically nilpotent (i.e. $G = \bigcup_{n\geq 0} \ker(F^n)$). In order to prove the theorem, it would help if one can decompose also on the side of Top-D_k-$\text{Mod}^{W(k)\{F\}\text{-profin}}$, such that $\underline{M}(-)$ and $\underline{G}(-)$ match the corresponding factors.

3.3.2. **Definition.** Let $M \in \text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-pro-Art}}$. We say that M is *étale* (*resp. connected*) if $FM = M$ (*resp.* for any $a \in M$, $F^n a = 0$).

3.3.3. **Proposition.** *Any* $M \in \text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-pro-Art}}$ *can be written uniquely as a direct sum* $M = M^{\text{ét}} \oplus M^c$ *of an étale component and a connected component.*

In other words, there is a decomposition

$$\text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-profin}} = \text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-profin,ét}} \times \text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-profin},c}.$$

Now it remains to prove the theorem on the étale and the connected parts, respectively.

3.4. **Proofs, II: the étale part.**

(A). *Exactness of* $\underline{M}(-)$ *on the étale part.* We start by the following observation:

3.4.1. **Proposition.** *The functor* $\underline{M}(-)$ *is exact on* $\mathrm{FGrp}_k^{\text{ét}}$, *i.e.* $\widehat{\mathrm{CW}}^{\text{ét}}$ *is an injective object of* $\mathrm{FGrp}_k^{\text{ét}}$.

The category $\mathrm{FGrp}_k^{\text{ét}}$ is not mysterious, since taking $\overline{k}$-points induces an equivalence of categories:

$$\mathrm{FGrp}_k^{\text{ét}} \to \{\text{discrete } \mathrm{Gal}(\overline{k}/k)\text{-modules}\}, \quad G \mapsto G(\overline{k}).$$

By (3.1.6.1), the object $\widehat{\mathrm{CW}}^{\text{ét}} \in \mathrm{FGrp}_k^{\text{ét}}$ is sent to $W(\overline{k})[\frac{1}{p}]/pW(\overline{k}) = \breve{K}/p\mathcal{O}_{\breve{K}}$. It suffices then to show that this is an injective object in the category of discrete $\mathrm{Gal}(\overline{k}/k)$-modules. This can be proved by an integral and unramified version of the Galois descent:

3.4.2. **Lemma.** *Let* k'/k *be a finite Galois extension. Let* M *be a pro-Artinian* $W(k')$*-module with a continuous semilinear* $\mathrm{Gal}(k'/k)$*-action. Then:*

(i) *The application* $W(k') \otimes_{W(k)} M^{\mathrm{Gal}(k'/k)} \to M$, $a \otimes x \mapsto ax$ *is bijective.*
(ii) *The* $\mathrm{Gal}(k'/k)$*-module* M *has vanishing* $H^{>0}(\mathrm{Gal}(k'/k), -)$.

The (i) says that the $\mathrm{Gal}(k'/k)$-module M is induced, hence the (ii) follows thanks to Shapiro's Lemma.

(B). *Tangent space.* For $G \in \mathrm{FGrp}_k$ (*resp.* $G \in \mathrm{Grp}_k$), we define its tangent space as

$$t_G(k) := \ker(G(k[\varepsilon]/\varepsilon^2) \to G(k)).$$

The Verschiebung vanishes, and the remaining Frobenius makes $t_G(k)$ a discrete (*resp.* profinite) topological $k\{F\}$ module. There is a natural identification of (profinite) topological $k\{F\}$-modules

$$\mathrm{Hom}_{\mathrm{FGrp}_k}(G, \widehat{\mathbf{G}}_a) \simeq t_{G^D}(k)$$

induced by sending $b \in \widehat{\mathbf{G}}_a(G) = B_G$ to $(1 \oplus \mathrm{ev}_b \cdot \varepsilon) \in \mathrm{Hom}_{k,\mathrm{cont}}(B^\vee, k \oplus k\varepsilon) \supset G^D(k[\varepsilon]/\varepsilon^2)$; the condition that b be a morphism of formal groups corresponds exactly to the condition that $1 \oplus \mathrm{ev}_b \cdot \varepsilon$ be a ring homomorphism.

The key to the proof of (3.1.22) in the étale case is the following identification:

3.4.3. **Proposition.** *Let us keep the notation B_G^W of (3.1.17). Then the map (3.1.18.1) induces an isomorphism of topological $k\{F\}$-modules:*

$$\gamma_G\colon \ker(\underline{M}(G) \xrightarrow{V} \underline{M}(G)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{FGrp}_k}(G, \widehat{\mathbf{G}_a}), \quad (\ldots, 0, \ldots, 0, 0, 0, a_0) \mapsto a_0 \in I_G$$

Proof. Let us use the shorthand $\ker(V)$. Since $p = FV$ vanishes on $\ker(V)$, the D_k-module $\ker(V)$ becomes in fact a $k\{F\}$-module. The map is easily seen to be additive, so it is a morphism of topological $k\{F\}$-modules.

The morphism γ_G is well-defined since it fits into the following commutative diagram of topological D_k-modules:

$$\begin{tikzcd} \ker(V) \arrow[r, dash, "\gamma_G"] \arrow[d, hook, "(3.1.18)"] & \mathrm{Hom}_{\mathrm{FGrp}_k}(G, \widehat{\mathbf{G}_a}) \arrow[d, hook] \\ B_G^W \arrow[r, "\simeq", "(3.1.17.1)"'] & I_G. \end{tikzcd}$$

It is then easy to see that γ_G is injective.

For surjectivity (as well as injectivitiy, alternatively but directly), noticing that

$$\ker(V) = \{a_0^W = (\ldots, 0, 0, 0, a_0) \in B_G^W \mid (\Delta_G(a_0))^W = (a_0 \otimes 1)^W + (1 \otimes a_0)^W \in \widehat{\mathrm{CW}}(B_G \hat{\otimes} B_G)\},$$

and that

$$\mathrm{Hom}_{\mathrm{FGrp}_k}(G, \widehat{\mathbf{G}_a}) = \{a \in I_G \mid \Delta_G(a_0) = a_0 \otimes 1 + 1 \otimes a_0 \in B_G \hat{\otimes} B_G\}.$$

The two conditions coincide, as indeed $V_{G\times G}(a_0 \otimes 1) = V_G(a_0) \otimes 1 = 0$ as well as similarly $V_{G\times G}(1 \otimes a_0) = 0$, for which the Witt covector addition is just adding the 0-th components. □

In summary, we obtain isomorphisms of profinite $k\{F\}$-modules:

$$\ker(\underline{M}(G) \xrightarrow{V} \underline{M}(G)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{FGrp}_k}(G, \widehat{\mathbf{G}_a}) \simeq t_{G^D}(k).$$

It follows immediately that:

- If $G \in \mathrm{FinGrp}_k^{p^\infty\text{-tors}}$ has $V = 0$, then $\underline{M}(G) \in D_k\text{-Mod}^{\mathrm{flen}}$ and has $V = 0$, and $\mathrm{ord}(G) = p^{\mathrm{length}(\underline{M}(G))}$.
- If $M \in D_k\text{-Mod}^{\mathrm{flen}}$ has $V = 0$, then $\underline{G}(M) \in \mathrm{FinGrp}_k$ and has $V = 0$, and $\mathrm{ord}(\underline{G}(M)) = p^{\mathrm{length}(M)}$.

(C). *Reduction to the finite case.* The proof of (3.1.22) in the étale case can be easily reduced to the case of finite groups and finite modules, by the following observations:

- on the side of formal groups, any $G \in \mathrm{FGrp}_k^{\text{ét}}$ is the union of its finite subgroups;
- on the side of Dieudonné modules, one has

  $$\text{Top-}D_k\text{-Mod}^{W(k)\{F\}\text{-profin,ét}} = \text{Top-}D_k\text{-Mod}^{D_k\text{-profin,ét}}$$

  since informally (and in fact truly) $V = pF^{-1}$ on étale modules, hence any such M is the limit of its finite quotient D_k-modules.

In the finite case where $V=0$, the adjoint functors $\underline{M}(-)$ and $\underline{G}(-)$ are equivalences by the identification (3.4.3) as well as its consequences, and by dévissage to the simple case.

For the general finite case, one can show that the unit $\alpha\colon \mathrm{id}\to \underline{GM}(-)$ is an equivalence on $\mathrm{FinGrp}_k^{p^\infty\text{-tors,ét}}$ by dévissage to simple objects (or simply by dévisage by V-filtrations) and diagram chasing, using the left exactness of $\underline{G}(-)$ and the exactness of $\underline{M}(-)$ on the étale part (3.4.1).

Similarly, for β, one can make the same dévissage to simple objects, but the equivalence of β does not follow by induction directly from diagram chasing. Let us explain it. For a (non-trivial) short exact sequence in D_k-$\mathrm{Mod}^{\mathrm{flen,ét}}$

$$0\to M'\to M\to M''\to 0,$$

we obtain a commutative diagram with exact rows:

(3.4.3.1)
$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\beta_{M'}"] & M \arrow[r] \arrow[d, "\beta_M"] & M'' \arrow[r] \arrow[d, "\beta_{M''}"] & 0 \\ & \underline{MG}(M') \arrow[r] & \underline{MG}(M) \arrow[r] & \underline{MG}(M'') \arrow[r] & 0. \end{tikzcd}$$

By induction hypothesis, $\beta_{M'}$ and $\beta_{M''}$ are isomorphisms. The lack of left exactness in the second row makes the diagram chasing only yield the surjectivity, not yet bijectivity, of β_M. Fontaine's original argument for proving β_M to be an isomorphism does not seem to work[9].

Nevertheless, there is an alternative way to prove the equivalence of $\underline{M}(-)$ on the étale part, without looking at β.

Our alternative way is to show the essential surjectivity. Indeed, the equivalence that we have just proven of the unit α amounts to the fully faithfulness of $\underline{M}\colon \mathrm{FinGrp}_k^{p^\infty\text{-tors,ét}}\to D_k\text{-}\mathrm{Mod}^{\mathrm{flen,ét}}$, hence it suffices to prove the essential surjectivity of $\underline{M}(-)$. To prove this, we may proceed as in the proof of (2.1.5) with little modification[10]:

- *Length comparison.* By the equivalences $\underline{M}(-)$ and $\underline{G}(-)$ on simple objects of $\mathrm{FinGrp}_k^{p^\infty\text{-tors}}$, the left exactness of $\underline{M}(-)$ and $\underline{G}(-)$ implies that they are both length decreasing[11]; $\underline{M}(-)$ is even

[9]It is [Fon77, p. 141–142, Chapitre III, Proposition 3.4, iii)]. The "error" appears in the second to last paragraph of the proof, the last sentence, "Comme $\underline{G}(-)$ est exact à gauche et comme $\underline{G}(v_M)$ est un épimorphisme, on voit que $\underline{G}(N)=0$." Here, v_M is our unit $\beta_M\colon M\to \underline{MG}(M)$, which was already shown there to be an epimorphism by a dévissage, and $N:=\ker(v_M)$. The reason why this argument does not work, by left exactness of the *contravariant* functor $\underline{G}(-)$ and the definition of N, we obtain a left exact sequence $0\to \underline{GMG}(M)\xrightarrow{\underline{G}(v_M)}\underline{G}(M)\to\underline{G}(N)$. The surjectivity of $\underline{G}(v_M)$ shows that the map $\underline{G}(M)\to\underline{G}(N)$ is zero. This does not seem to imply $\underline{G}(N)=0$.

[10]The D_k-module D_k/D_kV^n used in the proof of (2.1.5) is not $W(k)\{F\}$-profinite, hence will not be in the target of Fontaine's Dieudonné functor $\underline{M}(-)$.

[11]Concretely, it means that
- for $G\in\mathrm{FinGrp}_k^{p^\infty\text{-tors}}$, we have
$$\mathrm{ord}(G)\geq p^{\mathrm{length}(\underline{M}(G))};$$
- for $M\in D_k\text{-}\mathrm{Mod}^{\mathrm{flen}}$, we have
$$p^{\mathrm{length}(M)}\geq \mathrm{ord}(\underline{G}(M)).$$

length preserving on $\mathrm{FGrp}_k^{p^\infty\text{-tors,ét}}$ since it is exact there as just proven above. Therefore, as direct consequences:

– For $M \in D_k\text{-Mod}^{\text{flen,ét}}$, we have comparison

$$p^{\operatorname{length}(M)} \geq \operatorname{ord}(\underline{G}(M)) \geq p^{\operatorname{length}(\underline{MG}(M))} \geq \operatorname{ord}(\underline{GMG}(M)) \stackrel{\alpha}{=} \operatorname{ord}(\underline{G}(M)).$$

Thus, we have equality

$$\operatorname{ord}(\underline{G}(M)) = p^{\operatorname{length}(\underline{MG}(M))}.$$

– For $G \in \mathrm{FGrp}_k^{p^\infty\text{-tors,ét}}$, we have

$$\operatorname{ord}(G) \geq p^{\operatorname{length}(\underline{M}(G))} \geq \operatorname{ord}(\underline{GM}(G)) \stackrel{\alpha}{=} \operatorname{ord}(G).$$

Thus $\operatorname{ord}(G) = p^{\operatorname{length}(\underline{M}(G))}$; hence,

$$\underline{M}(G) \in D_k\text{-Mod}^{\text{flen,ét}} \implies G \in \mathrm{FinGrp}_k^{p^\infty\text{-tors,ét}}. \tag{3.4.3.2}$$

- *A resolution.* Let $M \in D_k\text{-Mod}^{\text{flen,ét}}$. Then V is nilpotent, so there exists a resolution in $D_k\text{-Mod}^{\text{flen}}$.

$$\bigoplus_{i=r+1}^{r+s} D_k/(D_kP_i(F) + D_kV^n) \xrightarrow{\varphi} \bigoplus_{i=1}^{r} D_k/(D_kP_i(F) + D_kV^n) \to M \to 0$$

for some $r, s \in \mathbf{N}$, and some (non-commutative) polynomial $P_i(F) = \sum_{j=1}^{d_i} a_{ij}F^i \in W(k)\{F\}$ of degree d_i in F, with $a_{i0} \in W(k)$ invertible[12]. In particular, one has

$$D_k/(D_kP_i(F) + D_kV^n) \in D_k\text{-Mod}^{\text{flen,ét}}. \tag{3.4.3.3}$$

- *Preimage of resolving terms.* Let us show that $D_k/(D_kP_i(F) + D_kV^n)$ is isomorphic to $\underline{M}(G_i)$ for some $G_i \in \mathrm{FinGrp}_k^{p^\infty\text{-tors,ét}}$. Let

$$G_i := \ker(\widehat{W}_n \xrightarrow{P_i(F)} \widehat{W}_n).$$

Since $P_i(F) = \sum_{j=1}^{d_i} a_{ij}F^i$ with $a_{i0} \in W(k)^\times$, the Frobenius F on G_i is an isomorphism, thus G_i is étale. So, taking the connected components, one obtains $G_i = \ker(\widehat{W}_n^{\text{ét}} \xrightarrow{P_i(F)} \widehat{W}_n^{\text{ét}})$. By the

[12]One may take a set of generators $x_1, \dots, x_r$ of M as a $W(k)$-module. Then the sequence $x_i, Fx_i, F^2x_i, F^3x_i, \dots$ stops to be $W(k)$-linear independent at some $F^{d_i}x_i$, and $P_i(F) = \sum_{j=0}^{d_i} a_{ij}F^j$ is defined as the annihilator of x_i. Since $M \in D_k\text{-Mod}^{\text{flen,ét}}$, F is surjective, hence an isomorphism by counting lengths; this allows to show that $a_{i0} \in W(k)$ is not divided by p, i.e. $a_{i0} \in W(k)^\times$. Considering kernel of the surjection $\bigoplus_{i=1}^r D_k/(D_kP_i(F) + D_kV^n) \twoheadrightarrow M$ given by $x_1, \dots, x_r$, one may find a set of its generators $x_{r+1}, \dots, x_{r+s}$ and similarly get $P_i(F)$ for $i = r+1, \dots, r+s$ with $a_{ij} \in W(k)^\times$.

exactness of $\underline{M}(-)$ on $\mathrm{FGrp}_k^{\text{ét}}$ (3.4.1), one gets

$$
\begin{aligned}
\underline{M}(G_i) &= \operatorname{coker}(\underline{M}(\widehat{W}_n^{\text{ét}}) \xrightarrow{\underline{M}(P_i(F))} \underline{M}(\widehat{W}_n^{\text{ét}})) \\
&= \operatorname{coker}(\underline{M}(\widehat{W}_n)^{\text{ét}} \xrightarrow{\underline{M}(P_i(F))} \underline{M}(\widehat{W}_n)^{\text{ét}}) \\
&\overset{\simeq}{\leftarrow} \operatorname{coker}((D_k/D_kV^n)^{\wedge,\text{profin},\text{ét}} \xrightarrow{\cdot P_i(F)} (D_k/D_kV^n)^{\wedge,\text{profin},\text{ét}}) \\
&\overset{\simeq}{\rightarrow} \operatorname{coker}((D_k/D_kV^n)^{\wedge,\text{profin}} \xrightarrow{\cdot P_i(F)} (D_k/D_kV^n)^{\wedge,\text{profin}}) \\
&\overset{\simeq}{\rightarrow} D_k/(D_kP_i(F) + D_kV^n);
\end{aligned} \tag{3.4.3.4}
$$

Here, the third line is by the computation (3.1.13), the fourth line is because, $a_{i0} \in W(k)$ being invertible, the element 1 is topologically nilpotent on $\operatorname{coker}((D_k/D_kV^n)^{\wedge,\text{profin}} \xrightarrow{\cdot P_i(F)} (D_k/D_kV^n)^{\wedge,\text{profin}})$, hence this module is equal to zero.

- *Finiteness of G_i.* By (3.4.3.3) and the above computation (3.4.3.4), one has $\underline{M}(G_i) \in D_k\text{-}\mathrm{Mod}^{\text{flen},\text{ét}}$. Then by (3.4.3.2), we see that $G_i \in \mathrm{FinGrp}_k^{p^\infty\text{-tors},\text{ét}}$.
- *Final step: preimage of M.* Since $G_i \in \mathrm{FinGrp}_k^{p^\infty\text{-tors},\text{ét}}$, it follows from the fully faithfulness of $\underline{M}(-)$ on $\mathrm{FinGrp}_k^{p^\infty\text{-tors},\text{ét}}$ (by the equivalence of the unit α proven above) that the map φ is of the form $\varphi = \underline{M}(f)$ for some $f \in \mathrm{Hom}_{\mathrm{FGrp}_k}(\bigoplus_{i=1}^r G_i, \bigoplus_{i=r+1}^{r+s} G_i)$. In particular, one obtains an object $\ker(f) \in \mathrm{FinGrp}_k^{p^\infty\text{-tors},\text{ét}}$. The exactness of $\underline{M}(-)$ on $\mathrm{FinGrp}_k^{\text{ét}}$ (3.4.1) then implies that $M \simeq \underline{M}(\ker f)$.

3.5. **Proofs, III: the connected part.**

(A). *Cotangent space.* For $G \in \mathrm{FGrp}_k$, we define its cotangent space as

$$t_G^*(k) \coloneqq I_G/\overline{I_G^2}.$$

The obvious Frobenius vanishes, and the remaining Verschiebung makes $t_G^*(k)$ a (separated and) complete profinite topological $k\{V\}$-module[13].

The key to the proof of (3.1.22) in the connected case is the following identification:

3.5.1. **Proposition.** *The map (3.1.18.1) induces an isomorphism of topological $k\{F\}$-modules:*

$$\eta_G \colon \operatorname{coker}(\underline{M}(G) \xrightarrow{F} \underline{M}(G)) \overset{\simeq}{\rightarrow} t_G^*(k), \quad [a^W] \mapsto [a].$$

Proof. Let us use the shorthand $\underline{M}(G)/F\underline{M}(G) = \operatorname{coker}(F)$. Since $p = VF$ vanishes on $\operatorname{coker}(F)$, the D_k-module $\operatorname{coker}(F)$ becomes in fact a $k\{V\}$-module.

Let us start with the surjectivity of η_G.

Consider the Hochschild complex $C^\bullet(G, \widehat{\mathrm{CW}}) = \widehat{\mathrm{CW}}(G^\bullet)$ with continuous boundary maps ∂. It preserves the chain of subsets $B_{G^\bullet}^W \subset \widehat{\mathrm{CW}}(G^\bullet)$ by compatibility with the Verschiebung. The corresponding maps on $I_{G^\bullet}$ (3.1.17.1) are denoted by ∂_W. It is not additive in general.

[13] As the notation suggests, $t_G^*(k)$ can be identified as the topological dual $\mathrm{Hom}_{k,\text{cont}}(t_G(k), k)$, via the evaluation pairing $I_G \times \mathrm{Hom}_{k,\text{cont}}(B_G, k\varepsilon) \to k$, and exchanging V with F.

Consider another Hochschild complex $C^\bullet(G, \widehat{\mathbf{G}_a}) = B_{G^\bullet} = B_G^{\hat{\otimes}\bullet}$ with continuous boundary maps ∂_a. It is clear that $\partial_a(\overline{I^r_{G^m}}) \subset \overline{I^r_{G^{m+1}}}$.

Here are some approximation lemmata:

(o) If $a \in c + \overline{I^r_G}$, then $a -_W c := \eta_G(a^W - c^W) \in \overline{I^r_G}$ by description of the addition law of Witt covectors.

In particular, the map $\underline{M}(G) \to t^*_G(k)$ sending a^W to a is additive, hence a morphism of topological $k\{V\}$-modules. Since $t^*_G(k)$ has vanishing F-action, the map η_G is easily seen to be well-defined and s morphism of topological $k\{V\}$-modules.

(i) If $a \in I_{G^m}$ and $c \in \overline{I^r_{G^m}}$, then $\partial_W(a - c) \in \partial_W(a) - \partial_W(c) + \overline{I^{r+1}_{G^{m+1}}}$.
(ii) If $c \in \overline{I^r_{G^m}}$, then $\partial_W(c) \in \partial_a(c) + \overline{I^{r+1}_{G^{m+1}}}$. In particular, $\partial_W(\overline{I^r_{G^m}}) \subset \overline{I^r_{G^{m+1}}}$.

Using the structure theorem of the Hochschild cohomology groups of $C^\bullet(G, \widehat{\mathbf{G}_a})$, one can deduce from above the following technical approximation result:

(iii) If $b \in I_G$ such that $\partial_W(b) \in \overline{I^r_{G^2}}$ for some $r \geq 2$, then there exists $b' \in \overline{I^r_G}$ such that $\partial_W(b) \in \partial_W(b') + \overline{I^{r+1}_{G^2}}$. In particular, $\partial_W(b - b') \in \overline{I^{r+1}_{G^2}}$ by (ii).

Assume that $b \in I_G$. We want to show that $b = \eta_G(a^W)$ for some $a \in b + \overline{I^2_G}$, which amounts to finding a $a \in b + \overline{I^2_G}$ such that $a^W \in \underline{M}(G)$, i.e. $\Delta_G(a^W) = a^W \otimes 1 + 1 \otimes a^W$, or equivalently by definition of ∂, that

$$\partial_W(a) = 0.$$

This can be shown by approximation as follows.

We start with $\partial_a(b) = 1 \otimes b - \Delta_G(b) + b \otimes 1$. We have $\partial_a(b) \in \overline{I_{G^2}}$[14]. Indeed, if G is étale, then $I_G = 0$, so there is nothing to prove; if G is connected, then G is a also formal Lie group, hence $\Delta_G(b) \equiv b \otimes 1 + 1 \otimes b \bmod \overline{I^2_{G^2}}$. By applying (ii), we find $\partial_W(b) \in \partial_a(b) + \overline{I^2_{G^2}} = \overline{I^2_{G^2}}$. Applying now (i) and (iii) recursively, one can find a sequence $b_r \in \overline{I^r_G}$, $r \geq 2$ such that

$$\partial_W(b - b_2 - \cdots - b_r) \in \partial_W(b - b_2 - \cdots - b_{r-1}) - \partial_W(b_r) + \overline{I^{r+1}_{G^2}} = \overline{I^{r+1}_{G^2}}.$$

Finally, take

$$a := b - \sum_{r \geq 2} b_r.$$

Then by continuity of ∂ (hence that of ∂_W), one gets $\partial_W(a) = 0$. This proves the surjectivity of η_G.

Now, let us turn to the injectivity.

The injectivity would follow by induction from the following approximation lemma:

(iv) Let $b^W \in \underline{M}(G)$ such that $b \in \overline{I^r_G}$ for some $r \geq 2$, we want to find some $c^W \in \underline{M}(G)$ such that $b \in F(c) + \overline{I^{r+1}_G} \subset \overline{I^r_G}$. In particular, $b'^W := b^W - F(c)^W \in \underline{M}(G)$ satisfies $b' \in \overline{I^{r+1}_G}$ by the point (o) above, so that we can proceed by induction to write $b^W = \sum_{r \geq 2} F(c_r)$ such that $F(c_r) \in \overline{I^r_G}$[15].

[14]In other words, Δ_G becomes linearised as the usual addition on the cotangent space.

[15]This suffices, because for any profinite topological $W(k)\{F\}$-module M, the subspace $FM \subset M$ is closed. Indeed, if $M = \varprojlim_i M_i$ can be written as inverse limit of discrete $W(k)\{F\}$-modules, then $FM = \varprojlim_i FM_i$.

We omit the induction process, but we are going to spend a few words explaining this approximation. The belonging $b^W \in \underline{M}(G)$ means $\partial_W b = 0$. Since $b \in \overline{I_G^r}$, we have $\partial_a(b) \in \overline{I_{G^2}^{r+1}}$. The structure theorem of the Hochschild cohomology groups of $C^\bullet(G, \widehat{\mathbf{G}}_a)$ in characteristic p helps now:

- If r is not a power of p, then there exists $u \in \overline{I_{G^0}^r}$ such that $b \in \partial_a(u) + \overline{I_G^{r+1}}$. But $I_{G^0} = 0$, so $u = 0$ and $b \in \overline{I_G^{r+1}}$. In this case, simply $c \coloneqq 0$ is suitable.
- If $r = p^s$ $(s \geq 1)$, then one can write b as a finite k-linear combination of p^s-th powers up to higher terms:
$$b \in \sum_j \lambda_j x_j^{p^s} + \overline{I_G^{r+1}}, \quad \lambda_j \in k,\ x_j \in I_G.$$
Since k is a perfect field in characteristic p, every λ_j is a p^s-th power, so $b \in d^{p^s} + \overline{I_G^{r+1}}$ for some $d \in I_G$. It is clear now that one can set $c \coloneqq d^{p^{s-1}} \in I_G^{r/p}$.

This finishes the proof of (iv), whence the proof of the injectivity. □

It follows that:

- If $G \in \mathrm{FinGrp}_k$ has $F = 0$, then $\underline{M}(G) \in D_k\text{-Mod}^{\mathrm{flen}}$ has $F = 0$, and $\mathrm{ord}(G) = p^{\mathrm{length}(\underline{M}(G))}$.
- If $M \in D_k\text{-Mod}^{\mathrm{flen}}$ has $F = 0$, then $\underline{G}(M) \in \mathrm{FinGrp}_k$ and has $F = 0$, and $\mathrm{ord}(\underline{G}(M)) = p^{\mathrm{length}(M)}$.

3.5.2. **Corollary.** *The functor $\underline{M}(-)$ is exact on the connected part.*

Proof. Let $G' \to G$ be a monomorphism in FGrp_k^c. Using the identification (3.5.1), it is easy to show that $\underline{M}(G) \to \underline{M}(G')/F\underline{M}(G')$ is surjective. Since $\underline{M}(G') \in \mathrm{Top}\text{-}D_k\text{-Mod}^{W(k)\{F\}\text{-profin},c}$, this implies that $M(G) \to M(G')$ is surjective, whence the right exactness of $\underline{M}(-)$ on the connected part. □

(B). *Reduction to the case $F = 0$.* The proof of (3.1.22) in the connected case can now be finished analogously to that in the étale case. But instead of reducing to the finite case, we reduces to the case where F is nilpotent, by writing $G = \varinjlim_n \ker(F^n)$ and $M = \varprojlim_n M/F^n M$ respectively. Then one can make a dévissage by F-filtration to reduce to the case where $F = 0$.

In the case where $G \in \mathrm{FGrp}_k$ has $F = 0$, it is enough to show that the adjunction unit $\alpha\colon G \to \underline{GM}(G)$ induces an isomorphism on the tangent spaces. The latter is however true in general (without assumption $F = 0$): writing $M \coloneqq \underline{M}(G)$, by (3.5.1), we have

$$\begin{aligned}
t_G(k) &\xrightarrow{\simeq} \mathrm{Hom}_{k,\mathrm{cont}}(M/FM, k) \\
&\xleftarrow{\simeq} \mathrm{Hom}_{D_k,\mathrm{cont}}(M/FM, \ker(\widehat{\mathrm{CW}}^c(k[\varepsilon]/\varepsilon^2) \to \widehat{\mathrm{CW}}^c(k))) \\
&= \ker(\mathrm{Hom}_{D_k,\mathrm{cont}}(M/FM, \widehat{\mathrm{CW}}^c(k[\varepsilon]/\varepsilon^2)) \to \mathrm{Hom}_{D_k,\mathrm{cont}}(M/FM, \widehat{\mathrm{CW}}^c(k))) \\
&= \ker(\underline{G}(M)(k[\varepsilon]/\varepsilon^2) \to \underline{G}(M)(k)) \\
&= t_{\underline{G}(M)}(k),
\end{aligned}$$

where the second to last equality is due to the definition of functor $\underline{G}(-)$ (3.1.19.1). One checks readily that the composite identification is the one provided by the adjunction unit α.

Part 2. Rapoport–Zink spaces: the definitions and the representability

The goal of [RZ96] is to generalise Drinfeld's construction (Drinfeld's moduli problem and p-adic uniformisation by the Drinfeld half spaces) to other p-adic groups, using the moduli theory of p-divisible groups with a fixed (quasi-)isogeny[16] type.

Roughly, given (EL) data $(B, V; b, \mu; \mathcal{O}_B, \mathcal{L})$ or (PEL) data $(B, *, V, (\cdot,\cdot); b, \mu; \mathcal{O}_B, \mathcal{L})$, one can define a moduli functor $\breve{\mathcal{M}}\colon \mathrm{Nilp}_{\mathcal{O}_{\breve{E}}} \to \mathrm{Set}$ parametrising quasi-isogeneous liftings, with extra structures related to the (EL) or (PEL) data, of the p-divisible group $\mathbf{X} \in \mathrm{BT}(L)$ associated with these data. The functor $\breve{\mathcal{M}}$ turns out to be a locally Noetherian formal scheme which is formally locally of finite type over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$. The proof of the representability reduced to a universal and simpler case, the representatbility of a moduli functor $\mathcal{M}\colon \mathrm{Nilp}_{\mathcal{O}_E} \to \mathrm{Set}$, which parametrises quasi-isogeneous liftings, without extra structure, of the p-divisible group $\mathbf{X}$.

Let us say something about the field L of characteristic $p > 0$. In the (EL) and (PEL) cases, one has $L = \overline{L}$ (algebraically closed), while in the universal case, one needs to allow L to be arbitrary perfect fields in order to proceed the reduction steps.

4. Moduli of p-divisible groups

4.1. **Simple version (the universal case).** Let us start with the universal case, where no (EL) nor (PEL) data is required.

Let L be a perfect field of characteristic $p > 0$. Denote $K_0 = W(L)[\frac{1}{p}]$.

Fix a p-divisible group $\mathbf{X}$ over L (i.e. $\mathbf{X} \in \mathrm{BT}(L)$), although what we are going to see below depends only on its isogeny class.

4.1.1. **Definition.** Let $\mathrm{Nilp}_{W(L)}$ denote the category of locally Noetherian schemes over $\mathrm{Spec}(W(L))$ which are locally p-nilpotent. For any $S \in \mathrm{Nilp}_{W(L)}$, denote $\overline{S} = S_{p=0} = S \otimes_{W(L)} L$.

Let us define the moduli functor

$$\mathcal{M}\colon \mathrm{Nilp}_{W(L)} \to \mathrm{Set}$$

by

$$\mathcal{M}(S) \coloneqq \{(X, \rho) \mid X \in \mathrm{BT}(S), \rho \in \mathrm{QIsog}_{\overline{S}}(\mathbf{X}_{\overline{S}}, X_{\overline{S}})\}/\simeq,$$

where $(X_1, \rho_1) \simeq (X_2, \rho_2)$ if, by definition, the unique quasi-isogeny $\rho_{12} \in \mathrm{QIsog}_S(X_1, X_2)$ lifting $\rho_2\rho_1^{-1}$[17] comes from an actual isomorphism.

As remarked above, this moduli functor depends only on the isogeny class of $\mathbf{X} \in \mathrm{BT}(L)$.

[16]The isogenies do not establish an equivalence relation between p-divisible groups, the equivalence relation generated by them are quasi-isogenies (since the multiplication by p is an isogeny on them). Therefore, the *isogeny class* of $G \in \mathrm{BT}(S)$, by which we mean the smallest subclass of $\mathrm{BT}(S)$ containing G and "connected" by isogenies, is the same as the *quasi-isogeny class* of G, the subclass of $\mathrm{BT}(S)$ of all the objects quasi-isogeneous to G. For this reason, the isogeny class and the quasi-isogeny class mean the same class.

[17]The existence and the uniqueness follow from the rigidity of quasi-isogenies along locally nilpotent thickenings that are p-nilpotent.

4.1.2. **Definition.** The p-divisible group $\mathbf{X} \in \mathrm{BT}(L)$ is called *decent* if the associated isocrystal $(N, \Phi) \in \mathrm{Isoc}_{K_0}$ has N generated as a K_0-vector space by $\{n \in N \mid \Phi^s n = p^r n \text{ for some integers } n \geq 0 \text{ and } r > 0\}$[18].

4.1.3. **Example.** If $L = \overline{L}$, then $\mathbf{X} \in \mathrm{BT}(L)$ is always decent.

4.1.4. **Theorem.** *If $\mathbf{X} \in \mathrm{BT}(L)$ is decent, then the functor $\mathcal{M}$ is a locally Noetherian formal scheme which is formally locally of finite type over* $\mathrm{Spf}(W(L))$.

The decency is important in the proof, but it is no harm to the representatbility of $\breve{\mathcal{M}}$, which we shall now study.

5. Extra data

To obtain p-adic uniformisation of Shimura varieties, one needs to consider moduli problem with extra structures, such as (EL) data and (PEL) data, which are integral analogues of their Shimurian counterparts, the parametrisation of abelian varieties with extra structures. We shall introduce them progressively.

By Serre-Tate theorem, in positive characteristic, deforming an abelian variety A_0 over L is equivalent to deforming its p-divisible group $A_0[p^\infty]$, and the extra structures imposed on abelian varieties can in fact already be described in terms of their p-divisible groups. For example, the structures appear already as linear algebraic data (of a crystal) on the first crystalline cohomology group $H^1_{\mathrm{crys}}(A_0/W(L)) \simeq \mathbf{D}(A_0[p^\infty])_{W(L)}$, where the latter is the crystal given by the Grothendieck-Messing theory; when the abelian variety has an integral lift A over O_K with K having residue field L, one obtains moreover a Hodge filtration on the K-vector space $H^1_{\mathrm{crys}}(A_0/W(L)) \otimes_{W(L)} K \simeq \mathbf{D}(A_0[p^\infty]) \otimes_{W(L)} K$. As we will see, the data that we are going to introduce are also linear algebraic data on similar things, i.e. on filtered isocrystals.

From now on, let $L = \overline{L}$ be an algebraically closed field of characteristic $p > 0$. Denote $K_0 = W(L)[\frac{1}{p}]$.

5.1. Endomorphisms.
The data L is not relevant here.

5.1.1. **Data.** Let B be a finite semisimple algebra over $\mathbf{Q}_p$ and let V be a finite left B-module[19]

Let us elaborate more on the structure of the semisimple $\mathbf{Q}_p$-algebra B. Any such B can be written as a finite product

$$B = \prod_{i \in I} B_i$$

of simple $\mathbf{Q}_p$-algebras B_i. Each B_i is isomorphic to $M_{n_i}(D_i)$ for some integer $n_i > 0$ and some finite-dimensional division algebra D_i over $\mathbf{Q}_p$.Hence, the center F_i of D_i is a finite field extension of $\mathbf{Q}_p$, and the center of B is

$$F = \prod_{i \in I} F_i.$$

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[18]This is stronger than (N, Φ) being isotypial (i.e. isoclinic). The latter means that there exists $W(L)$-lattice $M \subset N$ such that $\Phi^s M = p^r M$ as sub-$W(L)$-modules of V for some integers $n \geq 0$ and $r > 0$, but it does not say anything about generation by Forbenius eigenvectors.

[19]In other words, the V is a finite-dimensional $\mathbf{Q}_p$-vector space with a $\mathbf{Q}_p$-linear left B-action.

5.1.2. **Example.** Let D be a central division algebra over $\mathbf{Q}_p$ with invariant $\frac{1}{d}$; hence $\dim_{\mathbf{Q}_p} D = d^2$. For $n \in \mathbf{N}$, we can form the simple $\mathbf{Q}_p$-algebra $B := M_n(D)$ acting on the left D-vector space $V := D^{\oplus n} \simeq \mathbf{Q}_p^{\oplus nd}$.

5.2. **Polarisation.** Let (B, V) be data as above. Let us introduce more data. (The L is still irrelevant.)

5.2.1. **Data.** Consider a $\mathbf{Q}_p$-linear anti-involution (i.e. a ring homomorphism such that $(-)^{**} = \mathrm{id}$)

$$(-)^*\colon B \to B^{\mathrm{op}}$$

and a non-degenerate antisymmetric $\mathbf{Q}_p$-bilinear form on V:

$$(\cdot,\cdot)\colon B \otimes_{\mathbf{Q}_p} B \to \mathbf{Q}_p,$$

such that the following holds:

$$(bv, v') = (v, b^* v'), \quad \forall v, v' \in V.$$

Let us introduce some abbreviations:

- "E" stands for endomorphisms. By "(E) case", we will refer to the data (B, V) without polarisation.
- "P" stands for polarisation. By "(PE) case", we will refer the data $(B, *, V, (\cdot,\cdot))$ with the polarisation included.

5.2.2. From the above data, let us consider the following group functors G on $\mathbf{Q}_p$-algebras.

- In the (E) case, consider the group scheme
$$G := \mathrm{GL}_B(V),$$
with functor of points on $\mathbf{Q}_p$-algebras R:
$$G(R) := \mathrm{GL}_{B \otimes_{\mathbf{Q}_p} R}(V \otimes_{\mathbf{Q}_p} R).$$
This G is an algebraic group over $\mathbf{Q}_p$.
- In the (PE) case, let
$$G := \mathrm{GSp}_B(V) \hookrightarrow \mathrm{GL}_B(V),$$
with functor of points on $\mathbf{Q}_p$-algebras R:
$$G(R) = \{g \in \mathrm{GL}_{B \otimes_{\mathbf{Q}_p} R}(V \otimes_{\mathbf{Q}_p} R) \mid \exists c(g) \in R^\times, \forall v, v' \in V, (gv, gv') = c(g)(v, v')\}.$$
This G is also an algebraic group over $\mathbf{Q}_p$, and one gets a morphism $c\colon G \to \mathbf{G}_m^\times$ of algebraic groups over $\mathbf{Q}_p$, called the associated *similitude factor*.

5.3. **Filtered isocrystals.** The L become relevant, as the base of the isocrystals. This part of the data is p-adic Hodge-theoretic, associated with p-divisible groups via the Grothendieck-Messing theory (and vice versa by Fontaine's conjecture, proved by Breuil and completely by Kisin).

Let us start with the crystalline data.

5.3.1. **Data.** Let $b \in G(K_0)$ be an element.

We have $b \in \mathrm{GL}_{K_0}(V \otimes_{\mathbf{Q}_p} K_0)$, hence one gets an isocrystal over L:

$$(N, \Phi) \coloneqq (V \otimes_{\mathbf{Q}_p} K_0, b(\mathrm{id} \otimes \sigma)).$$

By the definition of G, the action of b commutes with the B-action, hence endowing (N, Φ) with a B-action.

Moreover, similarly, in the (PE) case, (N, Φ) inherits a K_0-bilinear pairing

$$\psi \colon N \otimes_{K_0} N \to \mathbf{1}(n),$$

where $\mathbf{1}(n)$ is the one-dimensional isocrystal $(K_0, c(b)\sigma)$ over L, where the integer n is given by

$$n = v_p(c(b)).$$

5.3.2**. Requirement.** In the (PE) case, for the purpose of studying moduli of p-divisible group, we require that

$$v_p(c(b)) = 1.$$

Now we turn to the de Rham data (i.e. a filtration after a possibly ramified field extension).

Recall that G has been defined in the (E) or (PE) case; see (5.2.2).

5.3.3**. Data.** Let $\mu \colon \mathbf{G}_m \to G_{\overline{\mathbf{Q}_p}}$ be a cocharacter; actually, we consider its conjugacy class.

Let E be the *reflex field* of μ, i.e. the field of definition of the conjugacy class of μ; then E is a finite extension of $\mathbf{Q}_p$. Denote the composite field

$$\breve{E} \coloneqq EK_0.$$

5.3.4**. Data.** Let K be a finite extension of K_0 containing $\breve{E}$, i.e. such that $\mu_{\overline{K_0}}$ is defined over K.

The size of K is not essential once the following (weak) admissibility condition holds (see below); indeed, if K satisfies this condition, then so will any finite extension of K do[20].

The cocharacter μ induces a direct sum decomposition of K-vector spaces:

$$V_K (\coloneqq V \otimes_{\mathbf{Q}_p} K) = \bigoplus_{i \in \mathbf{Z}} V_{K,i},$$

where $V_{K,i}$ is the weight i eigenspace of the representation of $\mathbf{G}_m$ induced by the cocharacter μ. Again, by the definition of G, this decomposition respects the B-action, so that it is a direct sum of left B-modules. Consider the decreasing filtration

$$\mathrm{Fil}^\bullet V_K \coloneqq \bigoplus_{i \geq \bullet} V_{K,i},$$

which is a filtration by left sub-B-modules. The $i \in \mathbf{Z}$ such that $\mathrm{Fil}^i \neq 0$ are called the Hodge–Tate weights of the filtration.

5.3.5**. Requirement.** For the moduli of p-divisible group, we require that the filtered isocrystal $(N, \Phi, \mathrm{Fil}^\bullet V_K)$ over K is weakly admissible[21] with Hodge–Tate weights in $\{0, 1\}$.

[20] cf. if a $\mathbf{Q}_p$-representation of $\mathrm{Gal}(\overline{K_0}/K)$ is crystalline with Hodge-Tate weights in $\{0,1\}$, then so is it after restricting to the open subgroup $\mathrm{Gal}(\overline{K_0}/K')$ for any finite extension K' of K.

[21] Roughly speaking, weakly admissibility means that the Newton polygon of (N, Φ) lies above the Hodge polygon of $\mathrm{Fil}^\bullet V_K$ and shares the same end points.

In particular, the (Newton) slopes of the isocrystal (N, Φ) lies in $[0, 1]$.

The weakly admissibility is implied by the admissibility; the latter means that $(N, \Phi, \mathrm{Fil}^\bullet V_K)$ arises from a crystalline p-adic Galois representation V of $\mathrm{Gal}(\overline{K_0}/K)$. The fact that they are in fact equivalent is a theorem of Colmez–Fontaine. Moreover, by a theorem of Breuil, and of Kisin in general, the admissibility combined with the above Hodge–Tate weights condition shows that there exists a unique p-divisible $X \in \mathrm{BT}(O_K)$ (up to quasi-isogeny) such that

$$\begin{aligned}(N, \Phi, \mathrm{Fil}^\bullet V_K) \simeq (&\mathbf{D}(\mathbf{X})_{W(L)} \otimes_{W(L)} K_0,\\ &\mathbf{D}(\mathrm{Frob}_{\mathbf{X}})_{W(L)} \otimes_{W(L)} K_0,\\ &\mathrm{Fil}^1 \subset \mathbf{D}(\mathbf{X})_{W(L)} \otimes_{W(L)} K),\end{aligned}$$

where $\mathbf{X} \in \mathrm{BT}(L)$ is the special fibre of $X \in \mathrm{BT}(\mathcal{O}_K)$, and Fil^1 is given by

$$\mathrm{Fil}^1 = \mathrm{Lie}(X^\vee)^\vee_K.$$

Hence only the filtration part depends on the lifting X of $\mathbf{X}$, while other information depends only on its special fibre $\mathbf{X}$.

Under the weakly admissibility condition, the composition map $c \circ \mu \in \mathrm{Hom}_{\mathrm{Grp}}(\mathbf{G}_m, \mathbf{G}_m) \simeq \mathbf{Z}$ corresponds to $v_p(c(b)) \in \mathbf{Z}$, which is equal to 1 by the requirement (5.3.2).

5.4. **Rational data.** In summary, the rational data consist of an endormorphism structure together with a compatible filtered isocrystal structure.

Recall that L is a fixed algebraically closed field of characteristic p and $K_0 = W(L)[\frac{1}{p}]$.

Let us distinguish the two cases:

- In the (E) case, the rational data consist of $(B, V; b, \mu)$ satisfying the requirement (5.3.5) with a chosen finite extension K of K_0.
- In the (PE) case, the rational data consist of $(B, *, V, (\cdot, \cdot); b, \mu)$ satisfying the requirements (5.3.2) and (5.3.5) with a chosen finite extension K of K_0.

The finite field extension K/K_0 can be enlarged if needed.

As a consequence of the admissibility, we have seen that the data (b, μ) comes from a p-divisible group

$$X \in \mathrm{BT}(\mathcal{O}_K),$$

whose special fibre will be constantly used and will be denoted by

$$\mathbf{X} \in \mathrm{BT}(L).$$

5.5. **(Parahoric) level structure.** The level structure is an integral structure regarding lattices in V. To discuss them we need to choose an integral model of B.

5.5.1. **Data.** Choose a maximal order $\mathcal{O}_B$ of B.

In the (PE) case, we require moreover that $\mathcal{O}_B$ is stable under the anti-involution $(-)^*$.

5.5.2. **Example.** Let D be a finite-dimensional division algebra over $\mathbf{Q}_p$.

If $B = D$, then it has a unique maximal order $\mathcal{O}_D$, with a uniformiser $\Pi \in \mathcal{O}_D$.

If $B = M_n(D)$, then any unique maximal order $\mathcal{O}_B$ is conjugate with the order $M_n(\mathcal{O}_D)$.

Before talking about the (parahoric) level structure in general, we look at the case of a trivial level structure.

5.5.3. **Definition.** An $\mathcal{O}_B$*-lattice* of V is a left sub-$\mathcal{O}_B$-module $\Lambda \subset V$ such that $\Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = V$.

5.5.4**. Data.** Choose an $\mathcal{O}_B$-lattice $\Lambda \subset V$, i.e. a left sub-$\mathcal{O}_B$-module such that $\Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = V$.

In the (PE) case, we require additionally Λ to be self-dual with respect to the non-degenerate pairing $(\cdot, \cdot)$, i.e. that the dual $\Lambda^\vee \hookrightarrow V^\vee \simeq V$ is identified with Λ.

In this case, the stabliser of Λ is an integral model over $\mathbf{Z}_p$ of the G.

5.5.5. **Example.** Assume $B = \mathbf{Q}_p$. Then up to conjugation we have $\Lambda = \mathbf{Z}_p^{\oplus 2} \subset V = \mathbf{Q}_p^{\oplus 2}$, hence the stabliser of Λ is $\mathcal{GL}_2$ over $\mathbf{Z}_p$, with $\mathbf{Z}_p$-points $\mathrm{GL}_2(\mathbf{Z}_p)$.

However, there is another integral model $\mathcal{G}$ over $\mathbf{Z}_p$ of GL_2, which is *parahoric*, with $\mathbf{Z}_p$-points $\mathcal{G}(\mathbf{Z}_p)$ being the preimage under the map $\mathrm{GL}_2(\mathbf{Z}_p) \to \mathrm{GL}_2(\mathbf{F}_p)$ of the standard Borel $B(\mathbf{F}_p)$ (hence the terminology *parahoric*); in other words, $\mathcal{G}$ is the stabiliser of the pair of lattices $\mathbf{Z}_p \oplus p\mathbf{Z}_p \subset \mathbf{Z}_p^{\oplus 2}$. Indeed, $\mathcal{G}$ is an integral model of GL_2 because the chain of lattices $\mathbf{Z}_p \oplus p\mathbf{Z}_p \subset \mathbf{Z}_p^{\oplus 2}$ becomes identified with $\mathbf{Q}_p^{\oplus 2}$ after base changing to $\mathbf{Q}_p$, hence $\mathcal{G}$, the stabiliser of this chain, becomes identified with the stabiliser of the single $\mathbf{Q}_p^{\oplus 2}$, which is GL_2.

Briefly, the above discussion shows that, the chain of lattices $\mathbf{Z}_p \oplus p\mathbf{Z}_p \subset \mathbf{Z}_p^{\oplus 2}$ seems to refine the single lattice $\mathbf{Z}_p^{\oplus 2}$, giving $\mathcal{G}$ which seems finer than $\mathcal{GL}_2$, while they both become identified with GL_2 after base changing to over $\mathbf{Q}_p$. Hence they provide two different level structures for GL_2.

The level structure in general will then be a generalisation of the trivial level structure (5.5.4) and the above example (5.5.5).

5.5.6**. Data.** Choose a *parahoric* level structure, i.e. a (multi-)chain[22] $\mathcal{L}$ of $\mathcal{O}_B$-lattices of V that is closed under the $N_{B^\times}(O_B)$-action. Here,

$$N_{B^\times}(O_B) \subset B^\times$$

is the subset of elements normalising $\mathcal{O}_B$[23]

In the (PE) case, we ask $\mathcal{L}$ to be self-dual, i.e. $\Lambda \in \mathcal{L} \implies \Lambda^\vee \in \mathcal{L}$ (via the identificaiton $V^\vee \simeq V$).

If $B = M_n(D)$, $\mathcal{O}_B = M_n(\mathcal{O}_D)$ and $\Pi \in \mathcal{O}_D$ is a uniformiser, then $N_{B^\times}(O_B) = D^\times \cdot \mathrm{GL}_n(\mathcal{O}_D) = \Pi^{\mathbf{Z}} \cdot \mathcal{O}_B^\times$; the closedness of a (multi-)chain $\mathcal{L}$ of $\mathcal{O}_B$-lattices of V under the $N_{B^\times}(O_B)$-action translates to the condition:

$$\Lambda \in \mathcal{L} \implies \Pi^{\pm 1}\Lambda \in \mathcal{L}.$$

[22]A chain is a totally ordered set. Here, if B is simple, we consider the chain of lattices, where the order is given by inclusions; if $B = \prod_i B_i$ is semisimple, then $V = \prod_i V_i$ we consider the so-called *multi-chain* of lattices, which means that we are given a chain $\mathcal{L}_i$ on each V_i, and that we are taking all lattices whose projection to each factor V_i lies in $\mathcal{L}_i$.

[23]Otherwise, it makes few sense to talk about the stability/closedness.

In particular, the chain must be infinite.

5.5.7. **Example.** Assume $B = \mathbf{Q}_p$ and $V = \mathbf{Q}_p^{\oplus}2$. We have a (trivial) chain of lattices

$$\cdots \subset p\mathbf{Z}_p^{\oplus 2} \subset \mathbf{Z}_p^{\oplus 2} \subset p^{-1}\mathbf{Z}_p^{\oplus 2} \subset \cdots,$$

and a (non-trivial) parahoric chain of lattices

$$\cdots \subset p\mathbf{Z}_p^{\oplus 2} \subset \mathbf{Z}_p \oplus p\mathbf{Z}_p \subset \mathbf{Z}_p^{\oplus 2} \subset \cdots.$$

They are both infinite paths on the Bruhat-Tits tree for $\mathrm{GL}_{2,\mathbf{Q}_p}$: the second chain records all the vertices while the first keeps only half of them.

5.6. **Integral data.** Now we are ready to add integral data to the previous rational data. We are adding the letter "L" for level structures.

Let us distinguish the two cases:

- The (EL) data consist of $(B, V; b, \mu; \mathcal{O}_B, \mathcal{L})$ satisfying the requirement (5.3.5) with a chosen finite extension K of K_0, and such that $\mathcal{O}_B$ is stable under $(-)^*$.
- The (PEL) data consist of $(B, *, V, (\cdot,\cdot); b, \mu; \mathcal{O}_B, \mathcal{L})$ satisfying the requirements (5.3.2) and (5.3.5) with a chosen finite extension K of K_0, such that $\mathcal{O}_B$ is stable under $(-)^*$, and such that $\mathcal{L}$ is self-dual.

6. Moduli of p-divisible groups

6.1. **(EL) and (PEL) cases.** Let L be an algebraically closed field of characteristic p.

Given a set of (EL) data $(B, V; b, \mu; \mathcal{O}_B, \mathcal{L})$ or (PEL) data $(B, *, V, (\cdot,\cdot); b, \mu; \mathcal{O}_B, \mathcal{L})$, with associated p-divisible group $X \in \mathrm{BT}(\mathcal{O}_K)$ and its special fibre $\mathbf{X} \in \mathrm{BT}(L)$.

In the (PE) case, the pairing on V induces a pairing ψ on $\mathbf{X}$, hence a quasi-isogeny $\lambda_\psi \colon \mathbf{X} \to \mathbf{X}^\vee$.

Let E be the reflex field of (the conjugacy class of) the cocharacter μ.

Recall our diagram of field extensions:

$$\begin{array}{ccccc} K & \text{——} & \breve{E} = EK_0 & \text{——} & K_0 = W(L)[\frac{1}{p}] \\ & & | & & | \\ & & E & \text{————} & \mathbf{Q}_p, \end{array}$$

where the top row has residue field $L = \overline{L}$, and both rows are finite extensions.

6.1.1. **Definition.** Given (EL) or (PEL) data as above, let us define the moduli functor

$$\breve{\mathcal{M}} \colon \mathrm{Nilp}_{\mathcal{O}_{\breve{E}}} \to \mathrm{Set}$$

by

$$\breve{\mathcal{M}}(S) \coloneqq \{(X_\Lambda, \rho_\Lambda)_{\Lambda \in \mathcal{L}} \mid X_\Lambda \in \mathrm{BT}_{\mathcal{O}_B}(S), \rho_\Lambda \in \mathrm{QIsog}_{\overline{S},\mathcal{O}_B}(\mathbf{X}_{\overline{S}}, (X_\Lambda)_{\overline{S}}) + \text{conditions (i)–(v)}\}/_{\simeq},$$

where $(X_1, \rho_1) \simeq (X_2, \rho_2)$ as in the definition (4.1.1).

Before stating the conditions, let us make the following notations:

- For any $X_\Lambda \in \mathrm{BT}_{\mathcal{O}_B}(S)$, denote by EX_Λ its universal vector extension and then
$$M_\Lambda \coloneqq \mathrm{Lie}(EX_\Lambda),$$
which is a finite locally free $\mathcal{O}_S$-module over S. There is a short exact sequence of finite locally free $\mathcal{O}_S$-modules:
$$0 \to \mathrm{Lie}(X_\Lambda^\vee)^\vee \to M_\Lambda \to \mathrm{Lie}(X_\Lambda) \to 0.$$
- Denote by
$$\rho_{\Lambda,\Lambda'} \in \mathrm{QIsog}_{S,\mathcal{O}_B}(X_\Lambda, X_{\Lambda'})$$
the unique quasi-isogeny lifting $\rho_{\Lambda'}\rho_\Lambda^{-1} \in \mathrm{QIsog}_{\overline{S},\mathcal{O}_B}((X_\Lambda)_{\overline{S}}, (X_{\Lambda'})_{\overline{S}})$. By abuse of notation, we denote also by
$$\rho_{\Lambda,\Lambda'} \colon M_\Lambda \to M_{\Lambda'}$$
the corresponding map.
- For any $a \in N_{B^\times}(O_B)$, let us denote by
$$(X_\Lambda)^a \in \mathrm{BT}_{\mathcal{O}_B}(S)$$
the same p-divisible group $X_\Lambda \in \mathrm{BT}(S)$ but with a-conjugated $\mathcal{O}_B$-action, i.e. via the composition
$$\mathcal{O}_B \xrightarrow{a(\cdot)a^{-1}} \mathcal{O}_B \to \mathrm{End}(X_\Lambda).$$
There is then a natural quasi-isogeny
$$\theta_{a,\Lambda} \colon (X_\Lambda)^a \xrightarrow{a\cdot} X_\Lambda \xrightarrow{\rho_{\Lambda,a\Lambda}} X_{a\Lambda},$$
which is natural in $\Lambda \in \mathcal{L}$.

The definition of $\breve{\mathcal{M}}$ is still incomplete until now we finally state the conditions:

(i) For any $\Lambda \in \mathcal{L}$, M_Λ is locally isomorphic to $\Lambda \otimes_{\mathbf{Z}_p} \mathcal{O}_S$ as $\mathcal{O}_B \otimes_{\mathbf{Z}_p} \mathcal{O}_S$-modules.

(ii) For any adjacent $\Lambda \subset \Lambda'$ in $\mathcal{L}$, the cokernel of $\rho_{\Lambda,\Lambda'} \colon M_\Lambda \to M_{\Lambda'}$ is locally isomorphic to $(\Lambda'/\Lambda)\otimes_{\mathbf{Z}_p}\mathcal{O}_S$ as $\mathcal{O}_B \otimes_{\mathbf{Z}_p} \mathcal{O}_S$-modules.

(iii) For any $\Lambda \in \mathcal{L}$ and $a \in N_{B^\times}(O_B)$, the quasi-isogeny $(X_\Lambda)^a \xrightarrow{a\cdot} X_\Lambda \xrightarrow{\rho_{\Lambda,a\Lambda}} X_{a\Lambda}$ comes from an actual isomorphism.

(iv) (Kottwitz condition) For any $\Lambda \in \mathcal{L}$, we have
$$\det_{\mathcal{O}_S}(b; \mathrm{Lie}\, X_\Lambda) = \det_K(b; V_{K,0})$$
as polynomial functions of $b \in \mathcal{O}_B$ with values in $\mathcal{O}_S$[24][25].

In the (PEL) case, we need additionally:

[24] As μ is defined over E, one has $\det_K(b; V_{K,0}) \in E$. Besides, by the integrality of $b \in \mathcal{O}_B$, one has $\det_K(b; V_{K,0}) \in \mathcal{O}_K$. Therefore, $\det_K(b; V_{K,0}) \in \mathcal{O}_E$. And the determinant equality holds as section of $\mathcal{O}_S$ via the map $\mathcal{O}_E \to \Gamma(S, O_S)$, since S lies over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$.

[25] Recall that, in terms of $X \in \mathrm{BT}(\mathcal{O}_K)$ corresponding to (b, μ), we have $V_{K,0} = \mathrm{Lie}(X) \otimes_{\mathcal{O}_K} K$.

(v) There is a constant $\mathfrak{c} \in \mathbf{Q}_p^\times$ such that for any adjacent $\Lambda \subset \Lambda'$ in $\mathcal{L}$, the composite quasi-isogeny

$$(X_\Lambda)_{\overline{S}} \xleftarrow{\rho_\Lambda} \mathbf{X}_{\overline{S}} \xrightarrow{\lambda_\psi} \mathbf{X}^\vee_{\overline{S}} \xrightarrow{\rho^\vee_{\Lambda'}} (X_{\Lambda^\vee})^\vee_{\overline{S}}$$

comes from an actual isomorphism up to the constant $\mathfrak{c}$.

6.1.2. **Remark.** The functor $\breve{\mathcal{M}}$ depends only on the (homogeneously polarised) isocrystal:

$$(N, \Phi) \quad \textit{resp.} \quad (N, \Phi, \mathbf{Q}_p^\times \psi).$$

Let J be the automorphism (algebraic) group of this (homogeneously polarised) isocrystal compatible with B-action, namely, for any $\mathbf{Q}_p$-algebra R, we set

$$J(R) = \{g \in G(R \otimes_{\mathbf{Q}_p} K_0) \mid (\mathrm{id} \otimes \sigma)(g) = b^{-1} g b\}.$$

This defines a smooth affine algebraic group J over $\mathbf{Q}_p$ acting by quasi-isogenies on $\mathbf{X}$. The left action of $g \in J(\mathbf{Q}_p)$ on $\breve{\mathcal{M}}$ is as follows:

$$g \cdot (X_\Lambda, \rho_\Lambda) = (X_\Lambda, \rho_\Lambda \circ g^{-1}).$$

Let us comment on the conditions (i)–(v).

6.1.3. **Remark.** Under the Kottwitz condition (iv), the condition (ii) is equivalent to

($\mathrm{ii}^{\mathrm{bis}}$) For any adjacent $\Lambda \subset \Lambda'$ in $\mathcal{L}$, we have

$$\mathrm{ht}(X_\Lambda \xrightarrow{\rho_{\Lambda,\Lambda'}} X_{\Lambda'}) = \log_p [\Lambda' : \Lambda].$$

6.1.4. **Remark.** The Kottwitz condition (iv) is a (formal algebraic) analogue of the following analytic statement: "the underlying set of the Shimura variety X is the $G(\mathbf{R})$-*conjugacy classes* of the cocharacter $\mu \colon \mathbf{C}^\times \to G_{\mathbf{R}}$".

6.1.5. **Remark.** The Kottwitz condition (iv) can be thought of as a rank condition (see (6.1.5.2) below), which is precisely the case when the center F of B is an unramified extension of $\mathbf{Q}_p$.

In order to state the rank condition, let us first introduce some notations. For simplicity, let us assume that $B = M_n(D)$, where D is a division algebra of degree d and invariant $\frac{s}{d}$ over its center F, which is a finite field extension of $\mathbf{Q}_p$. Let $\varpi_F \in F$ and $\Pi \in \mathcal{O}_D$ be respective uniformisers. Then there exists an unramified (cyclic) extension $\widetilde{F}$ of F of degree d with Galois group isomorphic to $\langle\sigma\rangle$, independent of s and contained in D, such that D can be written in the form of a cyclic algebra as

$$D = \widetilde{F}\{\Pi\}; \quad \Pi^d = \varpi_F^s, \quad \Pi a = \sigma(a)\Pi, \ \forall a \in \widetilde{F}. \tag{6.1.5.1}$$

Let $\widetilde{F}_0$ and F_0 be the respective maximal unramified subfield (i.e. maximal unramified subextension over $\mathbf{Q}_p$) of $\widetilde{F}$ and F. Let us summarise notations in the following diagram:

$$\begin{tikzcd}[arrows=dash]
D \arrow[d] & & \\
\widetilde{F} \arrow[r] \arrow[d] & F \arrow[d] & \\
\widetilde{F}_0 \arrow[r] & F_0 \arrow[r] & \mathbf{Q}_p,
\end{tikzcd}$$

where the horizontal rows are unramified extensions.

It is no harm to enlarge the extension K/K_0 so that $\widetilde{F}$ embeds into K. So we get a decomposition of K-vector spaces:

$$V_{K,0} = \bigoplus_{\phi\colon \widetilde{F}_0 \to K} V_{K,0}^{\phi},$$

where ϕ runs through all possible embeddings (over $\mathbf{Q}_p$) and $V_{K,0}^{\phi}$ is the ϕ-typical subspace of $V_{K,0}$, i.e. the subset where the $\widetilde{F}$-action inherited from the B-action (or the D-action via diagonal embedding) coincides with that given by ϕ. In other words, with the notation $X \in \mathrm{BT}(\mathcal{O}_K)$ corresponding to (b, μ), one can rewrite the decomposition as

$$\mathrm{Lie}^{\phi}(X) \otimes_{\mathcal{O}_K} K = \bigoplus_{\phi\colon \widetilde{F}_0 \to K} \mathrm{Lie}^{\phi}(X) \otimes_{\mathcal{O}_K} K.$$

Note that $\phi(\mathcal{O}_{\widetilde{F}_0}) \subset \mathcal{O}_E \cdot \breve{\mathbf{Z}}_p \subset \mathcal{O}_{\breve{E}} (\subset \mathcal{O}_K)$. Similarly, there is a decomposition of $\mathcal{O}_S$-modules:

$$\mathcal{O}_{\widetilde{F}_0} \otimes_{\mathbf{Z}_p} \mathcal{O}_S = \prod_{\phi\colon \widetilde{F}_0 \to K} \mathcal{O}_S.$$

Since $\mathrm{Lie}(X_\Lambda)$ is an $\mathcal{O}_{\widetilde{F}_0} \otimes_{\mathbf{Z}_p} \mathcal{O}_S$-module, we obtain a decomposition of $\mathcal{O}_S$-modules:

$$\mathrm{Lie}(X_\Lambda) = \bigoplus_{\phi\colon \widetilde{F}_0 \to K} \mathrm{Lie}^{\phi}(X_\Lambda).$$

Finally, the Kottwitz condition (iv) implies the following rank equality:

$$\mathrm{rk}_{\mathcal{O}_S}(\mathrm{Lie}^{\phi}(X_\Lambda)) = \mathrm{rk}_K(\mathrm{Lie}^{\phi}(X) \otimes_{\mathcal{O}_K} K), \quad \forall \phi\colon \widetilde{F}_0 \to K,\ \forall \Lambda \in \mathcal{L}. \tag{6.1.5.2}$$

The converse is true when F is unramified over $\mathbf{Q}_p$.

6.1.6. **Remark.** The Kottwitz condition (iv) implies the condition (i). Indeed, roughly speaking, as M_Λ is already a finite locally free $\mathcal{O}_S$-modules, it is enough to check the statement on geometric points; then it could be proved using the rank counting (6.1.5.2) in the previous remark.

The condition (i) is a rank condition, and corresponds to the *special* condition in Drinfeld's moduli problem.

6.1.7. **Example** (Drinfeld case)**.** Let F be a finite field extension of $\mathbf{Q}_p$, and let D be the unique division algebra over F with invariant $\frac{s}{d}$. Recall from (6.1.5.1) that one can write

$$D = \widetilde{F}\{\Pi\}; \quad \Pi^d = \varpi_F^s, \quad \Pi a = \sigma(a)\Pi, \ \forall a \in \widetilde{F}.$$

There is even an anti-involution on D given by

$$\Pi^* = \Pi, \quad a^* = \sigma(a), \ \forall a \in \widetilde{F},$$

but we will not consider this datum; we will be consider the (EL) case.

In the Drinfeld moduli problem, we assume $s = 1$.

For simplicity, let us assume that $F = \mathbf{Q}_p$; in particular, F is an unramified extension of $\mathbf{Q}_p$ and $\widetilde{F} = \widetilde{F}_0 = \mathbf{Q}_{p^d}$.

Endormorphism: Consider $B = D$, $V = D$ with the natural left D-action. Then

$$G = (D^{\mathrm{op}})^\times$$

as algebraic groups over $\mathbf{Q}_p$.

Isocrystal: Let $b = p\Pi^{-1} \in G(\mathbf{Q}_p) \subset G(\breve{\mathbf{Q}}_p)$. Then the associated isocrystal over $\overline{\mathbf{F}_p}$ is isoclinic of slope $\frac{d-1}{d}$, and one can show that

$$J(\mathbf{Q}_p) \simeq \mathrm{GL}_d(\mathbf{Q}_p).$$

Filtration: The field $K = \widetilde{F} \cdot \breve{\mathbf{Q}}_p \supset \widetilde{F} = \mathbf{Q}_{p^d}$ splits D, and we have

$$D_K \simeq M_d(K) \simeq \bigoplus_{\phi \in \mathrm{Gal}(\mathbf{Q}_{p^d}/\mathbf{Q}_p)} D_K^\phi$$

as K-vector spaces, where $D_K^\phi \simeq K^{\oplus d}$ as K-vector spaces but with $\widetilde{F}$-action (inherited from that of D) twisted by ϕ. The Hodge cocharacter μ is given by endowing D_K^ϕ with weight 0 if $\phi = \mathrm{id}$, and with weight 1 if $\phi \neq \mathrm{id}$. Then μ is defined over $E = \breve{\mathbf{Q}}_{p^d}$ and the Hodge–Tate weights are 0 with multiplicity d, and 1 with multiplicity $(d-1)d$.

In particular, the associated formal $\mathcal{O}_D$-module X over $L = \overline{\mathbf{F}_p}$ is *special*, i.e. its Lie algebra $\mathrm{Lie}(X)$ is a finite projective (hence free) $\mathbf{Z}_{p^d} \otimes_{\mathbf{Z}_p} W(L)$-module of rank 1.

Level structure: We take the trivial level structure given by $\Lambda = \mathcal{O}_D$.

The Kottwitz condition (iv), together with its consequence the condition (i), says exactly that $\breve{\mathcal{M}}$ parametrising the *special* formal $\mathcal{O}_D$-modules over $S \in \mathrm{Nilp}_{\breve{\mathbf{Z}}_p}$, i.e. those formal $\mathcal{O}_D$-modules X_Λ over S such that $\mathrm{Lie}(X_\Lambda)$ is a locally free $\mathcal{O}_{\mathbf{Z}_{p^d}} \otimes_{\mathbf{Z}_p} \mathcal{O}_S$-module of rank 1.

Then the Drinfeld moduli problem $\breve{\mathcal{M}}$ on $\mathrm{Nilp}_{\breve{\mathbf{Z}}_p}$ (which is of (EL) type) is $\mathrm{GL}_d(\mathbf{Q}_p)$-equivariantly represented by the formal scheme $\coprod_{h \in 2\mathbf{Z}} (\widehat{\Omega}_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p)$ over $\mathrm{Spf}(\breve{\mathbf{Z}}_p)$, which is a p-adic formal scheme[26]; here $\widehat{\Omega}_{\mathbf{Z}_p}$ denotes the Deligne's formal model of $\Omega_{\mathbf{Z}_p}$, and the index $h \in 2\mathbf{Z}$ indicates the *height* of the quasi-isogenies parametrised by the corresponding component.

[26]That this formal scheme is actually p-adic is related to the fact that there is a *unique* special formal $\mathcal{O}_D$-module over $\overline{\mathbf{F}_p}$. In general, the formal scheme representing $\breve{\mathcal{M}}$ might not have p-adic topology.

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